Group theory – Exam 1

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) a) Let α, β be elements of the symmetric group S_n. Show that if α and β commute and $i \in \{1, 2, \cdots, n\}$ is fixed by α, i.e., $\alpha(i) = i$, then $\beta(i)$ is also fixed by α. (0.5 pt)
b) Show that, for $n > 2$, $Z_{S_n} = \{e\}$. (0.5 pt)
c) Show that, for $n > 3$, $Z_{A_n} = \{e\}$. (0.5 pt)
d) What is the center of A_3? (0.5 pt)

Solution. a) Since $\alpha(i) = i$, we have that

$$\alpha(\beta(i)) = \beta(\alpha(i)) = \beta(i).$$

b)Let $\beta \in Z_{S_n}$ and let $\alpha_i = (1\ 2\ \cdots\ i-1\ i+1\ \cdots\ n)$. Then, since β is in the center, it commutes with α_i, hence from the first part

$$\alpha_i(\beta(i)) = \beta(i)$$

But notice that for $n > 2$ the only number fixed by α_i is i. Since $\beta(i)$ is fixed by α_i we have $\beta(i) = i$. Since this is true for all i, we get $\beta = e$ and hence the center of S_n is trivial.

c) Let $\beta \in Z_{A_n}$. If n is even the permutations α_i used above belong to A_n and hence the same argument used above shows that $\beta = e$ and hence $Z_{A_n} = \{e\}$ for n even and greater than 2.

For n odd, consider $\alpha_{ij} = (1\ 2\ \cdots\ i-1\ i+1\ \cdots\ j-1\ j+1\ \cdots n-1\ n)$. Then $\alpha(i) = i$, $\alpha(j) = j$ and these are the only two points fixed by α_{ij} if $n > 3$. Using item (a) of this exercise we see that

$$\alpha(\beta(i)) = \beta(i)$$

so $\beta(i)$ is fixed by α_{ij}, hence $\beta(i) = i$ or $\beta(i) = j$.

Taing $k \neq i, j$, since β commutes also with $\alpha_{ik} = (1\ 2\ \cdots\ i-1\ i+1\ \cdots\ k-1\ k+1\ \cdots n-1\ n)$ the same argument used above show that $\beta(i) = i$ or $\beta(i) = k$.

Since ($\beta(i) = i$ or $\beta(i) = j$) and ($\beta(i) = i$ or $\beta(i) = k$) must be both true, we conclude that $\beta(i) = i$ for all i and hence $\beta = e$, showing that $Z_{A_n} = \{e\}$ for n odd.

d)A_3 is a group with 3 elements, hence isomorphic to $\mathbb{Z}_3$ which is Abelian, so $Z_{A_3} = A_3 = Z_3$.

2) For each of the lists below, determine which groups are isomorphic:
a) $\mathbb{Z}_4 \times \mathbb{Z}_9$, $\mathbb{Z}_6 \times \mathbb{Z}_6$, $\mathbb{Z}_{36}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. (0.75 pt)
b) $A_5 \times \mathbb{Z}_2$, S_5, D_{30}, $D_{15} \times \mathbb{Z}_2$. (0.75 pt)

Solution. a) We know from lectures and the book that $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if m and n are coprime. Hence

$$(\mathbb{Z}_2 \times \mathbb{Z}_3) \times (\mathbb{Z}_2 \times \mathbb{Z}_3) \cong \mathbb{Z}_6 \times \mathbb{Z}_6$$

$$\mathbb{Z}_9 \times \mathbb{Z}_4 \cong \mathbb{Z}_{36}$$

and

$$\mathbb{Z}_6 \times \mathbb{Z}_6 \ncong \mathbb{Z}_{36}.$$

b)$\#D_{15} \times \mathbb{Z}_2 = \#D_{30} = 60$ and $\#A_5 \times \mathbb{Z}_2 = \#S_5 = 5! = 120$, so the first two groups can not be isomorphic to the last two.

From lectures we know that $D_n \times \mathbb{Z}_2 \cong D_{2n}$ if and only if n is odd, therefore $D_{15} \times \mathbb{Z}_2 \cong D_{30}$.

From question 1 (and from lectures) we know that $Z_{S_n} = \{e\}$ for $n > 2$ and $Z_{A_n} = \{e\}$ for $n > 3$, hence

$$Z_{A_5 \times \mathbb{Z}_2} = Z_{A_5} \times Z_{\mathbb{Z}_2} = \mathbb{Z}_2 \neq Z_{S_5}.$$

Therefore $A_5 \times \mathbb{Z}_2$ is not isomorphic to S_5.

3) Let G be the group generated by

$$G = \langle a, b | a^n = b^m = e; bab^{-1} = a^l \rangle$$

Show that if $l^m \neq 1 \bmod n$ then the order of a is less than n. (1 pt)

Solution. Since b has order m, we have the following equality

$$a = b^m a b^{-m} = b^{m-1}(bab^{-1})b^{-m+1} = b^{m-1}a^l b^{-m+1} = (b^{m-1}ab^{-m+1})^l = \cdots = a^{l^m}$$

Hence $a^{l^m - 1} = e$ and therefore the order of a must divide $l^m - 1$. If $l^m - 1 \neq 0 mod n$ then a does not have order n. since $a^n = e$, the order of a must be a divisor of a, hence the order of a is less than n.

4) Given a group G, a subgroup $H < G$ is called *proper* if H is neither $\{e\}$ nor G. Find a group which is isomorphic to one of its proper subgroups. (Hint: this is only possible for infinite groups). (1 pt)

Let $G = \mathbb{Z}$ and consider $H < \mathbb{Z}$ the subgroup formed by the even numbers. Then

$$\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}_{even}, \qquad \varphi(n) = 2n$$

is a group isomorphism between $\mathbb{Z}$ and $\mathbb{Z}_{even}$.

Indeed, this map is clearly surjective. Computing $\varphi(m + n) = 2m + 2n = \varphi(m) + \varphi(n)$ we see that φ is a group homomorphism and finally if $\varphi(n) = 0$ then $2n = 0$ and hence $n = 0$, showing that φ is injective.

5) Let G be a group. Then the *conjugacy class* of an element $x \in G$ is the set

$$\mathcal{C}_x = \{gxg^{-1} : g \in G\}$$

and the centralizer of x, denoted by $C(x)$ is the set of all elements in G which commute with x, i.e.,

$$C(x) = \{g \in G : gxg^{-1} = x\}$$

a) Show that the centralizer of x is a subgroup of G. (0.75 pt)
b) Show that, if G is finite, then index of $C(x)$ in G, i.e., the number of elements in $G/C(x)$, is the number of elements in $\mathcal{C}_x$, the conjugacy class of x. (0.75 pt).

Solution. G acts on itself by conjugation. For a given $x \in G$, the stabilizer of x is precisely $C(x)$ defined above, hence $C(x)$ is a subgroup of G (all stabilizers are subgroups).

Further, the orbit of x by this action is the set $\mathcal{C}_x$, so, by the Orbit–Stabilizer theorem,

$$\#G = \#C(x) \cdot \mathcal{C}_x.$$

or equivalently, $\#(G/C(x)) = \#\mathcal{C}_x$.

6 a) Show that if S_n acts on a set with p elements and $p > n$ is a prime number then the action has more than one orbit (0.75 pt).
b) Let p be a prime. Show that the only action of $\mathbb{Z}_p$ on a set with $n < p$ elements is the trivial one (0.75 pt).

Solution. a) Assume that there is an action of S_n on a set with p elements which has only one orbit. Then, by the Orbit–Stabilizer theorem we have

$$n! = \#S_n = \#orbit \cdot \#stabiliser = p \cdot k,$$

for some $k \in \mathbb{N}$. In particular, we conclude that $p|n!$. Since p is prime, if it divides a product, it must divide one of the factors. But since $p > n$ all the factors in $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ are smaller than p and hence are not divisible by p.

This contradiction proves that there is more than one orbit.

b) An action of $\mathbb{Z}_p$ on a set with n-elements corresponds to a group homomorphism $\varphi : \mathbb{Z}_p \longrightarrow S_n$. Since every element of $\mathbb{Z}_p$ different of the identity is a generator of the group either the kernel of the map is trivial or it is the whole of $\mathbb{Z}_p$. In the second case the action is trivial. So we have to show that the first case can not happen. If the kernel of φ is trivial, the φ is an injection of $\mathbb{Z}_p$ into S_n, hence $\mathbb{Z}_p = im(\varphi) < S_n$. Since the order of any subgroup must divide the order of the group we see that $p|n!$, but since$n < p$ this can not happen.

7) Let G be a group, S a set and $\varphi : G \times S \longrightarrow S$ be an action. Let H be the stabilizer of a point $s \in S$. Show that the stabilizer of $g \cdot s$ is gHg^{-1}. Conclude that H is a normal subgroup of G if and only if it is the stabilizer of all the points in the orbit of s. (1.5 pt)

Solution. Let J be the stabiliser of $g \cdot s$. Then for $j \in J$ we have

$$j \cdot g \cdot s = g \cdot s \Rightarrow (g^{-1}jg) \cdot s = s$$

Therefore $g^{-1}jg \in H$ for all $j \in J$ hence $g^{-1}Jg \subset H$ or, equivalently, $J \subset gHg^{-1}$.

Conversely, given $j \in gHg^{-1}$, there is $h \in H$ such that $j = ghg^{-1}$ and

$$j \cdot g \cdots = ghg^{-1}g \cdot s = gh \cdot s = gs$$

so $j \in J$, showing the reverse inclusion.

Therefore $J = gHg^{-1}$.

Points in the orbit of s are of the form $g \cdot s$ for some $g \in G$ and by the above the stabiliser of such point is gHg^{-1}. Therefore H is the stabilizer of all points in the orbit of s if and only if $H = gHg^{-1}$ for all $g \in G$ which happens if and only if H is normal.