Group theory: Orbit–Stabilizer Theorem

1 The orbit stabilizer theorem

Definition 1.1. Given a group G and a set S, an *action* of G on S is a map

$$G \times S \longrightarrow S \qquad (g, s) \mapsto g \cdot s$$

which satisfies

- $e \cdot s = s$ for all $s \in S$,
- $g \cdot (h \cdot s) = (gh) \cdot s$.

Given an action of G on S and an element $s \in S$, there are two sets one can define:

Definition 1.2. The *orbit* of s is the set

$$\mathcal{O}_s = \{g \cdot s | g \in G\} \subset S.$$

The *stabilizer* of s is the set

$$\mathcal{B}_s = \{g \in G | g \cdot s = s\}.$$

Lemma 1.3. The stabilizer of a point $s \in S$ is a subgroup of G.

Proof. We will check the three properties required for subgroups.

Check that the identity is in $\mathcal{B}_s$. By definition of group action $e \cdot s = s$ for all $s \in S$, so this follows from the definition of group action.

Check that $\mathcal{B}_s$ is closed under inversion. Let $g \in \mathcal{B}_s$. Then

$$s = e \cdot s = (g^{-1}g) \cdot s = g^{-1} \cdot (g \cdot s) = g^{-1}s.$$

Hence $g^{-1} \in \mathcal{B}_s$ and $\mathcal{B}_s$ is closed under inversion.

Check that $\mathcal{B}_s$ is closed under products. Let $g, h \in \mathcal{B}_s$, then

$$(gh) \cdot s = g \cdot (h \cdot s) = g \cdot s = s.$$

Hence $\mathcal{B}_s$ is closed under multiplication and therefore it is a subgroup. □

Given a subgroup $H < G$ one can always consider sets the form

$$gH = \{a \in G | a = gh \text{ for some } h \in H\}$$

for $g \in G$. These are called left H-cosets. By the proof of Lagrange's theorem we have that

$$\#(\text{left } H\text{-cosets}) \cdot \#H = \#G, \tag{1}$$

and in particular $\#H$ divides $\#G$.

This statement takes the following form for subgroups obtained as stabilizers

Theorem 1.4. *Let G be a finite group, S be a set and $G \times S \longrightarrow S$ be a G-action. Let $s \in S$ then*

$$\#O_s \cdot \#B_s = \#G.$$

Proof. Due to equation (1), to prove this theorem it suffices to prove that the number of left $\mathcal{B}_s$-cosets is the same as the number of elements in the orbit of s. We achieve this with two observations.

Observation 1: Any two elements in the same left $\mathcal{B}$-coset map s into the same element. Let a_1, a_2 be two elements in $g\mathcal{B}_s$. Then there are h_1 and h_2 in $\mathcal{B}_s$ such that $a_1 = gh_1$ and $a_2 = gh_2$. Therefore

$$a_1 \cdot s = g \cdot (h_1 \cdot s) = g \cdot s = g \cdot (h_2 \cdot s) = a_2 \cdot s.$$

Observation 2: If a_1 and a_2 are not in the same coset, then $a_1 \cdot s \neq a_2 \cdot s$. Indeed, assume that $a_2 \notin a_1\mathcal{B}_s$ (i. e., they are not in the same coset) but still $a_1 \cdot s = a_2 \cdot s$. Then

$$a_1^{-1} \cdot (a_2 \cdot s) = a_1^{-1} \cdot (a_1 \cdot s) = (a_1^{-1}a_1) \cdot s = s,$$

and hence $a_1^{-1}a_2 \in \mathcal{B}_s$ or equivalently $a_2 \in a_1\mathcal{B}_2$, which is a contradiction. Hence $a_1 \cdot s \neq a_2 \cdot s$.

With these two observations, we have established a bijection between the left $\mathcal{B}_s$-cosets and the points in the orbit of s, hence the number of points in the orbit of s equals the number of left B_s-cosets in G, which finishes the proof. □