Differentiable manifolds – hand-in sheet 5

Hand in by 12/Dec

Exercise

Let $\mathcal{A}$ be the space of all $\mathbb{R}$-linear endomorphisms of $\Omega^\bullet(M)$,

$$\mathcal{A} = \{A : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M) | A \text{ is linear}\}.$$

We can make $\mathcal{A}$ into a $\mathbb{Z}$-graded vector space by declaring that and element of $\mathcal{A}$ has degree k if

$$A : \Omega^l(M) \longrightarrow \Omega^{l+k}(M) \qquad \forall l \in \mathbb{Z}.$$

1. Define the graded commutator in $\mathcal{A}$ as the degree zero bracket

$$\{\cdot,\cdot\} : \mathcal{A}^k \times \mathcal{A}^l \longrightarrow \mathcal{A}^{k+l}; \qquad \{A, B\} = AB + (-1)^{kl+1}BA.$$

   Show that $\mathcal{A}$ with the graded commutator is a graded Lie algebra.

2. Let d be the exterior derivative, X be a vector field and ξ be a 1-form. Show that $d \in \mathcal{A}^1$, $\xi\wedge \in \mathcal{A}^1$ and $\iota_X \in \mathcal{A}^{-1}$. Further show that $\{d, d\} = 0$.

3. Following the construction from exercise 2 from hand-in sheet 3, show that for vector fields X and Y

$$\{\iota_X, \iota_Y\}_d := \{\{\iota_X, d\}, \iota_Y\} = \iota_{[X,Y]}.$$

   Remark: The conclusion from this part is that

   - the exterior derivative determines the Lie bracket of vector fields and
   - that the Jacobi identity for the Lie bracket of vector fields is a consequence of the fact that $d^2 = 0$.

4. We can also see $\mathcal{A}$ as a $\mathbb{Z}_2$-graded vector space, where $\mathcal{A}^{ev}$ are the maps which preserve the parity of forms and $\mathcal{A}^{od}$ the ones that reverse:

$$A \in \mathcal{A}^{ev} \text{ if and only if } A : \Omega^{ev/od}(M) \longrightarrow \Omega^{ev/od}(M),$$

$$A \in \mathcal{A}^{od} \text{ if and only if } A : \Omega^{ev/od}(M) \longrightarrow \Omega^{od/ev}(M),$$

   Following the construction from exercise 2 from hand-in sheet 3, show that for vector fields X and Y and 1-forms ξ and η

$$\{\iota_X + \xi\wedge, \iota_Y + \eta\wedge\}_d := \{\{\iota_X + \xi\wedge, d\}, \iota_Y + \eta\wedge\} = \iota_{[X,Y]} + (\mathcal{L}_X\eta) \wedge -(\iota_Y d\xi) \wedge .$$

About notation: Feel free to drop the ι from the interior product and the $\wedge$ from the exterior product and just write $X\varphi$ for $\iota_X\varphi$ and $\xi\varphi$ for $\xi \wedge \varphi$.