Differentiable manifolds – homework 11

Solve exercises 1, 2, 13, 14, 15 from Chapter 4.

Read the proof of Stokes theorem or just prove it yourself using the following steps:

Exercise 1 (Stokes Theorem)**.** Let M^n be a compact oriented manifold with boundary $\iota : \partial M \longrightarrow M$ and let $\omega \in \Omega^{n-1}(M)$. Then

$$\int_M d\omega = \int_{\partial M} \iota^* \omega.$$

1. (From global to local) Let ψ_α be a partition of unity by functions with compact support such that $\mathrm{supp}(\psi_\alpha) \Subset U_\alpha$, where U_α is the domain of a coordinate chart. By writing $\omega = \sum_\alpha \psi_\alpha \omega$ we get

$$d\omega = \sum d(\psi_\alpha \omega)$$

   Use this to show that Stokes theorem holds for all $(n-1)$-forms ω if and only if it holds for all $\psi_\alpha \omega$, that is, it is enough to prove the theorem for forms ω with compact support in a coordinate chart.

   Since M has boundary, there are two types of charts either just open sets in $\mathbb{R}^n$ or open sets in $\mathbb{R}^n_-$ which intersect $\{0\} \times \mathbb{R}^{n-1}$.

2. (Open sets in $\mathbb{R}^n$) Let $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ be a form with compact support in $\mathbb{R}^n$. Then, since $\mathbb{R}^n$ has no boundary, one of the sides of Stokes theorem simply states

$$\int_{\partial \mathbb{R}^n} \omega = 0.$$

   For the other side, write

$$\omega = \sum_i \omega_i dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots dx_n,$$

   for some functions ω_i of compact support. Then

$$d\omega = \sum_i (-1)^{i-1} \frac{\partial \omega_i}{\partial x_i} dx_1 \wedge \cdots \wedge \cdots dx_n.$$

   Now use Fubini's theorem to change the order of integration and the fundamental of calculus plus the fact that each ω_i has compact support to prove that the integral of each summand above vanishes.

3. (Open sets in $\mathbb{R}^n_-$) Let $\omega \in \Omega^{n-1}(\mathbb{R}^n_-)$ be a form with compact support in $\mathbb{R}^n_-$ intersecting $\{0\} \times \mathbb{R}^{n-1}$ and again write

$$\omega = \sum_i \omega_i dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots dx_n,$$

   for some functions ω_i of compact support. Then, since at the boundary $x_1 \equiv 0$, we get

$$\iota^* dx_1 = d\iota^* x_1 = d0 = 0$$

   and hence the pull back of ω to the boundary is

$$\iota^* \omega = \omega_1 dx_2 \wedge \cdots \wedge dx_n.$$

   And one side the equality in Stokes theorem is simply

$$\int_{\partial M} \iota^* \omega = \int_{\mathbb{R}^{n-1}} \omega_1 dx_2 \wedge \cdots \wedge dx_n.$$

   As for the other side use the same argument from previous part (Fubini's theorem and the fundamental of calculus) but now notice that the integral in the direction of the first coordinate gives a contribution which equals to one above while the remaining integrals do not contribute to the result.