Differentiable manifolds – sheet 1

Exercise 1 (The birth of long exact sequences)**.** A *cochain complex* is a collection, U, of Abelian groups U^k, $k \geq 0$, together with a coboundary operator, $\partial_U : U^k \longrightarrow U^{k+1}$ which is a group homomorphism for all k satisfying $\partial_U^2 = 0$. A *cocycle* is an element in the kernel of the ∂_U and a *coboundary* is an element in the image. We define cohomology as usual, e.g.,

$$H^k(U) = \frac{\ker(\partial_U : U^k \longrightarrow U^{k+1})}{\operatorname{im}(\partial_U : U^{k-1} \longrightarrow U^k)}.$$

Below we let (U, ∂_U), (V, ∂_V) and (W, ∂_W) be cochain complexes.

1. Let $f : U \longrightarrow V$ be a group homomorphism (of degree zero), that is $f : U^k \longrightarrow V^k$, is a group homomorphism for all k. Show that if f commutes with the coboundary operators, i.e., the diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_U} & U^k & \xrightarrow{\partial_U} & U^{k+1} & \xrightarrow{\partial_U} & \cdots \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\ \cdots & \xrightarrow{\partial_V} & V^k & \xrightarrow{\partial_V} & V^{k+1} & \xrightarrow{\partial_V} & \cdots \end{array}$$

   commutes, then f sends coboundaries to coboundaries and cocycles to cocycles. In particular, f induces a map in cohomology:

$$f^* : H^k(U) \longrightarrow H^k(V), \qquad f_*[u] = [f(u)], \quad \text{for all } k.$$

2. Let $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be group homomorphisms (of degree zero) which commute with the coboundary operators. Assume that f is injective, g is surjective and $\operatorname{im}(f) = \ker(g)$, that is, we have a short exact sequence of cochain complexes

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0.$$

   Show that the induced maps

$$H^k(U) \xrightarrow{f^*} H^k(V) \xrightarrow{g^*} H^k(W)$$

   satisfy $\operatorname{im}(f^*) = \ker(g^*)$.

3. Next we try to define a map $\delta : H^k(W) \longrightarrow H^{k+1}(U)$ as follows. Given a cocycle $w \in W^k$ ($\partial_W w = 0$), let $v \in V^k$ be such that $g(v) = w$. Then

$$0 = \partial_W w = \partial_W g(v) = g(\partial_V v),$$

   that is $\partial_V v \in \ker(g) = \operatorname{im}(f)$, hence there is $u \in U^{k+1}$ such that $f(u) = \partial_V v$. Show that $\partial_U u = 0$ and set $\delta[w] = [u]$. Show that δ is well defined.

4. Show that the sequence

$$\cdots \xrightarrow{\delta} H^k(U) \xrightarrow{f^*} H^k(V) \xrightarrow{g^*} H^k(W) \xrightarrow{\delta} H^{k+1}(U) \xrightarrow{f^*} \cdots$$

   is exact at every point, that is, the image and kernel of consecutive maps agree.