Geometry and Topology – Extra hand-in sheet

Hand in by 13/June
Worth up to 0.5 points extra in the final mark of the course.

Prelude to the exercise

Definition 1. A map $p : E \longrightarrow B$ has the *homotopy lifting property with respect to a space* X if given a homotopy $H : I \times X \longrightarrow B$ and a map $\tilde{h} : \{0\} \times X \longrightarrow E$ such that the following diagram commutes

$$\begin{array}{ccc} & & E \\ & \overset{\tilde{h}}{\nearrow} & \big\downarrow p \\ \{0\} \times X & \xrightarrow{H|_{\{0\}\times X}} & B \end{array}$$

then there is a map $\tilde{H} : I \times X \longrightarrow E$ which lifts H and agrees with $\tilde{h}$ on $\{0\} \times X$, i.e., the following diagram commutes:

$$\begin{array}{ccc} & & E \\ & \overset{\tilde{H}}{\nearrow} & \big\downarrow p \\ I \times X & \xrightarrow{H} & B \end{array}$$

Exercise 2. Assume that $p : E \longrightarrow B$ has the homotopy lifting property with respect to $D^k \cong I^k$ for all $k \geq 0$. Assume further that E and B are path connected. Let $x_0 \in E$, $b_0 = p(x_0)$ and $F = p^{-1}(b_0)$. Show that there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(F, x_0) \xrightarrow{\iota_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, x_0) \xrightarrow{\iota_*} \cdots,$$

where $\iota : F \longrightarrow E$ is the natural inclusion. Part of the exercise is to define the map $\pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, x_0)$.

Hint: Often you will have a homotopy, say $H : I \times I \times I \longrightarrow B$ for which you can find by hand a lift $\tilde{h}$ to all but one of the faces of $\partial(I \times I \times I)$. Argue that since the sides of the cube with one face removed is homotopic to the disc, the homotopy lifting property property can be used to find a lift $\tilde{H}$ of H to the whole cube which agrees with $\tilde{h}$ where both maps are defined.