Formality in generalized Kähler geometry

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Abstract

We prove that no nilpotent Lie algebra admits an invariant generalized Kähler structure. This is done by showing that a certain differential graded algebra associated to a generalized complex manifold is formal in the generalized Kähler case, while it is never formal for a generalized complex structure on a nilpotent Lie algebra.

Introduction

Generalized Kähler manifolds, as introduced by Gualtieri [9], have received recently a lot of attention from both physicists and mathematicians. From the physics point of view, they are the general solution to the $(2,2)$ supersymmetric sigma model [8], while for mathematicians they represent interesting examples of bihermitian structures [9]. A classification of manifolds which admit such structures was achieved in four dimensions [1] and finding concrete examples of such structures has been a driving force in this area [3, 12, 13, 15].

However very little is known about the differential topology of generalized Kähler manifolds. This is despite of the fact that their better known relatives, Kähler manifolds, have very restrictive topological properties, e. g., they have even 'odd Betti numbers', satisfy the strong Lefschetz property and are formal in the sense of Sullivan [16, 7]. The last property, for example, can be easily used to prove that no nilpotent Lie algebra has a Kähler structure [11]. The point of this paper is to prove the analogous result for generalized Kähler structures. We achieve this goal by providing a formality result for generalized Kähler manifolds.

Generalized Kähler manifolds are a special type of generalized complex manifolds, and as such many of their properties stem from general properties of generalized complex structures. For example, every generalized complex structure induces a decomposition of the forms analogous to the (p,q)-decomposition of forms on a complex manifold, and this decomposition causes the exterior derivative to decompose as $d = \partial + \overline{\partial}$. By studying Hodge theory on generalized Kähler manifolds, Gualtieri showed in [10] that in a generalized Kähler manifold a series of $\partial\overline{\partial}$-lemma-like hold.

Given that formality of Kähler manifolds is a consequence of the $\partial\overline{\partial}$-lemma, one might expect that Gualtieri's result implies formality of generalized Kähler manifolds by the same argument from [7]. However, in a generalized complex manifold, the operators ∂ and $\overline{\partial}$ are not derivations (an important fact in for the formality theorem for Kähler manifolds) and

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not only does that spoil the proof, but there are examples of nonformal generalized complex manifolds which *satisfy* the $\partial\overline{\partial}$-lemma [4].

In this paper we advance on this problem by abandoning the differential algebra $(\Omega^\bullet(M), d)$, and hence the question of whether M is formal, and looking at a different DGA.

A generalized complex structure can be described in terms of a Lie algebroid $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ and hence the exterior algebra $(\Gamma(\wedge^\bullet L^*), d_L)$ is a DGA. The key observation for our result is that on a generalized complex manifold with holomorphically trivial canonical bundle, this algebra is isomorphic, as a differential complex, to $(\Omega^\bullet_{\mathbb{C}}(M), \overline{\partial})$. In a generalized Kähler manifold, the operator $\overline{\partial}$ decomposes further $\overline{\partial} = \delta_+ + \delta_-$, giving rise to a decomposition $d_L = \partial_L + \overline{\partial}_L$, with the advantage that, unlike ∂, $\overline{\partial}$, δ_+ or δ_-, the operators ∂_L and $\overline{\partial}_L$ are derivations. Hence, in this setting, using the correspondence between the different operators and Gualtieri's $\partial\overline{\partial}$-lemmas we can prove formality of $(\Gamma(\wedge^\bullet L^*), d_L)$.

This result provides concrete differential-topological obstructions and allows us to prove that there are no invariant generalized Kähler structures on nilpotent Lie algebras.

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1 Differential graded algebras

In this section we give a lightening review formality for differential graded algebras and recover the well known fact that the DGA associated to a nontrivial nilpotent Lie algebra is not formal.

Definition 1. A *differential graded algebra*, or *DGA* for short, is an $\mathbb{N}$ graded vector space $\mathcal{A}^\bullet$, endowed with a product and a differential d satisfying:

1. The product maps $\mathcal{A}^i \times \mathcal{A}^j$ to $\mathcal{A}^{i+j}$ and is graded commutative:
$$a \cdot b = (-1)^{ij} b \cdot a;$$
2. The differential has degree 1, $d : \mathcal{A}^k \longrightarrow \mathcal{A}^{k+1}$, and squares to zero;
3. The differential is a derivation: for $a \in \mathcal{A}^i$ and $b \in \mathcal{A}^j$
$$d(a \cdot b) = da \cdot b + (-1)^i a \cdot db.$$

The cohomology of a DGA is defined in the standard way and naturally inherits a grading and a product, making it into a DGA on its own with $d = 0$. A *morphism* of differential graded algebras is a map preserving the structure above, i.e., degree, products and differentials. Any morphism of DGAs $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ gives rise to a morphism of cohomology $\varphi^* : H^\bullet(\mathcal{A}) \longrightarrow H^\bullet(\mathcal{B})$. A morphism φ is a *quasi-isomorphism* if the induced map in cohomology is an isomorphism.

Given a DGA, $\mathcal{A}$, for which $H^k(\mathcal{A})$ is finite dimensional for every k, one can construct another differential graded algebra that captures all the information about the differential and which is minimal in the following sense.

Definition 2. A DGA $(\mathcal{M}, d)$ is *minimal* if it is free as a DGA (i.e. polynomial in even degree and skew symmetric in odd degree) and has generators $e_1, e_2 \ldots, e_n, \ldots$ such that

1. The degree of the generators form a weakly increasing sequence of positive numbers;
2. There are finitely many generators in each degree;

3. The differential satisfies $de_i \in \wedge\mathrm{span}\{e_1, \ldots, e_{i-1}\}$.

A *minimal model* for a differential graded algebra $\mathcal{A}$ is a minimal DGA, $\mathcal{M}$, together with a quasi-isomorphism $\psi : \mathcal{M} \longrightarrow \mathcal{A}$.

Since the cohomology of a DGA is also a DGA we can also construct its minimal model. The minimal models for $\mathcal{A}$ and $H^\bullet(\mathcal{A})$ are not the same in general.

Definition 3. A DGA $\mathcal{A}$ is *formal* if it has the same minimal model as its cohomology, or equivalently, there is a quasi-isomorphism $\psi : \mathcal{M} \longrightarrow H^\bullet(\mathcal{A})$, where $\mathcal{M}$ is the minimal model of $\mathcal{A}$. A manifold M is *formal* if $(\Omega^\bullet(M), d)$ is formal.

Example 1 (Nilpotent Lie algebras [11]). A typical example of nonformal DGA can be obtained from a finite dimensional nilpotent Lie algebra $\mathfrak{g}$ with nontrivial bracket. The Lie bracket induces a differential d on $\wedge^\bullet\mathfrak{g}^*$ making it into a DGA. Furthermore, $\mathfrak{g}^*$ is filtered by $\mathfrak{g}^*_1 = \ker d$ and

$$\mathfrak{g}^*_i = \{v \in \mathfrak{g}^* : dv \in \wedge^2\mathfrak{g}^*_{i-1}\}.$$

Nilpotency implies that $\mathfrak{g}^*_s = \mathfrak{g}^*$ for some s. Let $\{e^1, \cdots, e^n\}$ be a basis for $\mathfrak{g}^*$ compatible with this filtration. Then

$$de^i \in \wedge^2\mathrm{span}\{e^1, \cdots, e^{i-1}\}.$$

showing that $(\wedge^\bullet\mathfrak{g}^*, d)$ is minimal.

Since the bracket is nontrivial, $de^n \neq 0$ and hence one can see that $e^1 \wedge \cdots \wedge e^{n-1}$ is exact and $e^1 \wedge \cdots \wedge e^n$ is a volume element and therefore represents a nontrivial cohomology class. If $(\wedge\mathfrak{g}^*, d)$ was formal, there would be a map $\psi : (\wedge^\bullet\mathfrak{g}^*, d) \longrightarrow H^\bullet(\mathfrak{g})$, but

$$0 \neq \psi(e^1 \wedge \cdots \wedge e^n) = \psi(e^1 \wedge \cdots \wedge e^{n-1}) \cdot \psi(e^n) = 0 \cdot \psi(e^n) = 0.$$

So there is no such ψ and $\wedge^\bullet\mathfrak{g}^*$ is not formal.

2 Generalized complex structures and Lie algebroids

In this section we recall the definition of generalized complex structures and their relation to Lie algebroids, following [9].

Given a closed 3-form H on a manifold M, we define the *Courant bracket* of sections of $T \oplus T^*$, the sum of the tangent and cotangent bundles, by

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(\eta(X) - \xi(Y)) + i_Y i_X H.$$

The bundle $T \oplus T^*$ is also endowed with a natural symmetric pairing of signature (n, n):

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\eta(X) + \xi(Y)).$$

Definition 4. A *generalized complex structure* on a manifold with closed 3-form (M, H) is a complex structure on the bundle $T \oplus T^*$ which preserves the natural pairing and whose i-eigenspace is closed under the Courant bracket.

A generalized complex structure can be fully described in terms of its i-eigenspace L, which is a maximal isotropic subspace of $T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ satisfying $L \cap \overline{L} = \{0\}$.

Two extreme examples of generalized complex structures, with $H = 0$, are given by complex and symplectic structures: in a complex manifold we let $L = T^{0,1} \oplus T^{*1,0}$ and in a symplectic manifold we let $L = \{X - i\omega(X) : X \in T_{\mathbb{C}}\}$, where ω is the symplectic form. What distinguishes these structures is their *type* which is the dimension of the kernel of $\pi : L \longrightarrow T_{\mathbb{C}}$. So, a complex structure on M^n has type n at all points and symplectic structures have type zero at all points.

The Courant bracket does not satisfy the Jacobi identity. Instead we have the relation for the Jacobiator

$$\mathrm{Jac}\,(A,B,C) := [\![[\![A,B]\!],C]\!] + c.p. = \tfrac{1}{3}d(\langle[\![A,B]\!],C\rangle + c.p.),$$

where *c.p.* stands for cyclic permutations. However, the identity above also shows that the Courant bracket induces a Lie bracket when restricted to sections of any involutive isotropic space L. This Lie bracket together with the projection $\pi_T : L \longrightarrow TM$, makes L into a Lie algebroid and allows us to define a differential d_L on $\Omega^\bullet(L^*) = \Gamma(\wedge^\bullet L^*)$ making it into a DGA. If L is the i-eigenspace of a generalized complex structure, then the natural pairing gives an isomorphism $L^* \cong \overline{L}$ and with this identification $(\Omega^\bullet(\overline{L}), d_L)$ is a DGA.

If a generalized complex structure has type zero over M, i.e., is of symplectic type, then $\pi : L \xrightarrow{\cong} T_{\mathbb{C}}$ is an isomorphism and the Courant bracket on $\Gamma(L)$ is mapped to the Lie bracket on $\Gamma(T_{\mathbb{C}})$. Therefore, in this particular case, $(\Omega^\bullet(\overline{L}), d_L)$ and $(\Omega^\bullet_{\mathbb{C}}(M), d)$ are isomorphic DGAs.

2.1 Decomposition of forms

A generalized complex structure can also be described using differential forms. Recall that the exterior algebra $\wedge^\bullet T^*$ carries a natural spin representation for the metric bundle $T \oplus T^*$; the Clifford action of $X + \xi \in T \oplus T^*$ on $\rho \in \wedge^\bullet T^*$ is

$$(X+\xi)\cdot\rho = i_X\rho + \xi\wedge\rho.$$

The subspace $L \subset T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ annihilating a spinor $\rho \in \wedge^\bullet T^*_{\mathbb{C}}$ is always isotropic. If L is maximal isotropic, then ρ is called a *pure spinor* and it must have the following algebraic form at every point:

$$\rho = e^{B+i\omega}\wedge\Omega, \tag{1}$$

where B and ω are real 2-forms and Ω is a decomposable complex form. Pure spinors annihilating the same space must be equal up to rescaling, hence a maximal isotropic $L \subset T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ may be uniquely described by a line subbundle $U \subset \wedge^\bullet T^*_{\mathbb{C}}$.

For a complex manifold $U = \wedge^{n,0}T^*$ and for a symplectic manifold U is generated by the globally defined closed form $e^{i\omega}$. In general we have the following definition.

Definition 5. Given a generalized complex structure $\mathcal{J}$, the line subbundle $U \subset \wedge^\bullet T^*_{\mathbb{C}}$ annihilating its i-eigenspace is the *canonical bundle* of $\mathcal{J}$.

Note that the condition $L \cap \overline{L} = \{0\}$ at the fiber of E over $p \in M$ is equivalent to the requirement that

$$\Omega\wedge\overline{\Omega}\wedge\omega^{n-k} \neq 0 \tag{2}$$

for a generator $\rho = e^{B+i\omega}\wedge\Omega$ of U at p, where $k = \deg(\Omega)$ and $2n = \dim(M)$.

By letting $\wedge^\bullet\overline{L} \subset \mathrm{Cliff}\,(L\oplus\overline{L})$ act on the canonical line bundle we obtain a decomposition of the differential forms on M^{2n}:

$$\wedge^\bullet T^*_{\mathbb{C}}(M) = \oplus_{k=-n}^{n} U^k, \qquad \text{where } U^k = \wedge^{n-k}\overline{L}\cdot\rho.$$

one can also describe the spaces U^k as the ik-eigenspaces of $\mathcal{J}$ acting on forms.

Letting $\mathcal{U}^k = \Gamma(U^k)$ and $d_H = d + H\wedge$, Courant integrability of the generalized complex structure is equivalent to

$$d_H : \mathcal{U}^k \longrightarrow \mathcal{U}^{k+1}\oplus\mathcal{U}^{k-1},$$

which allows us to define operators $\partial : \mathcal{U}^k \longrightarrow \mathcal{U}^{k+1}$ and $\overline{\partial} : \mathcal{U}^k \longrightarrow \mathcal{U}^{k-1}$ by composing d_H with the appropriate projections.

Given a local section ρ of the canonical bundle the operator $\overline{\partial}$ is related to d_L by

$$\overline{\partial}(\alpha \cdot \rho) = (d_L \alpha) \cdot \rho + (-1)^{|\alpha|} d_H \rho,$$

where $\alpha \in \Omega^\bullet(\overline{L})$ and $|\alpha|$ is the degree of α. In the particular case when $(M, \mathcal{J})$ has *holomorphically trivial canonical bundle*, i.e., there is a nonvanishing d_H-closed global section ρ of the canonical bundle, the above becomes

$$\overline{\partial}(\alpha \cdot \rho) = (d_L \alpha) \cdot \rho \tag{3}$$

and hence ρ furnishes an isomorphism of differential complexes.

3 Generalized Kähler manifolds

In this section we introduce generalized Kähler manifolds. For these manifolds both $\Omega^\bullet_{\mathbb{C}}(M)$ and $(\Omega^\bullet(\overline{L}), d_L)$ admit a bigrading and, in certain conditions, some differential operators $\Omega^\bullet_{\mathbb{C}}(M)$ correspond to differential operators on $\Omega^\bullet(\overline{L})$. This correspondence was also used by Yi Li to study the moduli space of a generalized Kähler structure [14] and is the key ingredient for our formality theorem.

Definition 6. A *generalized Kähler structure* on a manifold M^{2n} is a pair of commuting generalized complex structures $\mathcal{J}_1$, $\mathcal{J}_2$ on M such that

$$\langle \mathcal{J}_1 \mathcal{J}_2 v, v \rangle > 0 \qquad \text{for } v \in T \oplus T^* \backslash \{0\}.$$

Let L_i be the i-eingenspace of $\mathcal{J}_i$. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ communte, $\mathcal{J}_2$ furnishes a complex structure on L_1 with i-eigenspace $L_1 \cap L_2$. Using the fact that the natural pairing has signature (n, n) and that $\langle \mathcal{J}_1 \mathcal{J}_2 \cdot, \cdot \rangle$ is positive definite one can show $\dim(L_1) = 2 \dim(L_1 \cap L_2)$. Since L_2 is closed under the Courant bracket, we see that $L_1 \cap L_2$ is closed under the bracket in the Lie algebroid L_1, and hence $\mathcal{J}_2|_{L_1}$ is an integrable complex structure on L_1. Using this complex structure we can decompose

$$\wedge^\bullet \overline{L_1} = \oplus_{p,q} \wedge^{p,q} \overline{L_1} \qquad \text{and} \qquad d_{L_1} = \partial_{L_1} + \overline{\partial}_{L_1},$$

As in a complex manifold, the operators ∂_{L_1} and $\overline{\partial}_{L_1}$ are derivations, in the sense that they satisfy the Leibniz rule.

A generalized Kähler structure also gives a refinement of the deposition of forms into the spaces U^k. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute one immediately obtains that the space of differential forms can be decomposed in terms of the eigenspaces of $\mathcal{J}_1$ and $\mathcal{J}_2$: $U^{p,q} = U^p_{\mathcal{J}_1} \cap U^q_{\mathcal{J}_2}$. This allows us to decompose d_H further in 4 components

$$d_H : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p+1,q+1} + \mathcal{U}^{p+1,q-1} + \mathcal{U}^{p-1,q+1} + \mathcal{U}^{p-1,q-1}.$$

In this case, the operator $\overline{\partial}$ for the generalized complex structure $\mathcal{J}_1$ corresponds to the sum of the last two terms:

$$\overline{\partial}_1 : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p-1,q+1} + \mathcal{U}^{p-1,q-1},$$

and we can define δ_+ and δ_- as the projections of $\overline{\partial}_1$ into each of the components

$$\delta_+ : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p-1,q+1} \qquad \delta_- : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p-1,q-1}.$$

By studying the Hodge theory of a generalized Kähler manifold, Gualtieri proved the following

Theorem 3.1. (Gualtieri [10]) **$\delta_+\delta_-$-lemma**. *In a compact generalized Kähler manifold*

$$\mathrm{Im}\,\delta_+ \cap \mathrm{Ker}\,\delta_- = \mathrm{Im}\,\delta_- \cap \mathrm{Ker}\,\delta_+ = \mathrm{Im}\,(\delta_+\delta_-)$$

If $\mathcal{J}_1$ has holomorphically trivial canonical bundle, then the correspondence between $\overline{\partial}_1$ and d_{L_1} given in equation (3) also furnishes a correspondence between the operators ∂_{L_1} and $\overline{\partial}_{L_1}$ on $\Omega^\bullet(\overline{L})$ and the operators δ_+ and δ_- on $\Omega^\bullet_{\mathbb{C}}(M)$. So, as a consequence of Theorem 3.1, the operators ∂_{L_1} and $\overline{\partial}_{L_1}$ satisfy the $\partial_{L_1}\overline{\partial}_{L_1}$-lemma and since they are derivations the same argument from [7] gives:

Theorem 3.2. *If $(M, \mathcal{J}_1, \mathcal{J}_2)$ is a compact generalized Kähler manifold and $\mathcal{J}_1$ has holomophically trivial canonical bundle, then the DGA $(\Omega^\bullet(\overline{L_1}), d_{L_1})$ is formal.*

In the case when $\mathcal{J}_1$ is a symplectic structure, then not only does it have a holomorphically trivial canonical bundle, but $(\Omega^\bullet(\overline{L_1}), d_{L_1})$ is isomorphic to $(\Omega^\bullet_{\mathbb{C}}(M), d)$. Therefore we have:

Corollary 1. *If $(M, \mathcal{J}_1, \mathcal{J}_2)$ is a compact generalized Kähler manifold and $\mathcal{J}_1$ is a symplectic structure, then M is formal.*

This corollary generalizes the original theorem of formality of Kähler manifolds [7].

4 Nilpotent Lie algebras

In this section we use Theorem 3.2 to prove that no nilpotent Lie algebra admits a generalized Kähler structure. Before we state the theorem we should stress that a generalized complex structure on a Lie algebra $\mathfrak{g}$ with closed 3-form $H \in \wedge^3\mathfrak{g}^*$ is just an integrable linear complex structure on $(\mathfrak{g}\oplus\mathfrak{g}^*, [\![\ ,\ ]\!])$, orthogonal with respect to the natural pairing, where the Courant bracket is defined by

$$[\![X+\xi, Y+\eta]\!] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + i_Y i_X H,$$

and is a Lie bracket in this situation.

We also recall that a complex structure on a Lie algebra $\mathfrak{g}$ is called *abelian* if its i-eigenspace, $\mathfrak{g}^{1,0}$, is an abelian subalgebra of $\mathfrak{g}\otimes\mathbb{C}$ [2, 6]. By analogy, we say that a generalized complex structure on $\mathfrak{g}$ is *abelian* if the corresponding complex structure on $\mathfrak{g}\oplus\mathfrak{g}^*$ is abelian. Before we state our theorem on generalized Kähler structures on nilpotent Lie algebras we need a little lemma:

Lemma 1. *If a Lie algebra $\mathfrak{g}$ admits an abelian generalized complex structure, then $\mathfrak{g}$ is abelian.*

Proof. Let L be the i-eigenspce of an abelian generalized complex structure on $\mathfrak{g}$. Since L is abelian, so is its projection over $\mathfrak{g}\otimes\mathbb{C}$. Further, if $v \in \pi(L)\cap\pi(\overline{L})$ then v is a central element in $\mathfrak{g}_{\mathbb{C}}$. Indeed for such a v there is a $\xi \in \mathfrak{g}^*_{\mathbb{C}}$ such that $\mathcal{J}(v+\xi) \in \mathfrak{g}^*_{\mathbb{C}}$ so, for $w \in \mathfrak{g}_{\mathbb{C}}$

$$\begin{aligned}
4[v,w] &= 4\pi([\![v+\xi, w]\!]) \\
&= \pi([\![v+\xi+i\mathcal{J}(v+\xi)+v+\xi-i\mathcal{J}(v+\xi), w+i\mathcal{J}w+w-i\mathcal{J}w]\!]) \\
&= \pi([\![v+\xi+i\mathcal{J}(v+\xi), w+i\mathcal{J}w]\!] + [\![v+\xi+i\mathcal{J}(v+\xi), w-i\mathcal{J}w]\!] + \\
&+ [\![v+\xi-i\mathcal{J}(v+\xi), w+i\mathcal{J}w]\!] + [\![v+\xi-i\mathcal{J}(v+\xi), w-i\mathcal{J}w]\!]) \\
&= \pi([\![v+\xi+i\mathcal{J}(v+\xi), w-i\mathcal{J}w]\!] + [\![v+\xi-i\mathcal{J}(v+\xi), w+i\mathcal{J}w]\!]) \\
&= \pi([\![v+\xi-i\mathcal{J}(v+\xi), w-i\mathcal{J}w]\!] + [\![v+\xi+i\mathcal{J}(v+\xi), w+i\mathcal{J}w]\!]) = 0,
\end{aligned}$$

where we have used that L and $\overline{L}$ are abelian in the fourth and in the last equalities and in the fifth equality we used that $\mathcal{J}(v+\xi) \in \mathfrak{g}^*_{\mathbb{C}}$, hence the change of signs does not affect the projection of the bracket onto $\mathfrak{g}_{\mathbb{C}}$.

If we let $e^{B+i\omega} \wedge \Omega$, with $\Omega = \theta^1 \wedge \cdots \wedge \theta^k$, be a generator for the canonical bundle of $\mathcal{J}$, then $\pi(L) \cap \pi(\overline{L})$ is the annihilator of $\Omega \wedge \overline{\Omega}$. Since $\theta^i \in L$ there are $\overline{\partial_j} \in \overline{L}$ such that $\langle \theta^i, \overline{\partial_j} \rangle = \delta^i_j$ and we can compute

$$\theta^i([\pi(\partial_j), \pi(\overline{\partial_k})]) = d\theta^i(\pi(\partial_j), \pi(\overline{\partial_k})) = \langle [\![\theta^i, \partial_j]\!], \overline{\partial_k} \rangle = 0$$

since $\theta^i, \partial_j \in L$. Analogously we see that $[\pi(\partial_j), \pi(\overline{\partial_k})]$ also annihilates $\overline{\theta^i}$ and hence $[\pi(\partial_j), \pi(\overline{\partial_k})] \in \pi(L) \cap \pi(\overline{L})$, hence $\mathfrak{g}$ is either abelian or 2-step nilpotent.

If $\mathfrak{g}$ was 2-step nilpotent there would be an element $\xi \in \mathfrak{g}^*$ with $d\xi \neq 0$. Since the only nonvanishing brackets are of the form $[\pi(\partial_i), \pi(\overline{\partial}_j)]$ and ξ is real, we see that there is a ∂_i for which $d\xi(\pi(\partial_i), \pi(\overline{\partial}_i)) = \xi([\pi(\partial_i), \pi(\overline{\partial}_i)]) \neq 0$. Since all the θ^i are closed, we can further assume that $\xi = \mathcal{J}(v - B(v))$, for some $v \in \mathfrak{g}$, therefore $v - B(v) - i\xi \in L$ and

$$0 = \langle [\![v - B(v) - i\xi, \partial_i]\!], \overline{\partial}_i \rangle = -id\xi(\pi(\partial_i), \pi(\overline{\partial}_i)) + (H + dB)(v, \pi(\partial_i), \pi(\overline{\partial}_i)).$$

Observe that the first term is real and nonzero while the second is purely imaginary, hence the equation above can never hold and $\mathfrak{g}$ is abelian. □

Theorem 4.1. *If a nilpotent Lie algebra* $\mathfrak{g}$ *admits a generalized Kähler structure, then* $\mathfrak{g}$ *is abelian.*

Proof. According to [5], Theorem 3.1, every generalized complex structure on a nilpotent Lie algebra $\mathfrak{g}$ has holomorphically trivial canonical bundle. Further, for any closed $H \in \wedge^3\mathfrak{g}^*$, $\mathfrak{g} \oplus \mathfrak{g}^*$ with the Courant bracket is again a nilpotent Lie algebra, hence the i-eigenspace, L, of any generalized complex structures is a nilpotent Lie subalgebra of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathbb{C}$. According to Lemma 1, if $\mathfrak{g}$ has nontrivial bracket, then L has a nontrivial bracket. Then, Example 1 shows that $(\wedge^\bullet \overline{L}, d_L)$ is not formal and hence, by Theorem 3.2, can not be part of a generalized Kähler pair. □

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