Topology and Geometry – exercise sheet 9

Solve exercises 1, 3, 4, 5, 11 and 14 from Section 2.1 in Hatcher's book.

Exercise 1 (Homology with coefficients in an Abelian group)**.** Let G be an Abelian group and X be a Δ-complex. In the definition of simplicial homology observe that one can take the space of n-chains to be finite formal sums of the form $\sum_\alpha n_\alpha e^n_\alpha$, with $n_\alpha \in G$. Define an appropriate boundary operator ∂ and show that $\partial^2 = 0$. Conclude that one can define homology with any Abelian group as coefficients: $H_*(X; G)$.

Exercise 2 (Homology with coefficients in an Abelian group II)**.** In the exercise above, show that if G is a ring then $H_n(X; G)$ is a G-module. Conclude that if G is a field then both the space of n-chains and the corresponding homology group, $H_n(X; G)$, are G-vector spaces. In particular if $G = \mathbb{R}$, $H_n(X, \mathbb{R})$ is a real vector space.

Exercise 3. Let V_i, $i = 0, \cdots, d$ be finite dimensional real vector spaces and let $\partial_i : V_i \longrightarrow V_{i-1}$ be linear maps such that $\partial_i \circ \partial_{i+1} = 0$ for all i. Define the corresponding "homology groups" by

$$H_i = \frac{\ker(\partial_i)}{\operatorname{im}(\partial_{i+1})}.$$

Show that

$$\sum (-1)^i \dim(V_i) = \sum (-1)^i \dim H_i.$$

Exercise 4 (Classification of surfaces revisited)**.**

1. Show that every compact surface without boundary can be realized as a Δ-complex.
2. For each such surface compute its homology (with respect to a fixed realization as a Δ-complex of your choosing) with $\mathbb{Z}$ and with $\mathbb{R}$ coefficients (use the classification theorem). A few notable things emerge from these computations:
   - No two nondiffeomorphic surfaces have the same homology group $H_1(\Sigma, \mathbb{Z})$.
   - For the sphere and for connected sums of tori $H_2(\Sigma; \mathbb{Z}) = \mathbb{Z}$ and $H_2(\Sigma; \mathbb{R}) = \mathbb{R}$ while if Σ is a connected sum of projective spaces $H_2(\Sigma, \mathbb{Z}) = \{0\}$ and $H_2(\Sigma; \mathbb{R}) = \{0\}$.
   - In all cases $H_0(\Sigma, \mathbb{Z}) = \mathbb{Z}$.
3. We define the Euler characteristic of a Δ-complex, X, to be the alternating sum

   $$\chi_X = \sum (-1)^i \dim H_i(X, \mathbb{R}).$$

   Show that if $\Sigma = \#g\, T^2$, then

   $$\chi_\Sigma = 2 - 2g$$

   and that this still holds if $g = 0$, i.e., if Σ is a sphere.
4. Show that if $\Sigma = \#g\, \mathbb{R}P^2$, then

   $$\chi_\Sigma = 2 - g$$
5. Conclude that the numbers χ_Σ and $\dim(H^2(\Sigma; \mathbb{R}))$ fully determine the diffeomorphism type of Σ.
6. Use Exercise 3 to conclude that the Euler characteristic of a compact surface realized as a Δ-complex is

   $$\chi_\Sigma = V - E + F,$$

   where V is the number of vertices (0-simplices), E is the number of edges (1-simplices) and F the number of faces (2-simplices) in the Δ-complex.