Differentiable manifolds – hand-in sheet 5

Hand in by 22/Jan

Before solving the exercise below, recall the definitions and results from hand-in exercise sheet 2.

Exercise 1. Let M be a manifold, $\Omega^\bullet(M)$ be the (infinite dimensional) graded vector space of smooth forms on M and $\mathcal{A}$ be the set of all $\mathbb{R}$-linear endomorphisms of $\Omega^\bullet(M)$, i.e., elements of $\mathcal{A}$ are $\mathbb{R}$-linear maps which send forms to forms. Examples of elements of $\mathcal{A}$ are

- Given a vector field $X \in \mathfrak{X}(M)$, interior product by X, $\iota_X \in \mathcal{A}$, $\iota_X : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ for all k;
- Given a 1-form ξ, exterior product by ξ is in $\mathcal{A}$, $\xi\wedge : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, for all k;
- The exterior derivative d is an element of $\mathcal{A}$, $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, for all k.

We can introduce a grading in $\mathcal{A}$. Namely, we declare that an element $\alpha \in \mathcal{A}$ has degree l if $\alpha : \Omega^k(M) \longrightarrow \Omega^{k+l}(M)$ for all k, so the elements introduced above have degree -1, 1 and 1, respectively.

We introduce a bracket in $\mathcal{A}$ as follows. For $\alpha \in \mathcal{A}^l$, $\beta \in \mathcal{A}^m$, we define

$$[\alpha, \beta] = \alpha\beta + (-1)^{lm+1}\beta\alpha.$$

This is called the graded commutator of α and β and due the results in hand-in sheet 2, $(\mathcal{A}^\bullet, [\cdot,\cdot])$ is a graded Lie algebra with a bracket of degree 0.

1. Show that $[d, d] = 0$ and hence (from hand-in sheet 2) the *derived bracket*

$$[\![\alpha, \beta]\!] := [[\alpha, d], \beta] \tag{1}$$

   satisfies Jacobi.

2. For $X, Y \in \mathfrak{X}(M)$, show that $[\![X, Y]\!]$ is just the Lie bracket between the vector fields X and Y and hence $\mathfrak{X}(M)$ is closed with respect to the derived bracket. Conclude that the condition $d^2 = 0$ implies that $\mathfrak{X}(M)$ is a Lie algebra;

3. For $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$ show that

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi$$

4. Let $\langle\cdot,\cdot\rangle : TM \oplus T^*M \longrightarrow \mathbb{R}$ be the natural symmetric pairing corresponding to evaluation of forms on vectors:

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\eta(X) + \xi(Y)), \qquad X, Y \in T_pM,\ \xi, \eta \in T_p^*M.$$

   For $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$ compute

$$[\![X + \xi, Y + \eta]\!] + [\![Y + \eta, X + \xi]\!].$$

5. Let L be an isotropic subbundle of $TM \oplus T^*M$, i.e., if $X + \xi, Y + \eta \in L_p$, then $\langle X + \xi, Y + \eta\rangle = 0$ (for all $p \in M$). Conclude that if L is involutive with respect to the bracket (1), then the space of sections of L is a Lie algebra.