Group theory – Mock Exam 2

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) Is $(\mathbb{R}, +)$ isomorphic to $(\mathbb{R} - \{0\}, \cdot)$?

2) Show that if a finite group G has only two conjugacy classes, then $G \cong \mathbb{Z}_2$.

3) Let H be a subgroup of finite index of an infinite group G. Prove that G has a normal subgroup of finite index contained in H.

4) Given a group G, a subgroup H is a *maximal normal* subgroup if

i) H is normal and

ii) if $K < G$ is a normal subgroup and $H < K$ then $K = H$ or $K = G$, i.e., the only normal subgroup of G which contains H as a proper subgroup is G.

Show that a normal subgroup H is maximal normal subgroup if and only if G/H is a simple group.

5) Let G be a finite group and let p be the smallest prime which divides the order of G. Show that is $H < G$ is a subgroup of index p then H is normal.

6) Show that a group or order $2 \cdot 3 \cdot 5 \cdot 29^2$ is not simple.

7) Show that in a group of order $5 \cdot 7 \cdot 13$ the 7-Sylow and the 13-Sylow are normal. Show that such group has nontrivial center.

8) Show that every element in $SO(3)$ corresponds to rotation around an axis in $\mathbb{R}^3$.