Group theory – Hand in sheet 1

1) *We have seen in Lectures that for any* $n \in \mathbb{N}$, *the set* $\mathbb{Z}_n$ *of remainders of division by* n *is a group. Namely, for* $a, b \in Z$, *if we denote by* $[a]_n$ *the positive remainder of the division of* a *by* n, *the group operation is given by*

$$[a]_n + [b]_n := [a+b]_n.$$

a – **1 point**) *Show that the map* $\cdot : \mathbb{Z}_n \times \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$, *given by*

$$[a]_n \cdot [b]_n = [a \cdot b]_n$$

is also well defined.

b – **1 point**) *Is* $\cdot : \mathbb{Z}_n \backslash \{[0]_n\} \times \mathbb{Z}_n \backslash \{[0]_n\}$ *a group operation on* $\mathbb{Z}_n \backslash \{[0]_n\}$*? If not, give an example where it fails to be a group operation and find conditions when it is?*

2 – **2 points**) *Let* $n > 2$ *and define a group by the following generators and relations:*

$$G = \langle a, b : a^n = b^2 = e; bab^{-1} = a^k \rangle$$

Show that if $k^2 \neq [1]_n$ *then these relations imply that the order of* a *is less than* n.

Remark: The point of this exercise is to show that just writing a bunch of relations may not automatically lead to a group with the desired order. In this case, if $k^2 \neq 1 \mod n$ the group will have less than $2n$ elements.

3) *Recall that an automorphism of a group* G *is an isomorphism* $\varphi : G \longrightarrow G$ *and that the set of automorphisms is itself a group with composition of functions as a product and the identity map as the identity element. Given an element* g *of a group* G, *we can define the following map* $\varphi_g : G \longrightarrow G$

$$\varphi_g(a) = gag^{-1}.$$

called conjugation by g.

a – **1 point**) *Show that* φ_g *is an automorphism of* G.

An automorphism obtained this way is called an inner automorphism, *while an automorphism of* G *which is not equal to* φ_g *for any* $g \in G$ *is called an* outer automorphism.

b – **1 point**) *Show that the set of inner automorphisms is a normal subgroup of the set of automorphisms.*

c – **1 point**) *Are outer automorphisms a subgroup of the group of automorphisms?*

4) *The center of a group* G *is defined as the set*

$$Z = \{g \in G : gh = hg \quad \forall h \in G\}.$$

a – **1 point**) *Show that if* G *is Abelian, then* $Z = G$.

b – **1 point**) *Show that the center of a group is fixed by any inner automorphism.*

c –**1 point**) *Use this to find an example of a group which has an outer automorphism.*