CLASSIFICATION OF BOUNDARY LEFSCHETZ FIBRATIONS OVER THE DISC

STEFAN BEHRENS GIL R. CAVALCANTI RALPH L. KLAASSE

Dedicated to Professor Nigel Hitchin on the occasion of his 70th birthday

Abstract. Stable generalized complex structures can be constructed out of boundary Lefschetz fibrations. On four-manifolds, these are essentially genus-one Lefschetz fibrations over surfaces except that generic fibers can collapse to circles over a codimension-one submanifold, which is often the boundary of the surface. We show that a four-manifold admits a boundary Lefschetz fibration over the disc degenerating over its boundary if and only if it is diffeomorphic to $S^1 \times S^3 \# n\overline{\mathbb{C}P^2}$, $\# m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ or $\# m(S^2 \times S^2)$. We conclude that the four-manifolds $S^1 \times S^3 \# n\overline{\mathbb{C}P^2}$, $\#(2m+1)\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ and $\#(2m+1)S^2 \times S^2$ admit stable generalized complex structures whose type change locus has a single component.

Contents

1. Introduction

Generalized complex structures, introduced by Hitchin [14] and Gualtieri [11] in 2003, are geometric structures which generalize simultaneously complex and symplectic structures while at the same time providing the mathematical background for string theory. One feature of generalized complex geometry is that the structure is not homogenous. In fact, a single connected generalized complex manifold may have complex and symplectic points. This lack of homogeneity is governed by the *type* of the structure, an integer-valued upper semicontinuous function on the given manifold which tells "how many complex directions" the structure has at the given point. In particular, on a $2n$-dimensional manifold, points of type 0 are symplectic points, while points of type n are complex.

Among all type-changing generalized complex structures, one kind seems to deserve special attention: stable generalized complex structures. These are the structures whose canonical section of the anticanonical bundle vanishes transversally along a codimension-two submanifold, $\mathcal{D}$, endowing it with the structure of an elliptic divisor in the language of [8]. Consequently, the type of such a structure is 0 on $X \setminus \mathcal{D}$, while on $\mathcal{D}$ it is equal to two. Many examples of stable generalized complex structures were produced in dimension four [7, 10, 15, 16] and a careful study was carried out in [8]. One of the outcomes of that study was that it related stable generalized complex structures to symplectic structures on a certain Lie algebroid.

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Theorem ([8, Theorem 3.7])**.** *Let $\mathcal{D}$ be a co-orientable elliptic divisor on X. Then there is a correspondence between gauge equivalence classes of stable generalized complex structures on X which induce the divisor $\mathcal{D}$, and zero-residue symplectic structures on $(X, \mathcal{D})$.*

This results paves the way for the use of symplectic techniques to study stable structures. One result, due to the last named authors [5], that exemplifies that use is the following relation between stable generalized complex structures and boundary Lefschetz fibrations in dimension four. The latter are essentially genus-one Lefschetz fibrations over surfaces whose generic fibers can collapse to circles over a codimension-one submanifold, which is often the boundary of the surface (see Section 2 for details).

Theorem ([5, Theorem 7.1])**.** *Let X^4 be a closed connected and orientable four-manifold and let Σ be a compact connected and orientable two-manifold with boundary $Z = \partial\Sigma$. Let $f\colon X^4 \to \Sigma^2$ be a boundary Lefschetz fibration for which $\mathcal{D} = f^{-1}(\partial\Sigma)$ is a co-orientable submanifold of X, and with $0 \neq [f^{-1}(p)] \in H_2(X \setminus \mathcal{D}; \mathbb{R})$, where $p \in \Sigma$ is a regular value of f. Then X admits a stable generalized complex structure whose degeneracy locus is $\mathcal{D}$.*

This result is reminiscent of Gompf's original one [9], showing that Lefschetz fibrations give rise to symplectic structures. It is also is similar in content to a number of other results relating structures which are close to being symplectic to maps which are close to being Lefschetz fibrations. These include the relations between near-symplectic structures and broken Lefschetz fibrations [1], and between folded symplectic structures and real log-symplectic structures and achiral Lefschetz fibrations [2, 4, 6].

The upshot of these results is that they at the same time furnish (at least theoretically) a large number of examples of manifolds admitting the desired geometric structure, and provide us with a better grip on those structures. With this in mind, our aim here is to classify all four-manifolds which admit boundary Lefschetz fibrations over the disc degenerating over its boundary. Our main result is the following (Theorem 3.13).

Theorem. *Let $f\colon X^4 \to D^2$ be a relatively minimal boundary Lefschetz fibration and $\mathcal{D} = f^{-1}(\partial D^2)$. Then X is diffeomorphic to one of the following manifolds:*

(1) $S^1 \times S^3$*;*
(2) $\#m(S^2 \times S^2)$*, including* S^4 *for* $m = 0$*;*
(3) $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ *with* $m > n \geq 0$*.*

In all cases the generic fibre is nontrivial in $H_2(X \setminus \mathcal{D}; \mathbb{R})$. In case (1), $\mathcal{D}$ is co-orientable, while in cases (2) and (3), $\mathcal{D}$ is co-orientable if and only if m is odd.

These last two theorems equip the manifolds $S^1 \times S^3\#n\overline{\mathbb{C}P^2}$, $\#(2m+1)\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ and $\#(2m+1)S^2 \times S^2$ with a stable generalized complex structure whose type change locus has a single component. Further, they provide a complete list of four-manifolds whose stable generalized complex structures are obtained from boundary Lefschetz fibrations over the disc degenerating over its boundary. Note that the previous theorem does not address cases where $\mathcal{D}$ has multiple components.

We use essentially the same methods that were used by the first author in [3] and Hayano in [12, 13]. We translate the problem into combinatorics in the mapping class group of the torus, and then translate combinatorial results back into geometry using handle decompositions and Kirby calculus. Hayano's work turns out to be particularly relevant. In his classification of so-called genus-one simplified broken Lefschetz fibrations he was led to study monodromy factorizations of Lefschetz fibrations over the disc whose monodromy around the boundary is a signed power of a Dehn twist. It turns out that the same problem appears for boundary Lefschetz fibrations.

Organization of the paper. This paper is organised as follows. In Section 2 we introduce boundary fibrations and boundary Lefschetz fibrations, and summarise their basic properties. In Section 3 we start by studying boundary Lefschetz fibrations over $(D^2, \partial D^2)$ with few Lefschetz singularities. We then prove the main theorem using induction, by showing how to reduce the number of Lefschetz singularities.

2. Boundary Lefschetz fibrations

In view of our interest in stable generalized complex structures and the results mentioned in the Introduction, the basic object with which we will be dealing in this paper are boundary (Lefschetz) fibrations. In this section we review the relevant definitions and basic results regarding them. We will use the following language. A *pair* $(X, \mathcal{D})$ consists of a manifold X and a submanifold $\mathcal{D} \subseteq X$. A *map of pairs* $f\colon (X, \mathcal{D}) \to (\Sigma, Z)$ is a map $f\colon X \to \Sigma$ for which $f(\mathcal{D}) \subseteq Z$. A *strong map of pairs* is a map of pairs $f\colon (X, \mathcal{D}) \to (\Sigma, Z)$ for which $f^{-1}(Z) = \mathcal{D}$.

Definition 2.1 (Boundary Lefschetz fibrations)**.** Let $f\colon (X^{2n}, \mathcal{D}^{2n-2}) \to (\Sigma^2, Z^1)$ be a strong map of pairs which is proper and for which $\mathcal{D}$ and Z are compact.

- The map f is a *boundary map* if the normal Hessian of f along $\mathcal{D}$ is definite;
- The map f is a *boundary fibration* if it is a boundary map and the following two maps are submersions:
  a) $f|_{X\setminus\mathcal{D}}\colon X \setminus \mathcal{D} \to \Sigma \setminus Z$, and
  b) $f\colon \mathcal{D} \to Z$.

  The condition that f is a boundary fibration (in a neighbourhood of $\mathcal{D}$) is equivalent to the condition that for every $x \in \mathcal{D}$, there are coordinates $(x_1, \ldots, x_{2n})$ centred at x and (y_1, y_2) centred at $f(x)$ such that f takes the form

  $$f(x_1, \ldots, x_{2n}) = (x_1^2 + x_2^2, x_3), \tag{2.1}$$

  where $\mathcal{D}$ corresponds to the locus $\{x_1 = x_2 = 0\}$ and Z to the locus $\{y_1 = 0\}$;
- The map f is a *boundary Lefschetz fibration* if X and Σ are oriented, f is a boundary fibration from a neighbourhood of $\mathcal{D}$ to a neighbouhood of Z and $f|_{X^{2n}\setminus\mathcal{D}}\colon X \setminus \mathcal{D} \to \Sigma \setminus Z$ is a proper Lefschetz fibration, that is, for each critical point $x \in X \setminus \mathcal{D}$ and corresponding singular value $y \in \Sigma \setminus Z$, there are complex coordinates centred at x and y compatible with the orientations for which f acquires the form

  $$f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2. \tag{2.2}$$

Example 2.2 ($S^1 \times S^3$)**.** In this example we provide $X = S^1 \times S^3$ with the structure of a boundary fibration over the disc, as described in [5, Example 8.3]. The map $f\colon S^1 \times S^3 \to D^2$ is a composition of maps, namely

$$S^1 \times S^3 \to S^3 \to D^2,$$

where the first map is projection onto the second factor and the last is the projection from $\mathbb{C}^2$ to $\mathbb{C}$, $(z_1, z_2) \mapsto z_1$, restricted to the sphere. In Lemma 3.1 we will see that this is, in fact, the only example of a boundary fibration over $(D^2, \partial D^2)$.

A few relevant facts about boundary Lefschetz fibrations were established in [5]. Beyond the local normal form (2.1) for the map f around points in $\mathcal{D}$ there is also a semi-global form for f in a neighbourhood of $\mathcal{D}$:

Theorem 2.3 ([5, Proposition 5.15]). *Let $f\colon (X^{2n}, \mathcal{D}^{2n-2}) \to (\Sigma^2, Z^1)$ be a boundary map which is a boundary fibration on neighbourhoods of $\mathcal{D}$ and Z and for which Z is co-orientable. Then there are*

- *neighbourhoods U of $\mathcal{D}$ and V of Z and diffeomorphisms between these sets and neighbourhoods of the zero sections of the corresponding normal bundles, $\Phi_{\mathcal{D}}\colon U \to N_{\mathcal{D}}$ and $\Phi_Z\colon V \to \mathbb{R} \times Z$, and*
- *a bundle metric g on $N_{\mathcal{D}}$,*

such that the following diagram commutes, where $\pi\colon N_{\mathcal{D}} \to \mathcal{D}$ is the bundle projection:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\Phi_{\mathcal{D}}"] & V \arrow[d, "\Phi_Z"] \\ N_{\mathcal{D}} \arrow[r, "(\|\cdot\|_g^2,\, f|_{\mathcal{D}}\circ\pi)"] & \mathbb{R} \times Z \end{tikzcd}$$

The most obvious consequence of this theorem is that in the description above, the image of f lies on one side of Z, namely in $\mathbb{R}_+ \times Z$. At this stage this is a local statement, but if Z is separating (i.e., represents the trivial homology class) this becomes a global statement: the image of f lies in closure of one component of $\Sigma \setminus Z$ and hence we can equally deal with f as a map between X and a manifold with boundary, Σ, whose boundary is Z. In this paper we will be concerned with the case when Σ is the two-dimensional disc and Z is its boundary.

Corollary 2.4. *Let $f\colon (X^4, \mathcal{D}^2) \to (\Sigma^2, Z^1)$ be a boundary fibration with connected fibres, where Z is co-orientable and X is connected and orientable. Then its generic fibres are tori.*

Proof. From Theorem 2.3 we see that the level set $f^{-1} \circ \Phi_Z^{-1}(\varepsilon, y)$ with $\varepsilon > 0$ is a surface which fibres over the level set of $f^{-1} \circ \Phi_Z^{-1}(0, y)$, which is a circle, hence $f^{-1} \circ \Phi_Z^{-1}(\varepsilon, y)$ must be a torus or a Klein bottle. If X is orientable, $N_{\mathcal{D}} \setminus \mathcal{D}$ is also orientable and due to Theorem 2.3, $\Phi_Z \circ f \circ \Phi_{\mathcal{D}}^{-1}\colon U \subset N_{\mathcal{D}} \setminus \mathcal{D} \to \mathbb{R} \times Z \setminus \{0\} \times Z$ is a fibration, where U is a neighbourhood of $\mathcal{D}$, hence the fibres must be orientable. ☐

Remark 2.5. In the case when X is connected, Σ is a surface with boundary $Z = \partial\Sigma$, and $f\colon X \to \Sigma$ is surjective, we can lift f to a cover of Σ so that the fibres of the boundary Lefschetz fibration become connected. That is, this particular hypothesis is not really a restriction on the fibration (see [5, Proposition 5.23]). In what follows we will always assume this is the case.

Remark 2.6. As shown in [5, Proposition 6.8], a boundary Lefschetz fibration $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ satisfies $\chi(X) = \mu$, where μ is the number of Lefschetz singular fibres.

2.1. **Vanishing cycles and monodromy.** Lefschetz fibrations on four-manifolds can be described combinatorially in terms of their monodromy representations and vanishing cycles. We now extend this approach to boundary Lefschetz fibrations. For simplicity, we focus on fibrations over the disc (degenerating over its boundary) and assume they are injective on their Lefschetz singularities. The latter condition can always be achieved by a small perturbation and the generalization to general base surfaces is similar to the Lefschetz case.

Definition 2.7 (Hurwitz systems). Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration with ℓ Lefschetz singularities, and let $y \in D^2$ be a regular value. A *Hurwitz system* for f based at y is a collection of embedded arcs $\eta_0, \eta_1, \dots, \eta_\ell \subset D^2$ such that

(1) η_0 connects y to ∂D^2 and is transverse to ∂D^2,
(2) η_i connects y to a critical value y_i,

(3) the arcs intersect pairwise transversely in y and are otherwise disjoint, and
(4) the order of the arcs is counterclockwise around y.

Given a Hurwitz system, we obtain a collection of simple closed curves in the regular fibre $F_y = f^{-1}(y)$ as follows. For $i > 0$ we have the classical construction of *Lefschetz vanishing cycles*: as we move from y along η_i towards y_i, a curve $\lambda_i \subset F_y$ shrinks and eventually collapses into Lefschetz singularity over y_i, leading to a nodal singularity in F_{y_i}. For later reference, we also recall that the monodromy along a counterclockwise loop around y_i contained in a neighbourhood of η_i is given by a right-handed Dehn twist about λ_i. Along η_0 we see a slightly different degeneration: the boundary of a solid torus degenerates the core circle. Indeed, using the local model for f near $\mathcal{D}$ and the transversality of η_0 to ∂D^2 we can find a diffeomorphism $f^{-1}(\eta_0) \cong D^2 \times S^1$ and a parameterization of η_0 that takes f into the function $D^2 \times S^1 \to \mathbb{R} \times Z$ given by $(x_1, x_2, \theta) \mapsto (x_1^2 + x_2^2, z_0)$, where $z_0 = \eta_0(1)$. To summarize, $f^{-1}(\eta_0)$ is a solid torus whose boundary is F_y. Further F_y contains a well-defined isotopy class of meridional circles, represented in the model by $\partial D^2 \times \{\theta\}$ for arbitrary $\theta \in S^1$. We will henceforth refer to this isotopy class as the *boundary vanishing cycle* associated to η_0 and denote it by δ.

To make things even more concrete, we can fix an identification of the reference fibre F_y with T^2 and consider the vanishing cycles in the standard torus. To make a notational distinction, we denote the images in T^2 by $(a; b_1, \ldots, b_\ell)$.

Definition 2.8 (Cycle systems)**.** A collection of curves $(a; b_1, \ldots, b_\ell)$ in T^2 associated to f by a choices of a Hurwitz system and an identification of the reference fibre with T^2 is called a *cycle system* for f.

It is well known that the Lefschetz part of f can be recovered from the Lefschetz vanishing cycles. In the next section we will explain how this statement extends to boundary Lefschetz fibrations. Just as in the Lefschetz case, the cycle system is not unique but the ambiguities are easy to understand and provide some flexibility to find particularly nice cycle systems representing a given boundary Lefschetz fibration. The following is a straightforward generalization of the analogous statement for Lefschetz fibrations, see also Figure 1.

Proposition 2.9. *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration with ℓ Lefschetz singularities. Any two cycle systems for f are related by a finite sequence of the following modifications:*

$$\begin{aligned}\big(a; b_1, \ldots, b_\ell\big), &\sim \big(a; b_2, B_2(b_1), b_3, \ldots, b_\ell\big),\\ &\sim \big(a; B_1^{-1}(b_2), b_1, b_3, \ldots, b_\ell\big),\\ &\sim \big(B_1(a); b_2, \ldots, b_\ell, b_1\big),\\ &\sim \big(B_\ell^{-1}(a); b_\ell, b_1, \ldots, b_{\ell-1}\big),\\ &\sim \big(h(a); h(b_1), \ldots, h(b_\ell)\big).\end{aligned}$$

Here $B_i = \tau_{b_i}$ is a right-handed Dehn twist about b_i and h is any diffeomorphism of T^2.

Definition 2.10 (Hurwitz equivalence)**.** If two cycle systems are related by the modifications listed in Proposition 2.9, we say that they are *(Hurwitz) equivalent.*

It turns out that the curves in a cycle system are not completely arbitrary. Let $S^1_r \subset D^2$ be the circle of radius $r < 1$ such that all the Lefschetz singularities of f map to the interior

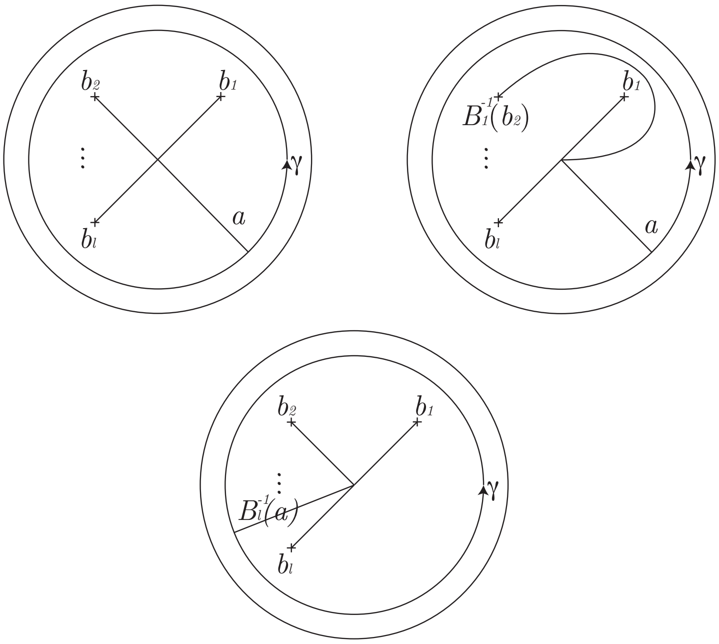

FIGURE 1. The origin of Hurwitz equivalence: here we illustrate how the equivalences $\big(a; b_1, \ldots, b_\ell\big) \sim \big(a; B_1^{-1}(b_2), b_1, b_3, \ldots, b_\ell\big) \sim \big(B_\ell^{-1}(a); b_\ell, b_1, \ldots, b_{\ell-1}\big)$ arise.

of D^2_r. Fix a reference point let $y \in S^1_r$ and let

$$\mu(f) \in \mathcal{M}(F_y) = \pi_0\mathrm{Diff}^+(F_y)$$

be the counterclockwise monodromy of f around S^1_r as measured in the mapping class group of F_y. Then for any cycle system for f derived from a Hurwitz system based at y the counterclockwise monodromy of f around S^1_r measured in F_y is given by the product of Dehn twists about the Lefschetz vanishing cycles $\lambda_i \subset F_y$,

$$\mu(f) = \tau_{\lambda_\ell} \circ \cdots \circ \tau_{\lambda_1} \in \mathcal{M}(F_y) = \pi_0\mathrm{Diff}^+(F_y). \tag{2.3}$$

On the other hand, we can also describe the monodromy using the boundary part of the fibration. Recall that $f^{-1}(S^1_r)$ is the boundary of a tubular neighbourhood $N\mathcal{D}$ of $\mathcal{D}$ and that the fibration structure over S^1_r essentially factors through the projection $N\mathcal{D} \to \mathcal{D}$. This exhibits $f^{-1}(S^1_r)$ as a circle bundle over $\mathcal{D}$, which is itself a circle bundle over S^1. It follows that the monodromy of f around S^1_r must fix the circle fibres of $f^{-1}(S^1_r) \to \mathcal{D}$, and the circle fibre contained in F_y is precisely the boundary vanishing cycle δ of the Hurwitz system. To conclude, $\mu(f)$ fixes δ as a set, but not necessarily pointwise. Indeed, it can (and does) happen that $\mu(f)$ reverses the orientation of δ.

Remark 2.11. At this point, it is worthwhile to point out some perks of working on a torus. First, there is the fact that any diffeomorphism of T^2 is determined up to isotopy by its action on $H_1(T^2)$. Given any pair of oriented simple closed curves $a, b \subset$ with (algebraic) intersection number $\langle a, b\rangle = 1$ — called *dual pairs* from now on — we get an identification $\mathcal{M}(T^2) \cong SL(2,\mathbb{Z})$. Moreover, the right-handed Dehn twists $A, B \in \mathcal{M}(T^2)$ about a and b are the generators in a finite presentation with relations $ABA = BAB$ and $(AB)^6 = 1$. In particular, we have that $(AB)^3$ maps to $-1 \in SL(2,\mathbb{Z})$, which we will also denote by writing $-1 = (AB)^3 \in \mathcal{M}(T^2)$. Second, in a similar fashion, simple closed curves up to ambient isotopies are uniquely determined by their (integral) homology classes. Note that this involves a choice of orientation, since simple closed curves are a priori unoriented objects. However, it is true that essential simple closed curves in T^2 correspond bijectively with primitive elements of $H_1(T^2)$ up to sign. In what follows we adopt the common bad habit of identifying simple closed curves with elements of $H_1(T^2)$ without explicitly mentioning orientation. In particular, we will freely use the homological expression for a Dehn twist, i.e. write

$$\tau_c(d) = d + \langle c, d\rangle c \in H_1(T^2). \tag{2.4}$$

We record two facts that are important for our purposes:

(1) If $h \in \mathcal{M}(T^2)$ satisfies $h(a) = a$ for some essential curve a, then $h = \pm\tau_a^k$ for some k with a negative sign if and only if h is orientation-reversing on a;
(2) If oriented curves $a, b, c \subset T^2$ satisfy $\langle a, b\rangle = \langle a, c\rangle = 1$, then $c = \tau_a^k(b) = b + ka$ for some k.

Returning to the discussion of the monodromy $\mu(f)$, we can conclude that the vanishing cycles have to satisfy the condition

$$\mu(f) = \tau_{\lambda_\ell} \circ \cdots \circ \tau_{\lambda_1} = \pm\tau_\delta^k \in \mathcal{M}(F_y), \qquad k \in \mathbb{Z}.$$

It is easy to see from the above discussion that a negative sign appears if and only if $\mathcal{D}$ fails to be co-orientable. Moreover, the integer k is precisely the Euler number of the normal bundle of $\mathcal{D}$ in X. Here we remark that a vector bundle $E \to M$ with M compact has a well-defined integer Euler number if the total space of E is orientable, even if M is not orientable itself.

For practical purposes, it is more convenient to work with cycle systems in the model T^2. Here is the upshot of the above discussion:

Proposition 2.12. *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration. If $(a; b_1, \dots, b_\ell)$ is any cycle system for f, then*

$$B_\ell \circ \cdots \circ B_1 = \pm A^k \in \mathcal{M}(T^2) \tag{2.5}$$

for some $k \in \mathbb{Z}$, where the sign is positive if and only if $\mathcal{D}$ is co-orientable. The integer k agrees with the Euler number of the normal bundle of $\mathcal{D}$ in X.

This motivates an abstract definition without reference to boundary Lefschetz fibrations.

Definition 2.13 (Abstract cycle systems)**.** An ordered collection of curves $(a; b_1, \dots, b_\ell)$ in T^2 is called an *abstract cycle system* if it satisfies the condition in (2.5). *Hurwitz equivalence* is defined exactly as in Definition 2.10.

2.2. **Handle decompositions and Kirby diagrams.**

Next we discuss one can recover boundary Lefschetz fibrations from their cycle systems. Along the way, we exhibit useful handle decompositions of total spaces of boundary Lefschetz fibrations.

Proposition 2.14. *Any abstract cycle system $(a; b_1, \dots, b_\ell)$ is the cycle system of some boundary Lefschetz fibration over the disc.*

Proof. We will build a four-manifold obtained by attaching handle to $T^2 \times D^2$. We choose points $\theta_0, \dots, \theta_\ell \in \partial D^2$ which appear in counterclockwise order and consider a copy of a in $T^2 \times \{\theta_0\}$ and of b_i in $T^2 \times \{\theta_i\}$ for $i > 0$. Note that for all these curves there is a natural choice of framing determined by parallel push-offs inside the fibres of $T^2 \times S^1 \to S^1$. We first attach 2-handle along the copies of b_i for $i > 0$ with respect to the fibre framing -1 and call the resulting manifold Z. It is well known that the projection $T^2 \times D^2$ extends to a Lefschetz fibration on Z over a slightly larger disc, which we immediately rescale to D^2, such that the Lefschetz vanishing cycles along the straight line from θ_i to zero is b_i. By construction, the boundary fibres over S^1 and the counterclockwise monodromy measured in $T^2 \times \{\theta_0\}$ is $B_\ell \circ \cdots \circ B_1 = \pm A^k$. In particular, ∂Z is diffeomorphic as an oriented manifold to the circle bundle with Euler number k over the torus or the Klein bottle. Let $N^{\pm}_{-k}$ be the corresponding disc bundle with Euler number $-k$. Then $\partial N^{\pm}_{-k}$ is diffeomorphic to ∂Z with the orientation reversed so that we can form a closed manifold X by gluing Z and $N^{\pm}_{-k}$ together, and the orientation of Z extends. Moreover, it was shown in [5] that $N^{\pm}_{-k}$ admits a boundary fibration over the annulus which can be used to extend the Lefschetz fibration on Z to a boundary Lefschetz fibration on X, again over a larger disc which we recale to D^2, in such a way that the boundary vanishing cycle along the straight line from θ_0 to zero is a.

Thus we have found a boundary Lefschetz fibration together with a Hurwitz system which produces the desired cycle system. □

Remark 2.15 (Construction of the Kirby diagram)**.** Observe that the gluing of $N^{\pm}_{-k}$ also has an interpretation in terms of handles. It is well known that $N^{\pm}_{-k}$ has a handle decomposition with one 0-handle, two 1-handles, and a single 2-handle. Turning this decomposition upside down gives a relative handle decomposition on $-\partial N^{\pm}_{-k} \cong \partial Z$ with a single 2-handle, two 3-handles, and a 4-handle. Moreover, the 2-handle can be chosen such that its core disc is a fibre. In particular, since the gluing of $N^{\pm}_{-k}$ to Z preserves the circle fibration, can arrange that the 2-handle of $N^{\pm}_{-k}$ is attached along the copy of a in the fibre of ∂Z over θ_0. However, in contrast to the Lefschetz handles, this time the framing is actually the fibre framing.

To summarize, the closed four-manifold X is obtained from $T^2 \times D^2$ by attaching, in order, a 2-handle along the boundary vanishing cycle with the fibre framing, and then 2-handles along the Lefschetz vanishing cycles $b_i \subset T^2 \times \{\theta_i\}$ with fibre framing -1. The two 3-handles as well as the 4-handle attach uniquely as explained in [9, Section 4.4].

As an illustration of this procedure, Figure 2 shows the Kirby diagrams corresponding to the abstract cycle systems $(a; a)$ and $(a; b + 2a, b)$, where $\{a, b\}$ is a dual pair of curves.

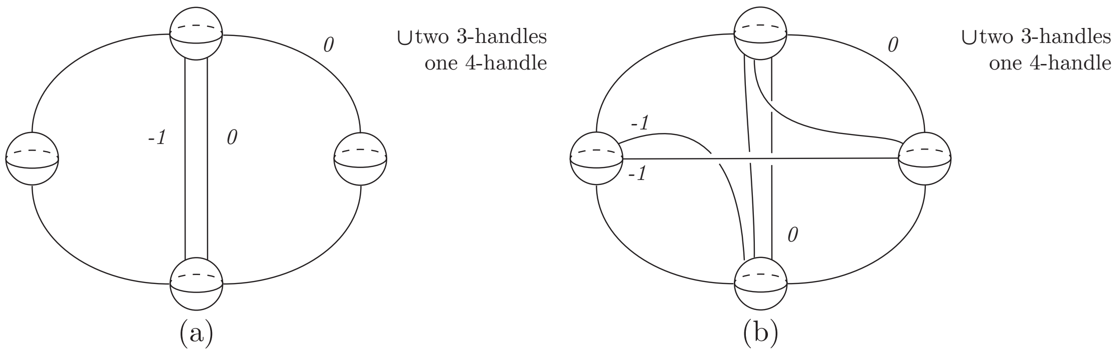

FIGURE 2. Kirby diagrams corresponding to the abstract cycle systems $(a; a)$ (Figure (a)) and $(a; b + 2a, b)$ (Figure (b)). The numbers indicate the blackboard framing of the corresponding 2-handles.

Next we show that the topology of the total space of a boundary Lefschetz fibration can be recovered from the cycle system.

Proposition 2.16. *If two boundary Lefschetz fibrations over $(D^2, \partial D^2)$ have equivalent cycle systems, then their total spaces are diffeomorphic.*

Proof. Elaborating on the proof of Proposition 2.14, one can show that, if a Hurwitz system and identification of the reference fibre with T^2 for a given boundary Lefschetz fibration produces the cycle system $(a; b_1, \dots, b_\ell)$, then its total space is diffeomorphic to the manifold constructed by attaching handles to $T^2 \times D^2$ as explained above. Similarly, one can then argue that the manifolds constructed from equivalent cycle systems are diffeomorphic. The details are somewhat tedious but straightforward and we leave them to the inclined reader. □

As a consequence, in order to classify closed four-manifolds admitting boundary Lefschetz fibrations over $(D^2, \partial D^2)$, it is enough to identify all four-manifolds obtained from abstract cycle systems as in the proof of Proposition 2.14. Moreover, as we argued in Remark 2.15, this problem is naturally accessible to the methods of Kirby calculus via the handle decompositions. For the relevant background about Kirby calculus we refer to [9] (Chapter 8, in particular).

3. Boundary Lefschetz fibrations over $(D^2, \partial D^2)$

As a warm-up to our main theorem, it is worth considering the following more basic question: Which oriented four-manifolds are boundary fibrations over $(D^2, \partial D^2)$? The answer is simple:

Lemma 3.1. *Let X be a compact, orientable manifold and let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary fibration. Then X is diffeomorphic to $S^1 \times S^3$ and $\mathcal{D}$ is co-orientable.*

Proof. Note that a boundary fibration is a boundary Lefschetz fibration without Lefschetz singularities. As such, its cycle systems consist of a single curve $a \subset T^2$ corresponding to the boundary vanishing cycle. Thus a is essential and we can therefore assume that $a = \{1\} \times S^1$. According to the discussion in Section 2.2, X is obtained from gluing $T^2 \times D^2$ together with a suitable disc bundle over a torus or Klein bottle, such that the boundary of a disc fibre is identified with a. Obviously, the only possibility is $N_0^+ = D^2 \times T^2$, the trivial disc bundle over the torus, and the gluing can be arranged such that $\partial D^2 \times \{(1,1)\} \subset N_0^+$ is identified with $a \times \{1\} \subset T^2 \times D^2$. Since this is achieved by the diffeomorphism of T^3 which flips the first two factors, we see that

$$\begin{aligned} X &\cong S^1 \times S^1 \times D^2 \cup_\varphi D^2 \times S^1 \times S^1 \\ &\cong S^1 \times S^1 \times D^2 \cup_{\mathrm{id}} S^1 \times D^2 \times S^1 \\ &\cong S^1 \times \left(S^1 \times D^2 \cup_{\mathrm{id}} D^2 \times S^1\right) \cong S^1 \times S^3, \end{aligned}$$

where the last diffeomorphism comes from the standard decomposition of S^3 considered as sitting in $\mathbb{C}^2$ and split into two solid tori by $S^1 \times S^1 \subset \mathbb{C}^2$. □

Now we move on to study honest boundary Lefschetz fibrations over the disc and eventually prove our classification theorem, Theorem 3.13. The proof of the theorem itself is done by induction on the number of singular fibres. So, in order to achieve our aim, we need to study the base cases, i.e., boundary Lefschetz fibrations with only a few singular fibres, and explain how to systematically reduce the number of singular fibres to bring us back to the base cases. It turns out that there is a step that appears frequently, namely, the blow-down of certain (-1)-spheres which is interesting on its own as it gives the notion of a relatively minimal boundary Lefschetz fibration. In the rest of this section, we will first study blow-downs and relatively minimal fibrations. We then move on to study the cases with one and two singular fibres and finally prove Theorem 3.13.

3.1. **The blow-down process and relative minimality.** Given a usual Lefschetz fibration $f\colon X^4 \to \Sigma^2$, we can perform the blow-up in a regular point $x \in X$ with respect to a local complex structure compatible with the orientation of X. The result is a manifold $\widetilde{X}$ together with a blow-down map $\sigma\colon \widetilde{X} \to X$ and it turns out that the composition $\widetilde{f} = f \circ \sigma\colon \widetilde{X} \to \Sigma$ is a Lefschetz fibration with one more critical point than f in the fibre over $y = f(x)$. Moreover, the exceptional divisor sits inside the (singular) fibre $\widetilde{f}^{-1}(y)$ as a sphere with self-intersection -1. Conversely, given any (-1)-sphere in a singular fibre of a Lefschetz fibration this process can be reversed: the (-1)-sphere can be blown down producing a Lefschetz fibration with one critical point less. For that reason it is enough to study *relatively minimal* Lefschetz fibrations: fibrations whose fibres do not contain any (-1)-spheres. Equivalently, a Lefschetz fibration is relatively minimal if no vanishing cycle bounds a disc in the reference fibre; and on the level of cycle systems the blow-up and blow-down procedures simply amount to adding or removing null-homotopic vanishing cycles.

For a boundary Lefschetz fibration $f\colon (X^4, \mathcal{D}^2) \to (\Sigma^2, Z^1)$ there is another way a (-1)-sphere can occur in relation to the fibration. These spheres arise if there is a simple path connecting a Lefschetz singular value of f to a component of Z with the property that the Lefschetz vanishing cycle in one end of the path agrees with the boundary vanishing cycle. In this case, we can form the corresponding Lefschetz thimble from the Lefschetz singularity which then closes up at the other end of the path to give rise to a (-1)-sphere, E, which intersects the divisor $\mathcal{D}$ at one point, as observed in [7]. Note that, in the case where $(\Sigma, Z) = (D^2, \partial D^2)$ this is equivalent to a cycle system $(a; b_1, \dots, b_\ell)$ such that some b_i agrees with a. From this description, it is clear that we can blow E down to obtain a new manifold, X'. What is not immediately clear is that X' admits the structure of a boundary Lefschetz fibration.

Proposition 3.2. *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration which has a cycle system $(a; b_1, \dots, b_\ell)$ such that $b_i = a$ for some i. Then there exists a boundary Lefschetz fibration $f'\colon (X', \mathcal{D}') \to (D^2, \partial D^2)$ with the cycle system $(a; A(b_1), \dots, A(b_{i-1}), b_{i+1}, \dots, b_\ell)$. Moreover, we have $X \cong X' \# \overline{\mathbb{C}P^2}$ and $\mathcal{D}'$ has the same co-orientability as $\mathcal{D}$.*

Proof. This is our first exercise in Kirby calculus. Using Hurwitz moves we have the equivalence of cycle systems:

$$\big(a; b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_\ell\big) \cong \big(a; a, A(b_1), \dots, A(b_{i-1}), b_{i+1}, \dots, b_\ell\big),$$

so we may assume without loss of generality that $b_1 = a$. Further, we can take a to be the first cycle of a dual pair $\{a, b\}$, that is, we may assume that $a = S^1 \times \{1\} \subset T^2$. We now compare the Kirby diagrams obtained from the cycle systems $(a; a, b_2, \dots, b_\ell)$ and $(a; b_2, \dots, b_\ell)$.

As we mentioned in Remark 2.15, to draw a Kirby diagram for a boundary Lefschetz fibration corresponding to a cycle system, we start with the Kirby diagram of $D^2 \times T^2$ and add cells corresponding to the boundary vanishing cycle followed by the Lefschetz vanishing cycles ordered counterclockwise. Therefore, the Kirby diagram for $(a; a, b_2, \dots, b_\ell)$ is the Kirby diagram for $(a; a)$ with a number of 2-handles on top of it representing the cycles $b_2, \dots, b_\ell$. The Kirby move we use next does not interact with these last $(l-1)$ 2-handles, therefore we will not represent them in the diagram. With this in mind, the relevant part of the Kirby diagram of $(a; a, b_2, \dots, b_\ell)$ is the Kirby diagram of $(a; a)$ as drawn in Figure 2.(a). Sliding the

-1-framed 2-handle corresponding to the first Lefschetz singularity over the 0-framed 2-handle corresponding the boundary vanishing cycle produces a -1-framed unknot which is unlinked from the rest (see Figure 3). The remaining Kirby diagram is precisely that corresponding to the cycle system $(a; b_2, \ldots, b_\ell)$. Since an isolated -1-framed unknot represents a connected sum with $\overline{\mathbb{C}P^2}$, the result follows. □

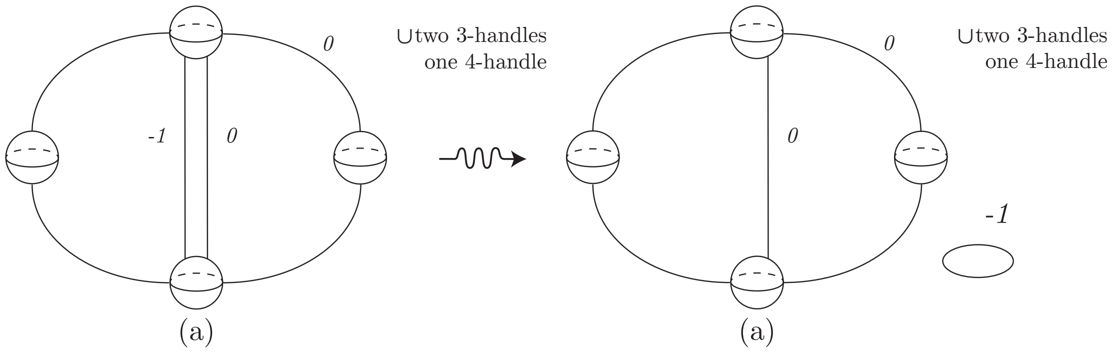

FIGURE 3. Figure showing the relevant part of the Kirby diagram of the cycle system $(a; a, b_2, \ldots, b_\ell)$ and the result of sliding the -1-framed 2-handle over the 0-framed one.

The previous proof is prototypical for much of what follows from now on. In light of Proposition 3.2 we make the following definition.

Definition 3.3 (Relative minimality)**.** Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration. Then f is called *relatively minimal* if it does not have any cycle system $(a; b_1, \ldots, b_\ell)$ in which some Lefschetz vanishing cycle b_i is either null-homotopic or isotopic to a.

3.2. **Boundary Lefschetz fibrations over $(D^2, \partial D^2)$ with few Lefschetz singularities.** The next step is to determine which manifolds admit boundary Lefschetz fibrations with only one or two Lefschetz singularities.

Lemma 3.4. *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration with one Lefschetz singularity. Then f is not relatively minimal, we have $X \cong (S^1 \times S^3)\#\overline{\mathbb{C}P^2}$, and $\mathcal{D}$ is co-orientable.*

Proof. This is [5, Example 8.4], but in light of our discussion about blow-ups in terms of cycle systems we can determine it directly. Indeed, any cycle system of f has the form $(a; b_1)$ such that $B_1 = \pm A^k$. Clearly this is only possible when b_1 is either null-homotopic or parallel to a. In either case, f is not relatively minimal and can be blown down to a boundary fibration, which, by Lemma 3.1, is diffeomorphic to $S^1 \times S^3$. □

Lemma 3.5. *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a relatively minimal boundary Lefschetz fibration with two Lefschetz singularities. Then $X \cong S^4$ and $\mathcal{D}$ is not co-orientable.*

Proof. All cycle systems of f have the form $(a; b_1, b_2)$ with b_1 and b_2 essential and not parallel to a. At this level of difficulty one can still perform direct computations. This was done by Hayano in [12]. The outcome is that for suitable orientations we must have that $b_1 = A^2(b_2) =$

$b_2 + 2a$ and $\langle a, b_1\rangle = \langle a, b_2\rangle = 1$. Using the relation $AB_2A = B_2AB_2$ and $(AB_2)^3 = -1$ in $\mathcal{M}(T^2)$ we find that

$$\mu(f) = B_2B_1 = B_2A^2B_2A^{-2} = B_2A(AB_2A)AA^{-4} = B_2A(B_2AB_2)AA^{-4} = -A^{-4}.$$

The corresponding Kirby diagram for X is given in Figure 2.(b). This particular type of Kirby diagram will appear repeatedly in this paper, so we deal with it in a separate claim.

Just as we mentioned in the proof of Proposition 3.2, when drawing the Kirby diagram for a boundary Lefschetz fibration we must draw, from bottom to top, a 0-framed 2-handle corresponding to the boundary vanishing cycle and then -1-framed 2-handles for each Lefschetz singularity ordered counterclockwise. We will often want to make simplifications to the diagram which involve only the bottom two or three 2-handles.

Lemma 3.6. *Let $a, b \subset T^2$ be a dual pair of curves. Then the Kirby diagram associated to a cycle system of the form $(a; A^k(b), b, \dots) = (a; b + ka, b, \dots)$ is equivalent to that in Figure 4.*

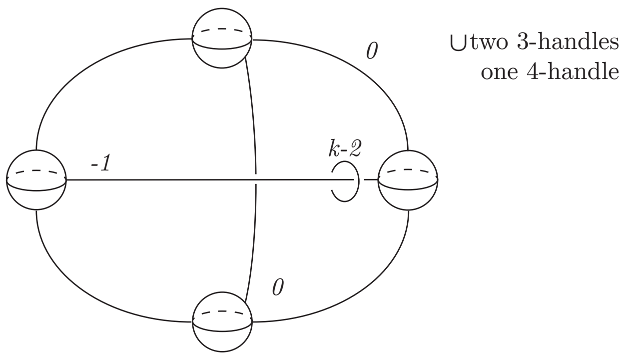

FIGURE 4. A Kirby diagram for cycle systems $(a; b + ka, b, \dots)$ after handle slides. Only the first three 2-handles are shown, the other handles appear above the diagram in their standard form. In particular, they are unlinked from the $(k-2)$-framed unknot.

Proof. The proof is a simple exercise: slide the 2-handle corresponding to $b + ka$ k times over the 0-framed 2-handle representing a and once over the 2-handle corresponding to b. None of these manoeuvres interacts with the other handles. $\square$

Proof of Lemma 3.5 continued. Using Lemma 3.6 we see that the boundary Lefschetz fibration is equivalent to the one depicted in Figure 4 with $k = 2$. If we slide the outer 2-handle over the 'a-handle' twice we get the diagram depicted in Figure 5. There, a few things happen: the outer 0-framed 2-handle can be pushed out of the 1-handle and cancels a 3-handle. The 'a-handle' cancels one of the 1-handles, and the 'b-handle' cancels the other so we are left with a 0-framed unknot which cancels the remaining 3-handle. After all this cancellation we are left only with the 0-handle and the 4-handle, hence X is S^4. $\square$

3.3. **Proof of the main theorem.** In this section we will prove our main theorem, Theorem 3.13, using an inductive procedure on the number of Lefschetz singular fibres. The key for the induction is a structural result about cycle systems of boundary Lefschetz fibrations

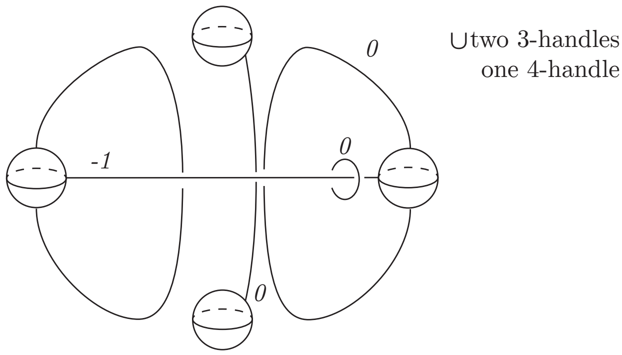

FIGURE 5. Kirby diagram for X after two handle slides. Now everything cancels.

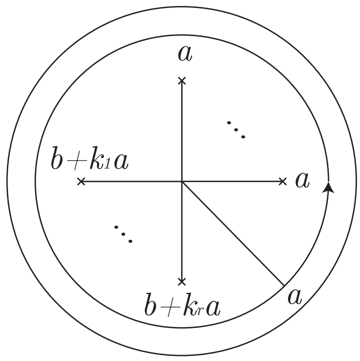

FIGURE 6. Factorisation for a boundary Lefschetz fibration from Hayano's theorem.

that was obtained by Hayano [12, 13], albeit in the slightly different but closely related context of genus-one simplified broken Lefschetz fibrations. In what follows, $a, b \subset T^2$ is a fixed dual pair of curves, and $A, B \in \mathcal{M}(T^2)$ are the corresponding Dehn twists.

Theorem 3.7 (Hayano Factorisation Theorem)**.** *Any abstract cycle system $(a; b_1, \dots, b_\ell)$ in the sense of Definition 2.13 is Hurwitz equivalent to one of the form*

$$\big(a;\ a, \dots, a,\ b + k_1 a, \dots, b + k_r a\big). \tag{3.1}$$

Moreover, for some $1 \le i < r$ we must have $k_i - k_{i+1} \in \{1, 2, 3\}$.

Proof (by reference). This is a combination of [12, Theorem 3.11] and [13, Lemma 4]. □

As a consequence, any boundary Lefschetz fibration over D^2 admits a Hurwitz system as indicated in Figure 6. Moreover, for relatively minimal fibrations we can say even more.

Corollary 3.8. *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a relatively minimal boundary Lefschetz fibration. Then f has a cycle system of the form*

$$\big(a; b + ka, b, b + na, \dots\big), \tag{3.2}$$

with $k \in \{1, 2, 3\}$ and $n \in \mathbb{Z}$.

Proof. From Theorem 3.7 and Proposition 3.2 we can deduce that f has a cycle system of the form $\big(a; b + k_1 a, \dots, b + k_r a\big)$ with $k_i - k_{i+1} \in \{1, 2, 3\}$ for some $1 \le i < r$. By Hurwitz moves we can bring the cycle system to the form $\big(a; b + k_i, b + k_{i+1} a, b + k_{i+2}, \dots\big)$. Furthermore, by applying $A^{-k_{i+1}}$ we get $\big(a; b + (k_i - k_{i+1})a, b, b + (k_{i+2} - k_{i+1})a, \dots\big)$. □

The next step is to match the pattern in the cycle systems in (3.2) with topological operations in the same spirit as Proposition 3.2. This is similar to what Hayano does while studying simplified broken Lefschetz fibrations (c.f. [12, Theorem 4.6]). We first treat the cases $k = 1, 3$.

Proposition 3.9. *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration with cycle system of the form $\big(a; b + ka, b, b_3, \dots, b_\ell\big)$ with $k \in \{1, 3\}$.*

1. *If $k = 1$, then f is not relatively minimal, that is, $X = X' \# \overline{\mathbb{C}P^2}$ where X' carries a boundary Lefschetz fibration whose divisor has the same co-orientability as $\mathcal{D}$;*
2. *If $k = 3$, then there is a boundary Lefschetz fibration $f'\colon (X', \mathcal{D}') \to (D^2, \partial D^2)$ with one fewer Lefschetz singularity. We have $X = X' \# \mathbb{C}P^2$ and the co-orientability of $\mathcal{D}'$ is opposite to that of $\mathcal{D}$. A cycle system for f' is given by $(a; b - a, b_3, \dots, b_\ell)$.*

Proof. For $k = 1$ we have

$$(a; b + a, b, \dots) \sim (a; \tau_{a+b}^{-1} b, a + b, \dots) = (a; a, a + b, \dots)$$

by a single Hurwitz move, and we can then apply Proposition 3.2.

For $k = 3$ we compare Kirby diagrams as in the proof of Proposition 3.2. We can draw a Kirby diagram for this fibration in which we represent only the handles corresponding to $b+3a$ and b and the boundary vanishing cycle and keep in mind that the handles corresponding to the other Lefschetz cycles are on top of the ones we represent in this diagram. Using Lemma 3.6 we obtain the diagram in Figure 7.(a). Sliding the 'b-handle' over the 1-framed unknot, that unknot becomes unlinked from the rest of the diagram, thereby splitting off a copy of $\mathbb{C}P^2$. Moreover, we can manipulate the remaining diagram into the shape of a Kirby diagram of a boundary Lefschetz fibration by first creating an overcrossing for the -2-framed 2-handle, so that its blackboard framing becomes -1 (see Figure 7.(c)) and then subtracting the 0-framed 2-handle representing a from the -2-framed 2-handle representing b to obtain Figure 7.(d). The final effect on the fibration is the replacement of the singularities with vanishing cycles $b + 3a = A^3(b)$ and b by one with vanishing cycle $b - a = A^{-1}(b)$. In order to understand the effect on the divisor we compare the monodromies:

$$\begin{aligned}
\tau_{b-a}^{-1}(\tau_b \circ \tau_{b+3a}) &= A^{-1}B^{-1}ABA^3BA^{-3} = A^{-1}B^{-1}(ABA)A^2BA^{-3} \\
&= A^{-1}B^{-1}(BAB)A^2BA^{-3} = A^{-1}(ABA)ABA^{-3} \\
&= A^{-1}(ABA)^2A^{-4} = A^{-1}(BA)^3A^{-4} = -A^{-5}.
\end{aligned}$$

It follows that the co-orientability is reversed by the replacement. □

The case $k = 2$ in (3.2) is a bit more complicated since it explicitly involves the third Lefschetz vanishing cycle.

Proposition 3.10. *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a boundary Lefschetz fibration with cycle system of the form $\big(a; b + 2a, b, b + na, b_4, \dots, b_\ell\big)$ with $n \in \mathbb{Z}$.*

1. *If n is even, then there is a boundary Lefschetz fibration $f'\colon (X', \mathcal{D}') \to (D^2, \partial D^2)$ with two fewer Lefschetz singularities. We have $X = X' \# S^2 \times S^2$ and the co-orientability of $\mathcal{D}'$ is opposite to that of $\mathcal{D}$. A cycle system for f' is given by $(a; b + na, b_4, \dots, b_\ell)$;*

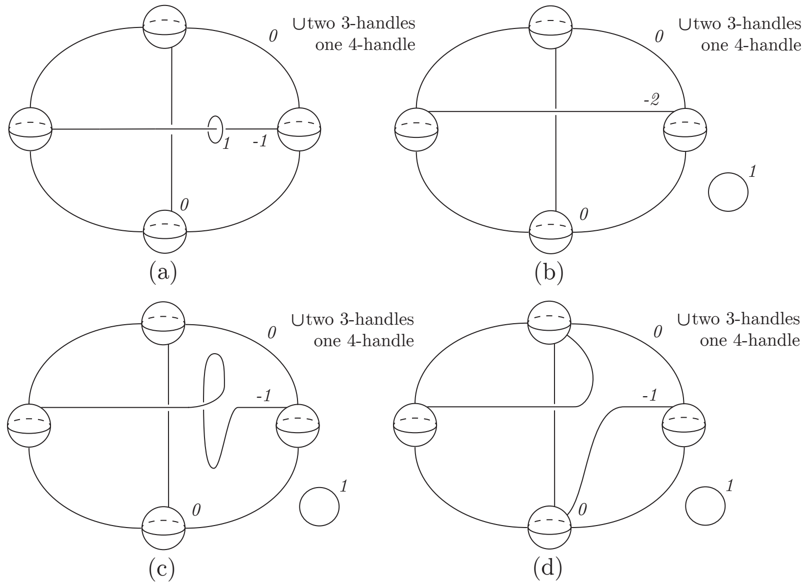

FIGURE 7. Case $k_1 - k_2 = 3$. Manipulation of the Kirby diagram to split off a copy of $\mathbb{C}P^2$.

(2) *If n is odd, the cycle system is equivalent to one of those covered by Proposition 3.9.*

Proof. Before we start drawing Kirby diagrams, we show that we can gain some more control over n, namely, we can can change it by arbitrary multiples of 4. This step is not strictly necessary for our aims, but may be of independent interest as it leads towards a classification of Lefschetz fibrations over the disc which have signed powers of Dehn twists as monodromy.

Lemma 3.11. *There is a Hurwitz equivalence of cycle systems*

$$\big(a; b+2a, b, b+na, b_4, \dots, b_\ell\big) \sim \big(a; b+2a, b, b+(n+4)a, A^4(b_4), \dots, A^4(b_\ell)\big).$$

Proof. As we saw in Lemma 3.5, the monodromy around the pair of singularities with vanishing cycles $b+2a$ and b is $-A^{-4}$, therefore, using Hurwitz moves and the fact that the vanishing cycles do not have a prefered orientation we have

$$\begin{aligned}\big(a; b+2a, b, b+na, b_4, \dots, b_\ell\big) &\sim \big(a; -A^4(b+na), -A^4(b_4), \dots, -A^4(b_\ell), b+2a, b\big)\\ &\sim \big(a; A^4(b+na), A^4(b_4), \dots, A^4(b_\ell), b+2a, b\big)\\ &= \big(a; b+(n+4)a, A^4(b_4), \dots, A^4(b_\ell), b+2a, b\big)\\ &\sim \big(a; b+2a, b, b+(n+4)a, A^4(b_4), \dots, A^4(b_\ell)\big). \qquad \square\end{aligned}$$

With this lemma at hand, we can arrange that in Hayano's factorisation as in Corollary 3.8 the cycle system is $(a; b+2a, b, b+na, \dots)$, where $n = -3, -2, -1$ or 0. It is worth looking at the four possibilities it yields. If $n = -1$, we note that

$$(a; b+2a, b, b-a, \dots) \sim (a; b, b-a, \dots) \sim (a; b+a, b, \dots),$$

which lands us back in case (1) of Proposition 3.9. Similarly, if $n = -3$, then we have

$$(a; b+2a, b, b-3a, \dots) \sim (a; b, b-3a, \dots) \sim (a; b+3a, b, \dots),$$

which lands us in case (2) of Proposition 3.9. What remains are the cases in which $n = 0$ or -2. We argue that these cases are, in fact, Hurwitz equivalent. A quick computation shows that $\tau_{b+2a}^{-1}(b) = -b - 4a$ which is just $b + 4a$ with the opposite orientation. Hence we have

$$\begin{aligned}(a; b + 2a, b, b, \dots) &\sim (a; \tau_{b+2a}^{-1}b, b + 2a, b, \dots)\\ &= (a; b + 4a, b + 2a, b, \dots)\\ &\sim (a; b + 2a, b, b - 2a, \dots).\end{aligned}$$

Now we can deal with the case $n = 0$ by drawing the Kirby diagram for the fibration. In what follows we will work only with the handles corresponding to the boundary vanishing cycle and the first three Lefschetz singularities, so we will omit the remaining 2-handles with the understanding that they remain unchanged and lay on top of the handles where the interesting part takes place. Using Lemma 3.6, this simplified Kirby diagram is drawn in Figure 8.(a). Sliding one 2-handle representing b over the other we obtain the diagram in Figure 8.(b) and we can slide the 2-handle representing b over the 0-framed 2-handle to split off a copy of $S^2 \times S^2$ from the diagram. Finally we observe that after removal of the $S^2 \times S^2$-factor, the remaining part is the Kirby diagram for the fibration with the singular fibres corresponding to $b+2a$ and b removed. Since the monodromy around these is $-A^{-4}$, the sign of the monodromy map for this new fibration is opposite to that of the original one. □

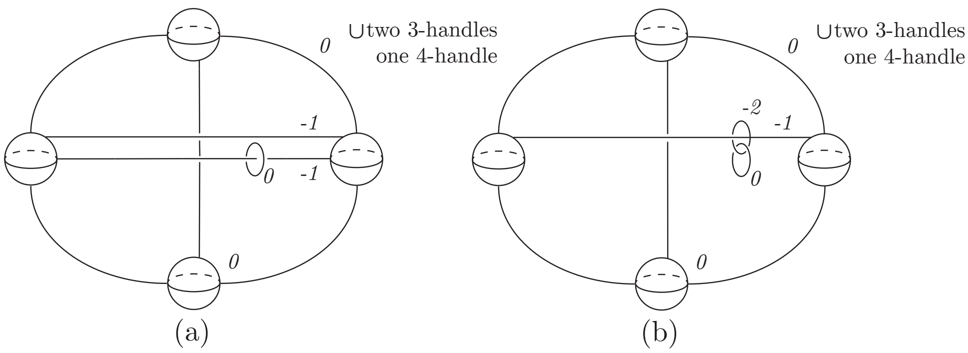

FIGURE 8. Case $k_2 = -2$. Lefschetz fibration and corresponding Kirby diagram.

Remark 3.12. For the cycle systems considered in Proposition 3.10, upon removal of the first two cycles $b + 2a$ and b, we always obtain a new cycle system $\big(a; b + na, b_4, \dots, b_\ell\big)$, regardless of the parity of n. This is because the monodromy around these two cycles is $-A^{-4}$. One can show directly using Kirby calculus that removing the first two cycles always splits off an S^2-bundle over S^2 from X, whose (non-)triviality is governed by the parity of n. In particular, in case (2), when n is odd, this exhibits X as a connected sum $X = X'\#(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$.

We now have the necessary tools to prove our main theorem:

Theorem 3.13. *Let $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ be a relatively minimal boundary Lefschetz fibration. Then X is diffeomorphic to one of the following manifolds:*

(1) $S^1 \times S^3$;
(2) $\#mS^2 \times S^2$, *including* S^4 *for* $m = 0$;

(3) $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ *with* $m > n \geq 0$.

In all cases the generic fibre is nontrivial in $H_2(X \setminus \mathcal{D}; \mathbb{R})$. *In case* (1), $\mathcal{D}$ *is co-orientable, while in cases* (2) *and* (3), $\mathcal{D}$ *is co-orientable if and only if* m *is odd.*

Proof. Firstly, recall from [5, Theorem 8.1] that the fibres of every boundary Lefschetz fibration over the disc are homologically nontrivial on $X\backslash\mathcal{D}$ because $X\backslash\mathcal{D}$ is obtained from the trivial fibration by adding Lefschetz singularities. Topologically, each of these added singularities corresponds to the addition of a 2-cell to $D^2 \times T^2$ which does not kill homology in degree 2.

The theorem is true for fibrations with at most two Lefschetz singularities by Lemma 3.1, Lemma 3.4, and Lemma 3.5. Finally, whenever there are three or more Lefschetz singularities, Hayano's factorisation theorem in the form of Corollary 3.8 shows that we can apply either Proposition 3.9 or Proposition 3.10 to pass to a boundary Lefschetz fibration with fewer Lefschetz points, whilst splitting off a copy of either $\overline{\mathbb{C}P^2}$, $\mathbb{C}P^2$, or $S^2 \times S^2$. As for the effect on the divisor, observe that each time we split off or add a connected summand that contributes to b_2^+, there is a change in co-orientability. The base case, S^4, has a negative sign (see Lemma 3.5), hence, if $f\colon (X^4, \mathcal{D}^2) \to (D^2, \partial D^2)$ is a boundary Lefschetz fibration with co-orientable $\mathcal{D}$, the number $b_2^+(X)$ must be odd and vice versa. □

As a final step, we observe that all the replacements used in the reduction process can be reversed. This allows us to produce boundary Lefschetz fibration on all the manifolds listed in Theorem 3.13.

Corollary 3.14. *Let* $(a; b_1, \dots, b_\ell)$ *be a cycle system.*

(1) *Passing to* $(a; a, b_1, \dots, b_\ell)$ *realizes a connected sum with* $\overline{\mathbb{C}P^2}$. *The co-orientability of the divisor is preserved;*
(2) *If* $\langle a, b_1\rangle = 1$, *then passing to* $(a; b_1 + 4a, b_1 + a, \dots, b_\ell)$ *realizes a connected sum with* $\mathbb{C}P^2$. *The co-orientability of the divisor is reversed;*
(3) *If* $\langle a, b_1\rangle = 1$, *then passing to* $(a; b_1, b_1 - 2a, b_1 \dots, b_\ell)$ *realizes a connected sum with* $S^2 \times S^2$. *The co-orientability of the divisor is reversed.*

Proof. This follows readily from Proposition 3.2, Proposition 3.9, and Proposition 3.10. □

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Stefan Behrens. Department of Mathematics, Utrecht University, 3508 TA Utrecht, The Netherlands
E-mail address: `s.behrens@uu.nl`

Gil R. Cavalcanti. Department of Mathematics, Utrecht University, 3508 TA Utrecht, The Netherlands
E-mail address: `g.r.cavalcanti@uu.nl`

Ralph L. Klaasse. Department of Mathematics, Utrecht University, 3508 TA Utrecht, The Netherlands
E-mail address: `r.l.klaasse@uu.nl`