# Group theory – Exam

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

**1) Compute the center of $D_n$ for $n \geq 3$. Analyse carefully the cases $n$ even and $n$ odd.**

As in lectures $D_n = \langle a, b : a^n = b^2 = e; bab^{-1} = a^{-1}\rangle$. Let $c = a^i b^j$ be an element of $D_n$. Then $c$ is in the center of $D$ if and only if $c$ commutes with the generators of $D_n$, i.e., if and only if the commutator of $c$ with $a$ and $b$ is the identity. So we compute the commutators

$$cac^{-1}a^{-1} = a^i b^j ab^{-j} a^{-i} a^{-1} = a^i a^{-1^j} a^{-i} a^{-1} = a^{-1+(-1)^j}$$

where in the second equality we used that

$$b^j ab^{-j} = b^{j-1}(bab^{-1})b^{-j+1} = b^{j-1}a^{-1}b^{-j+1} = b^{j-2}(ba^{-1}b^{-1})b^{-j+2} = b^{j-2}ba^{(-1)^2}b^{-j+2} = \cdots = a^{(-1)^j} \text{(recall quiz 2)}.$$

So we have that

$$cac^{-1}a^{-1} = a^{-1+(-1)^j} = \begin{cases} e & \text{if } j \text{ is even;} \\ a^{-2} & \text{if } j \text{ is odd.} \end{cases}$$

Since $n > 2$ we conclude that, if $c$ is in the center of $D_n$, $c$ must be of the form $a^i$.

Now we compute the commutator of $c = a^i$ with $b$:

$$cbc^{-1}b^{-1} = a^i ba^{-i}b{-}1 = a^i a^i = a^{2i}$$

Therefore $c$ is in the center if and only if $2i = 0 \mod n$. For $n$ odd the only solution is $i = 0$ as 2 is invertible in $\mathbb{Z}_n$ under multiplication, while for $n$ even we can have $i = 0$ or $i = n/2$.

Hence $Z_{D_n} = \{e\}$ if $n$ is odd and $Z_{D_n} = \{e, a^{n/2}\} = \mathbb{Z}_2$ if $n$ is even.

**2) For each list of groups a), b) and c) below, decide which of the groups within each list are isomorphic, if any:**

**a)** $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$**,** $\mathbb{Z}_9 \times \mathbb{Z}_2 \times \mathbb{Z}_2$**,** $\mathbb{Z}_{18} \times \mathbb{Z}_2$ **and** $\mathbb{Z}_6 \times \mathbb{Z}_6$**.**

**b)** $S_4$**,** $A_4 \times \mathbb{Z}_2$**,** $D_{12}$ **and** $\mathbb{H} \times \mathbb{Z}_3$**, where** $\mathbb{H}$ **is the quaternion group with 8 elements.**

**c)** $(\mathbb{Q}, +)$**,** $(\mathbb{R}, +)$**,** $(\mathbb{R}_+, \times)$**.**

a) As we saw in lectures $\mathbb{Z}_n \times \mathbb{Z}_m = \mathbb{Z}_{nm}$ if and only if $n$ and $m$ are coprimes. So $\mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_9 = \mathbb{Z}_{18}$ therefore

$$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2 = \mathbb{Z}_6 \times \mathbb{Z}_6$$

and

$$\mathbb{Z}_9 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_{18} \times \mathbb{Z}_2$$

Also, by the same result,

$$\mathbb{Z}_6 \times \mathbb{Z}_3 \neq \mathbb{Z}_{18}$$

hence

$$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \neq \mathbb{Z}_{18} \times \mathbb{Z}_2$$

Notice that to achieve this conclusion we used that fact that $G \times H = G' \times H$ implies $G = G'$ for *finite groups*.

If you don't want to use this result, notice that every nonidentity element in $\mathbb{Z}_6 \times \mathbb{Z}_6$ has order less than or equal to 6, while $\mathbb{Z}_{18} \times \mathbb{Z}_2$ has an element of order 18, so these two groups can not be isomorphic.

b) Firstly we look at the centers:

$$Z_{S_4} = \{e\}, \qquad Z_{A_4 \times \mathbb{Z}_2} = Z_{A_4} \times Z_{\mathbb{Z}_2} = \mathbb{Z}_2, \qquad Z_{D_{12}} = \mathbb{Z}_2, \qquad Z_{\mathbb{H} \times \mathbb{Z}_3} = Z_{\mathbb{H}} \times Z_{\mathbb{Z}_3} = \mathbb{Z}_2 \times \mathbb{Z}_3.$$

So the only two groups that can be isomorphic are $A_4 \times \mathbb{Z}_2$ and $D_{12}$. But $A_4 \times \mathbb{Z}_2$ has no elements of order 12 while $D_{12}$ does, so those groups are not isomorphic either.

c) The first group in the list is countable and the other two are not, so there is not bijection between $(\mathbb{Q}, +)$ and the other two groups. Finally, $(\mathbb{R}, +)$ is isomorphic to $(\mathbb{R}_+, \cdot)$ via the exponential map

$$exp : \mathbb{R} \longrightarrow \mathbb{R}_+ \qquad exp(x) = e^x$$

since $exp(x + y) = e^{x+y} = e^x e^y = exp(x)exp(y)$, showing that $exp$ is a group homomorphism. And we know from school that the exponential map is a bijection between $\mathbb{R}$ and $\mathbb{R}_+$ whose inverse is the logarithm.

**3) Show that** $(\mathbb{Q}, +)$ **is not finitely generated, i.e., if** $X \subset \mathbb{Q}$ **is a finite set, then the group generated by** $X$ **is not** $\mathbb{Q}$**.**

Let $X = \{p_1/q_1, \cdots p_k/q_k\}$ be a finite subset of $\mathbb{Q}$ where $p_i$ and $q_i$ are coprime integers. Then, the group generated by $X$ is given by

$$\langle X \rangle = \{\sum_i m_i p_i/q_i : m_i \in \mathbb{Z}\} = \{(\sum_i (\Pi_{j \neq i} q_j) m_i p_i)/\Pi_i q_i : m_i \in \mathbb{Z}\}.$$

In particular we see that every element in $\langle X \rangle$ is an *integer* multiple of $1/\Pi_i q_i$, so, in particular $\frac{1}{2\Pi_i q_i} \notin \langle X \rangle$, hence $\langle X \rangle \neq \mathbb{Q}$.

**4) Let $p$ be a prime and $X$ be a set with less than $p$ elements. Show that the only action of $\mathbb{Z}_p$ on $X$ is the trivial one.**

By the Orbit-Stabilizer theorem the number of elements in an orbit of a $\mathbb{Z}_p$ action must divide $p$, so the orbits must have size either 1 or $p$. Since $\#X < p$, there is no orbit of size $p$, so all orbits of this action have size 1 and hence the action is trivial.

**5) Let $G$ be a finite group and $H < G$ be a subgroup of index $n$, i.e., $\#G = n\#H$. Show that $g^{n!} \in H$ for every $g \in G$.**

$G$ acts on $G/H$ by left multiplication. This action corresponds to a group homomorphism

$$\varphi : G \longrightarrow S_n$$

According to Lagrange's theorem if $\sigma \in S_n$, $\sigma^{n!} = e$, hence, for any $g \in G$

$$\varphi(g^{n!}) = \varphi(g)^{n!} = e \in S_n$$

That is the left action of $g^{n!}$ is trivial, in particular it fixes the coset $H$, so $g^{n!}H = H$.Therefore $g^{n!} = g^{n!}e \in g^{n!}H = H$.

**6) Show that if a group $G$ has a conjugacy class with two elements then $G$ is not simple.**

Let $\mathcal{C}$ be the conjugacy class with 2 elements. Firstly we notice that $G$ has more than two elements, since the conjugacy class of $e$ has only one element, so $G$ must have at least three elements: $e$ and the two elements in $\mathcal{C}$.

Now, let $G$ act on $\mathcal{C}$ by conjugation. This corresponds to a group homomorphism

$$\phi : G \longrightarrow S_2 = \mathbb{Z}_2$$

Since the orbit of $g \in \mathcal{C}$ is $\mathcal{C}$, $\phi$ is a nontrivial group homomorphism. Since $G$ has more than 2 elements, the map is not injective, so $\ker(\phi) \neq G$ and $\ker(\phi) \neq \{e\}$. Since the kernel of a group homomorphism is a normal subgroup of the domain, we conclude that $\ker(\phi)$ is a normal subgroup of $G$ which is different from $G$ and $\{e\}$, hence $G$ is not simple.