# Group theory – Sheet 10

Recall Sylow's theorems:
If $G$ is a group of order $kp^n$ where $p$ is a prime number and $k$ and $p$ are coprimes then

1. There is a subgroup of $G$ of order $p^n$;
2. If $H_1$ and $H_2$ are subgroups of $G$ of order $p^n$ then there is $g \in G$ such that $gH_1g^{-1} = H_2$;
3. The number of subgroups of order $p^n$ is equal to 1 modulo $p$ and divides $k$.

**Exercises**

The exercises from the book are all exercises from chapter 20 except exercise 20.14.

1) Let $G$ be a group of order $203 = 7 \cdot 29$. Show that if $G$ has a normal subgroup of order 7 then $G$ is Abelian.

2) Let $G$ be a group of order $351 = 3^3 \cot 13$. Show that one of the Sylow subgroups of $G$ is normal.

3) Let $G$ be a group of order $231 = 3 \cdot 7 \cdot 11$. Show that the 7-Sylow and the 11-Sylow are normal. Show that the 11-Sylow is in the center of $G$.

4) Let $G$ be a group of order $3^3 \cdot 13^n$ with $n > 2$. Show that $G$ is not simple.
(Hint: in the case of existence of 27 13-Sylows, consider the action of $G$ on the set of 13-Sylows by conjugation. This action is given by a group homomorphism $\Phi : G \longrightarrow S_{27}$. Use Lagrange and isomorphism theorems to show that this action has nontrivial kernel.)