Group theory – Sheet 6

The exercises from the book are all exercises from chapter 6 and exercises 10.1, 10.2, 10.6, 10.7, 10.8, 10.11, 10.13, 11.2, 11.3, 11.4, 11.8, 11.9, 11.10, 11.11.

1) Show that $Z_{G_1 \times G_2} = Z_{G_1} \times Z_{G_2}$, where Z_G stands for the center of G .

2) Let G be the group whose elements are infinite sequences $(a_1, a_2, \cdots)$ of integers endowed with the following group operation:

$$(a_1, a_2, \cdots) \cdot (b_1, b_2, \cdots) = (a_1 + b_1, a_2 + b_2, \cdots).$$

Show that G is isomorphic to $\mathbb{Z} \times G$ and also to $G \times G$. Conclude that a group can be isomorphic to one of its proper subgroups[1].

3) Last exercise sheet we saw that given a group G, the adjoint map $Ad : G \longrightarrow Aut(G)$ is a group homomorphism between G and the automorphisms of G:

$$Ad : G \longrightarrow Aut(G); \qquad \begin{aligned} Ad_g &: G \longrightarrow G \\ Ad_g(x) &= gxg^{-1} \end{aligned}$$

The automorphisms in the image of Ad are called *inner automorphisms* while automorphisms not in the image of Ad are called *outer automorphisms*.

a) Show that G is Abelian, if and only if $Z_G = G$.

b) Show that the center of a group is fixed (pointwise) by any inner automorphism.

c) Use this to find an example of a group which has an outer automorphism.

4) Let S_5 act on itself by conjugation and consider the point $\sigma = (1\ 2\ 3\ 4\ 5) \in S_5$. What is the orbit of σ? What is the stabilizer of σ. Conclude from this example that S_5 can have orbits whose size is bigger than 5.

[1] a subgroup $H < G$ is proper if H is neither $\{e\}$ nor G