Group theory – Sheet 4

The exercises from the book are 6.3, 6.4, 6.7, 6.10, 6.12, 14.3, 14.7, 14.8, 16.1, 16.2, 16.8.

1) Let G be a group. For each $g \in G$ we define a map

$$Ad_g : G \longrightarrow G \qquad Ad_g(h) = ghg^{-1}$$

i) Show that Ad_g is a group automorphism, and hence Ad defines a map from G to $Aut(G)$.

ii) Show that the map $Ad : G \longrightarrow Aut(G)$ is a group homomorphism. Compute its kernel.

iii) Consider $G = \mathbb{Z}_3$ find an automorphism of G which is not in the image of the map Ad.

The image of G by the map $Ad : G \longrightarrow Aut(G)$ is the (sub)group of *inner automorphisms* of G.

2*) Following the steps below or otherwise prove that for $n > 6$ the automorphism group of S_n is equal to the group of inner automorphisms.

i) Let $\varphi : G \longrightarrow H$ be an isomorphism. Show that g and $\varphi(g)$ have the same order and that the number of elements in the conjugacy class of g is the same as the number of elements in the conjugacy class of $\varphi(g)$.

ii) Let $\alpha \in S_n$ be a permutation of order 2. Show that α must be a product of disjoint transpositions. If α is the product of k-disjoint transpositions, compute the number of elements in the conjugacy class of α.

iii) Using the previous two items, conclude that, for $n > 6$, any automorphism of S_n must send transpositions to transpositions.

iv) Show that an automorphism which sends transpositions to transpositions is an inner automorphism.