Group theory – Sheet 5

For exercises from chapter 7, recall that

- a group *homomorphism* is a map $\varphi : H \longrightarrow G$ such that $\varphi(h_1 \cdot h_2) = \varphi(h_1) \cdot \varphi(h_2)$.
- a group *isomorphism* is a group homomorphism $\varphi : H \longrightarrow G$ which is also a bijective map.
- a group *automorphism* is a group isomorphism between G and itself: $\varphi : G \longrightarrow G$

Two isomorphic groups have the same multiplication table and hence are considered to be "the same".

Finally, a subgroup $K < G$ is *normal* if $gKg^{-1} = K$ for all $g \in G$. We have seen that the kernel of any group homomorphism $\varphi : H \longrightarrow G$ is a normal subgroup of the domain, H.

The exercises from the book are all exercises from chapter 6 and exercises 7.4, 7.5, 7.6, 7.7, 7.9, 7.10, 7.12.

1) Given a group G, recall that the automorphisms of G, are the group homomorphisms $\varphi : G \longrightarrow G$ which are also bijections.
a) Show that $Aut(G)$, the set of group automorphisms of G, is itself a group where multiplication is given by function composition.
b) Adjoint map gives rise to a map $Ad : G \longrightarrow Aut(G)$:

$$g \mapsto Ad_g; \qquad where \qquad Ad_g(x) = gxg^{-1}.$$

Show that Ad is a group homomorphism.
c) What is the kernel of the map $Ad : G \longrightarrow Aut(G)$?
d) Show that the $\mathrm{Im}(Ad) < Aut(G)$ is a normal subgroup of $Aut(G)$.

2) (Group actions as group homomorphisms) Let $\varphi : G \times X \longrightarrow X$ be an action of the group G on the set X. Then for every $g \in G$, $\varphi(g, \cdot) : X \longrightarrow X$ is a bijection, hence can be regarded as an element in S_X, the group of bijections of X.

Show that the map $g \mapsto \varphi(g, \cdot) \in S_X$ is a group homomorphism between G and S_X.