

Geometry and Topology – hand-in sheet 2

Hand in by 3/March

Exercise 1. A *semigroup* is a set X endowed with a map $m : X \times X \rightarrow X$ and an element $e \in X$ such that $m(e, x) = m(x, e) = x$ for all x in X . A *topological semigroup* is topological space which is a semigroup and for which the multiplication m is continuous. Following the steps below or otherwise prove that if X is a topological semigroup, then $\pi_1(X, e)$ is an Abelian group.

- Define an operation on loops based at e by

$$\gamma_1 \tilde{\star} \gamma_2(t) := m(\gamma_1(t), \gamma_2(t)), \quad \text{for all } \gamma_i : (I, \partial I) \rightarrow (X, e).$$

Show that if γ'_i is homotopic to γ_i as loops based at e then $\gamma_1 \tilde{\star} \gamma_2$ is homotopic to $\gamma'_1 \tilde{\star} \gamma'_2$. Conclude that $\tilde{\star}$ defines an operation on $\pi_1(X, e)$:

$$\tilde{\star} : \pi_1(X, e) \times \pi_1(X, e) \rightarrow \pi_1(X, e).$$

- Letting \star denote concatenation of paths and e denote the constant loop, use that $\gamma_1 \simeq \gamma_1 \star e$ and $\gamma_2 \simeq e \star \gamma_2$ to conclude that $\tilde{\star}$ agrees with the usual product on $\pi_1(X, e)$.
- Using that $\gamma_1 \simeq e \star \gamma_1$ and $\gamma_2 \simeq \gamma_2 \star e$, conclude that $\pi_1(X, e)$ is Abelian.