Differentiable manifolds – homework 6

From previous sheets: solve exercise 9, 10 and 11 from Warner. From this time, solve exercises 16 and 18.

Exercise 1. Show that the sphere

$$S^n = \{v \in \mathbb{R}^{n+1} : \|v\| = 1\}$$

is an embedded submanifold of $\mathbb{R}^{n+1}$.

Exercise 2. Let $i : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be multiplication by the imaginary number i and for each $p \in C^n$ we let $t_p : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be translation by p, i.e.

$$t_p v = v + p.$$

For $p \in \mathbb{C}^n$ we define a map $I_p : T_p\mathbb{C}^n \longrightarrow T_p\mathbb{C}^n$ by

$$I_p v = (t_p \circ i \circ (t_p)^{-1})_*.$$

Show that $I_p^2 = -\mathrm{Id}$.

Exercise 3. We say that a function $f : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is *holomorphic* if $f_* \circ I_p = I_{f(p)} \circ f_*$. Show that if $f : U \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ is holomorphic and $p \in U$, then $f_*|_p : T_p\mathbb{C}^n \longrightarrow T_{f(p)}\mathbb{C}$ is either zero or onto. In particular, conclude that if f is holomorphic and $y \in \mathbb{C}$ is such that $f_*|_x \neq 0$ for all $x \in f^{-1}(y)$ then $f^{-1}(y)$ is an embedded submanifold of U.

Exercise 4. Let $p : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a homogeneous polynomial of degree m in three variables, i.e.,

$$p(X_0, X_1, X_2) = \sum_{i+j+k=m} a_{ijk} X_0^i X_1^j X_2^k.$$

Let $\Sigma \subset \mathbb{R}P^n$ be the set defined by the zeros of p, i.e.,

$$\Sigma = \{[X_0, X_1, X_2] \in \mathbb{R}P^2 : p(X_0, X_1, X_2) = 0\}.$$

1. Show that Σ is indeed a well defined subset of $\mathbb{R}P^2$, i.e., if two points are in the same line through the origin, then either both are zeros of p or neither is a zero of p.

2. Show that if the system
$$\begin{cases} p(X_0, X_1, X_2) & = 0, \\ \frac{\partial p}{\partial X_0}(X_0, X_1, X_2) & = 0, \\ \frac{\partial p}{\partial X_1}(X_0, X_1, X_2) & = 0, \\ \frac{\partial p}{\partial X_2}(X_0, X_1, X_2) & = 0, \end{cases}$$
has no solutions, then Σ is an embedded submanifold of $\mathbb{R}P^2$.

3. Extend this result to $n+1$ variables, namely, define $\Sigma \subset \mathbb{R}P^n$ as the zero set of a degree m homogeneous polynomial and find a condition for smoothness of Σ.

4. Extend this result to $n+1$ complex variables, namely, define $\Sigma \subset \mathbb{C}P^n$ as the zero set of a degree m homogeneous polynomial with complex coefficients and find a condition for smoothness of Σ