Differentiable manifolds – homework 2

Exercise 1. Show that the set of diffeomorphisms of a manifold is a group if endowed with composition of functions as group operation.

Exercise 2. Let $\mathbb{R}$ denote the real numbers with their usual smooth structure. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the map $f(t) = t^3$. Show that f is a smooth bijection whose inverse is not smooth.

Exercise 3. Let M_1 and M_2 be manifolds and $p \in M_2$. Endow $M_1 \times M_2$ with the manifold structure from example (1.5 g) Show that the maps

$$M_1 \times M_2 \longrightarrow M_1; \qquad (x, y) \mapsto x;$$

$$M_1 \longrightarrow M_1 \times M_2; \qquad x \mapsto (x, p)$$

are smooth

Exercise 4. Let G be a Lie group and $g \in G$. Show that the maps

$$l_g : G \longrightarrow G; \qquad l_g(h) = gh;$$

$$r_g : G \longrightarrow G; \qquad r_g(h) = hg$$

are smooth. Conclude that these maps are in fact diffeomorphisms of G.

Exercise 5. Solve question 3 from Warner.

Exercise 6 (Question 4 from Warner)**.** A normal topological space is one for which every two disjoint closed sets have F_1 and F_2 have disjoint open neighborhoods U_1 and U_2, i.e., U_i is open, $F_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$. Show that manifolds are normal topological spaces. (Hint: use the corollary to Theorem 1.11)

Exercise 7 (Čech cochains and differential)**.** Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an locally finite open cover of a connected manifold M. For $\alpha_1, \cdots, \alpha_n \in A$, we denote

$$U_{\alpha_0 \cdots \alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k},$$

or, equivalently, in (ordered) multiindex notation, if $\mathbf{a} = \{\alpha_0, \cdots, \alpha_k\}$

$$U_{\mathbf{a}} = \bigcap_{\alpha_i \in \mathbf{a}} U_{\alpha_i}.$$

A degree *k-Čech cochain with real coeficients* for the cover $\mathfrak{U}$ is a collection of functions

$$\check{f} := \{f_{\mathbf{a}} : U_{\mathbf{a}} \longrightarrow \mathbb{R} | \mathbf{a} \subset A; \|\mathbf{a}\| = k + 1\} \tag{1}$$

such that

- $f_{\alpha_0 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_k} = -f_{\alpha_0 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_k}$
- each $f_{\mathbf{a}}$ is constant.

Denote the set of all degree k Čech cochains with real coefficients by $\check{C}^k(M; \mathbb{R}; \mathfrak{U})$. Note that pointwise addition of functions and scalar multiplication make $\check{C}^k(M; \mathbb{R}; \mathfrak{U})$ into a real vector space.

Next we define the *Čech differential* as a linear map $\delta^{k-1} : \check{C}^{k-1}(M; \mathbb{R}; \mathfrak{U}) \longrightarrow \check{C}^k(M; \mathbb{R}; \mathfrak{U})$,

$$\delta^{k-1}(\check{f})_{\alpha_0 \cdots \alpha_k} = \sum_i (-1)^i f_{\alpha_0 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_k}.$$

1. Describe the elements of $\check{C}^0$;
2. Show that the elements of $\check{C}^0$ which are in the kernel of δ^0 correspond to a single constant defined on M;
3. Show that $\delta^k \circ \delta^{k-1} = 0$ for all k.

It is standard practice in mathematics that whenever one finds a sequence of linear maps between vector spaces

$$\delta^k : V^{k-1} \longrightarrow V^k$$

with $\delta^k \circ \delta^{k-1} = 0$ one defines *cohomology spaces*:

$$H^k := \frac{\ker(\delta^k)}{\mathrm{Im}(\delta^{k-1})}.$$

In our case, these spaces depend on M and the open cover $\mathfrak{U}$, se we write:

$$\check{H}^k(M; \mathbb{R}; \mathfrak{U}) = \frac{\ker(\delta^k)}{\mathrm{Im}(\delta^{k-1})}.$$

Exercise 8. Chance the definition of Čech cocycle so that each $f_{\mathbf{a}}$ in (1) is a smooth function instead of a constant. These are the Čech cochains with coefficients in $C^\infty(M)$. Assume that $\{\psi_\alpha : \alpha \in A\}$ is a partition of unity subbordinated to the cover $\mathfrak{U}$ with the same index set.

1. Show that $\check{H}^0(M; C^\infty(M), \mathfrak{U}) = \{f : M \longrightarrow \mathbb{R} : f \in C^\infty(M)\}$.
2. Let $\check{f} \in \check{C}^k$ with $k > 0$ be such that $\delta^k \check{f} = 0$. Define $\check{g} \in \check{C}^{k-1}$ by

$$g_{\alpha_1, \cdots \alpha_{k-1}} = \sum_{\beta \in A} \psi_\beta f_{\beta \alpha_1 \cdots \alpha_{k-1}}.$$

   Show that $\delta^{k-1}\check{g} = \check{f}$, i.e., $\ker \delta^k = \mathrm{Im}\delta^{k-1}$ and hence $\check{H}^k(M; C^\infty(M); \mathfrak{U}) = \{0\}$ for $k > 0$.