Differentiable manifolds – Mock Exam 2

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **allowed** to consult any text book and class notes but **not allowed** to consult colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Some definitions you should know, but may have forgotten.

- An n dimensional complex manifold is a manifold whose charts take values in $\mathbb{C}^n$ and for which the change of coordinates are holomorphic maps.
- A volume form on a manifold M^n is a nowhere vanishing n-form.

Questions

1) Show that $\mathbb{CP}^n$, the set of complex lines through the origin in $\mathbb{C}^{n+1}$, can be given the structure of a complex manifold, i.e. it can be covered by charts $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{C}^n$ for which the change of coordinates

$$\varphi_\beta \circ \varphi_\alpha^{-1} : V \subset \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

are holomorphic functions on their domain of definition. .

2) Given a manifold M, the space of sections of the bundle $TM \oplus T^*M$ is endowed with the natural pairing

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}(\eta(X) + \xi(Y))$$

and a bracket (the *Courant bracket*):

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi, \qquad X, Y \in \Gamma(TM);\ \xi, \eta \in \Gamma(T^*M).$$

a) Given a 2-form $B \in \Omega^2(M)$, let L be the subbundle of $TM \oplus T^*M$ given by

$$L = \{X - i_X B : X \in TM\}.$$

Show that L is involutive with respect to the Courant bracket if and only if B is closed.

b) Show that for $X, Y, Z \in \Gamma(TM)$ and $\xi, \eta, \mu \in \Gamma(T^*M)$ we have

$$\mathcal{L}_X \langle Y + \eta, Z + \mu \rangle = \langle [\![X + \xi, Y + \eta]\!], Z + \mu \rangle + \langle Y + \eta, [\![X + \xi, Z + \mu]\!] \rangle.$$

3a) Let V be a vector space. Show that if $\dim(V) = 3$, then every homogeneous element of degree greater than zero in $\wedge^\bullet V$ is decomposable, i.e., can be written as a product of 1-forms.

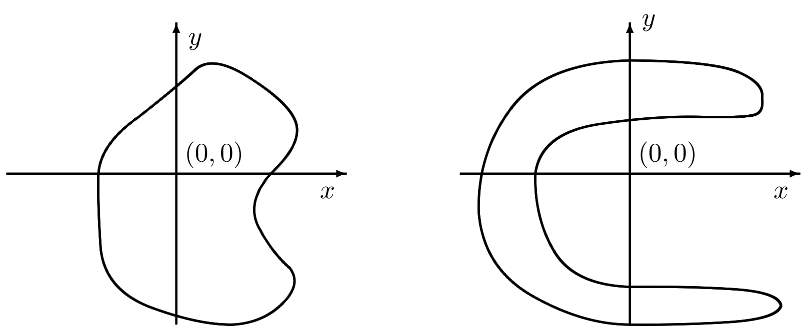
3b) Show that if $\dim(V) > 3$ there are indecomposable homogeneous elements in $\wedge^\bullet V$.

3c) Show that if $\alpha \in \wedge^k V$ is an odd form, then $\alpha \wedge \alpha = 0$. Show that if $\dim(V) > 3$, then there is $\alpha \in \wedge^2 V$ such that $\alpha \wedge \alpha \neq 0$.

4) Compute the integral of the 1-form

$$\theta = \frac{xdy - ydx}{x^2 + y^2}.$$

along the paths drawn below traced counterclockwise.



5) Compute the degree one de Rham cohomology of $\mathbb{R}^2 \backslash \{0\}$.