Generalized Kähler geometry of instanton moduli spaces

Henrique Bursztyn [*] Gil R. Cavalcanti [†] Marco Gualtieri [‡]

Abstract

We prove that Hitchin's generalized Kähler structure on the moduli space of instantons over a compact, even generalized Kähler four-manifold may be obtained by generalized Kähler reduction, in analogy with the usual Kähler case. The underlying reduction of Courant algebroids is a realization of Donaldson's μ-map in degree three.

Contents

[*]Instituto de Matemática Pura e Aplicada; henrique@impa.br

[†]Utrecht University; g.r.cavalcanti@uu.nl

[‡]University of Toronto; mgualt@math.toronto.edu

1 Introduction

The moduli space $\mathcal{M}$ of instantons over a four-manifold M often inherits geometric structures when M is endowed with more than the required conformal geometry. For example, if M is equipped with a complex, holomorphic Poisson, strong Kähler with Torsion (KT), Kähler, hypercomplex, strong hyper-Kähler with Torsion (HKT) or hyper-Kähler structure, $\mathcal{M}$ inherits the same structure, see e.g. [2, 13, 8, 11]. As proven by Hitchin [12], this also holds when M has a generalized Kähler structure of even type. The main goal of this paper is to provide a new approach to this result, which gives further insight into the geometry of the moduli space.

Hitchin's proof relies on combining the work of Lübke and Teleman [13], who establish the analogous result for strong KT structures, with a theorem from [10] which states that a generalized Kähler structure is equivalent to a compatible pair of strong KT structures. By an explicit computation, Hitchin establishes that the induced strong KT structures on $\mathcal{M}$ are compatible. In the special case that M is Kähler, there is a more direct approach to this construction: the space $\mathcal{A}$ of connections admits a natural gauge-invariant Kähler structure, and the induced Kähler structure on $\mathcal{M}$ can be understood as an infinite-dimensional symplectic reduction of a complex submanifold of $\mathcal{A}$ (see e.g., [8, Sec. 6.5]). This led Hitchin to ask, in [12], whether there was a generalized Kähler reduction procedure underlying his results.

In this paper, we apply the theory of generalized Kähler reduction, developed in [6], to answer this question in the affirmative. To outline our construction, we briefly recall the generalized reduction procedure, following [5, 6].

Classically, the reduction of a Kähler manifold N involves the action of a Lie group G by symmetries, infinitesimally described by a Lie algebra map

$$\psi : \mathfrak{g} \to \Gamma(TN), \tag{1.1}$$

admitting an equivariant moment map

$$\mu : N \to \mathfrak{g}^*. \tag{1.2}$$

Under appropriate conditions, the reduced space $\mu^{-1}(0)/G$ obtains the structure of a Kähler manifold.

In generalized geometry, we study structures on $TN \oplus T^*N$ that are compatible with the Courant algebroid structure determined by a closed 3-form $H \in \Omega^3(N)$. By the theory developed in [5], the reduction of any generalized geometry on N should be preceded by the reduction of its underlying Courant algebroid. This step of "Courant reduction" is independent of specific generalized geometric structures on N, and it presents some novelties: first, actions are allowed to have "cotangent components", i.e., usual actions (1.1) are lifted to maps

$$\widetilde{\psi} : \mathfrak{g} \to \Gamma(TN \oplus T^*N), \tag{1.3}$$

compatible with the Courant bracket on $TN \oplus T^*N$; second, a moment map

$$\mu : N \to \mathfrak{h}^* \tag{1.4}$$

may take values in a G-module $\mathfrak{h}^*$ which differs from the co-adjoint module $\mathfrak{g}^*$. Using these ingredients, Courant reduction produces, under usual smoothness assumptions, a Courant algebroid over the reduced space $\mu^{-1}(0)/G$. Once this reduction is in place, any generalized geometric structure on N, compatible with the action (1.3) and moment map (1.4), descends to $\mu^{-1}(0)/G$.

Our study of the instanton moduli space showcases all the above features of generalized reduction. Consider the instanton moduli space $\mathcal{M}$, obtained as a reduction of an open set in the space of connections $\mathcal{A}$ on a principal G-bundle E: we first impose the anti-self-dual condition $F_+^A = 0$, for $A \in \mathcal{A}$, and then quotient by the group $\mathscr{G}$ of gauge transformations. Any anti-self-dual connection A gives rise to an elliptic complex

$$0 \longrightarrow \Omega^0(M, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(M, \mathfrak{g}_E) \xrightarrow{d_{A+}} \Omega^2_+(M, \mathfrak{g}_E) \longrightarrow 0,$$

where $\mathfrak{g}_E$ is the adjoint bundle associated to E and d_{A+} is the projection of the exterior covariant derivative to the self-dual forms. The map $d_A : \Omega^0(M, \mathfrak{g}_E) \to \Omega^1(M, \mathfrak{g}_E)$ is interpreted as the infinitesimal gauge action $\psi : \mathrm{Lie}(\mathscr{G}) \to T\mathcal{A}$, and the kernel of d_{A+} is the infinitesimal counterpart of the anti-self-dual condition, so that the middle cohomology of the complex yields the tangent space $T_{[A]}\mathcal{M}$.

When M is Kähler, the kernel of d_{A+} may be viewed as the condition imposed by a symplectic moment map $\mathcal{A} \to \Omega^4(M, \mathfrak{g}_E) \cong \Omega^0(M, \mathfrak{g}_E)^*$ (see e.g. [8, Sec. 6.5.3]). Since this relies on the symplectic form on M, it does not immediately extend to the generalized Kähler case.

The generalized Kähler reduction procedure begins with the Courant reduction of the space of connections $\mathcal{A}$, endowed with the zero 3-form. For this reduction, it is enough to assume that M is endowed with a closed 3-form H, an orientation and a Riemmanian structure. The closed 3-form H is used to lift the infinitesimal gauge action to $\widetilde{\psi} : \Omega^0(M, \mathfrak{g}_E) \to \Gamma(T\mathcal{A} \oplus T^*\mathcal{A})$, via

$$\widetilde{\psi}(\gamma)|_A = d_A^H \gamma := (d_A + H\wedge)\gamma \ \in \ \Omega^1(M, \mathfrak{g}_E) \oplus \Omega^3(M, \mathfrak{g}_E),$$

where we identify $T_A^*\mathcal{A} \cong \Omega^3(M, \mathfrak{g}_E)$. The moment map for Courant reduction assigns to each connection the self-dual component of its curvature

$$\mu : \mathcal{A} \to \Omega^2_+(M, \mathfrak{g}_E), \ \ \mu(A) = F_+^A,$$

and it does not take values in the dual of the gauge Lie algebra. The corresponding reduced space is $\mathcal{M}$, and the Courant reduction identifies $T_{[A]}\mathcal{M} \oplus T^*_{[A]}\mathcal{M}$ with the middle cohomology of the elliptic complex

$$0 \longrightarrow \Omega^{ev}_+(M, \mathfrak{g}_E) \xrightarrow{d_A^H} \Omega^{od}(M, \mathfrak{g}_E) \xrightarrow{d_{A+}^H} \Omega^{ev}_+(M, \mathfrak{g}_E) \longrightarrow 0. \tag{1.5}$$

When M is endowed with an even generalized Kähler structure, the central question is whether the induced generalized Kähler structure on $\mathcal{A}$ is compatible with Courant reduction, so as to carry over to $\mathcal{M}$. We translate this compatibility condition into a Hodge-theoretic question, namely, whether the cohomology of the complex (1.5) inherits a (p, q)-decomposition from the corresponding decomposition of forms induced by the generalized Kähler structure. By extending the results of [9] on the Hodge theory of generalized Kähler manifolds to allow coefficients in $\mathfrak{g}_E$, we prove that the cohomology does decompose and the generalized Kähler structure on $\mathcal{M}$ obtained by Hitchin agrees with the one obtained by generalized Kähler reduction.

This paper is organized as follows. Section 2 recalls the basics of generalized complex and generalized Kähler geometry, while Section 3 reviews the relevant generalized reduction theorems.

In Section 4 we consider generalized reduction in the context of the moduli space of instantons, describing the reduced Courant algebroid, the induced generalized metric and 3-form, and proving that if M has an even generalized Kähler structure then $\mathcal{M}$ inherits a generalized Kähler structure via the reduction procedure.

Acknowledgements. We thank Nigel Hitchin, Ruxandra Moraru, and Andrei Teleman for assistance and helpful discussions. Henrique Bursztyn was supported by CNPq and FAPERJ. Gil Cavalcanti was supported by a Marie Curie Intra-European Fellowship. Marco Gualtieri was supported by a NSERC Discovery grant and an Ontario ERA.

2 Generalized complex and Kähler structures

Let M be an m-dimensional smooth manifold and $H \in \Omega^3(M)$ be a closed 3-form. In generalized geometry, one considers the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, endowed with the natural pairing

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\eta(X)+\xi(Y)), \qquad X, Y \in TM,\ \xi, \eta \in T^*M, \tag{2.1}$$

and the *Courant bracket* on its space of sections,

$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi - i_Y i_X H. \tag{2.2}$$

The bundle $\mathbb{T}M$ is also equipped with the natural projection

$$\pi_T : \mathbb{T}M \longrightarrow TM, \tag{2.3}$$

called the *anchor map*, which is bracket preserving. If the 3-form is clear from the context we write simply $[\cdot,\cdot]$ for the Courant bracket.

Given a 2-form $B \in \Omega^2(M)$, we can think of it as an endomorphism of $\mathbb{T}M$ given by $B(X+\xi) = -i_X B$. By exponentiating such maps,

$$e^B(X+\xi) = X + \xi - i_X B, \tag{2.4}$$

one obtains an action of the abelian group $\Omega^2(M)$ on $\mathbb{T}M$ by transformations which preserve the natural pairing (2.1), the anchor map π_T, and relate to the Courant bracket as follows:

$$[e^B v_1, e^B v_2]_{H-dB} = e^B[v_1, v_2]_H \qquad v_i \in \Gamma(\mathbb{T}M). \tag{2.5}$$

The action of a 2-form B preserves the subspace $T^*M \subset \mathbb{T}M$, but does not preserve TM, sending it to another isotropic complement of T^*M (with respect to (2.1)). Conversely, different choices of isotropic complements to T^*M are related to each other by the action of a 2-form.

Since the natural pairing on $\mathbb{T}M$ has split signature and T^*M is a maximal isotropic subspace, $\wedge^\bullet T^*M$ is naturally the space of spinors for $\mathrm{Clif}(\mathbb{T}M)$ and hence is endowed with a spin invariant bilinear form, the *Chevalley pairing*: for $\varphi = \sum \varphi_j, \psi = \sum \psi_j \in \wedge^\bullet T^*M$, with $\deg(\varphi_j) = \deg(\psi_j) = j$, we have

$$(\varphi, \psi)_{Ch} = -(\varphi \wedge \psi^t)_{top} = \sum_j (-1)^{\frac{(m-j)(m-j-1)}{2}+1} \varphi_j \wedge \psi_{m-j}, \tag{2.6}$$

where the superscript t denotes the Clifford transposition.

Definition 2.1. A *generalized metric* on M is an orthogonal and self-adjoint bundle automorphism $\mathbb{G} : \mathbb{T}M \longrightarrow \mathbb{T}M$ for which the bilinear form $\langle \mathbb{G}v, w\rangle$, $v, w \in \mathbb{T}M$, is positive definite.

Since $\mathbb{G}$ is orthogonal and self adjoint we have that $\mathbb{G}^{-1} = \mathbb{G}^t = \mathbb{G}$, hence $\mathbb{G}^2 = \mathrm{Id}$ and $\mathbb{G}$ splits $\mathbb{T}M$ into its ± 1 eigenbundles, denoted by $V_\pm$. Since T^*M is isotropic, $V_\pm \cap T^*M = \{0\}$ and the anchor map π_T restricts to isomorphisms between each of $V_\pm$ and TM. A generalized metric $\mathbb{G}$ induces a bona fide metric g on M, given by the restriction of the pairing (2.1) to V_+, identified with TM via the anchor map. It is also clear that $\mathbb{G}(T^*M)$, the metric orthogonal complement of T^*M, is an isotropic subspace of $\mathbb{T}M$ which is transverse to T^*M. Hence a metric determines a natural splitting, referred to as the *metric splitting*, of $\mathbb{T}M$ as $\mathbb{G}(T^*M) \oplus T^*M$. By identifying TM with $\mathbb{G}(T^*M)$ (through the action (2.4) of a uniquely defined 2-form on M), the generalized metric has the form

$$\mathbb{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}. \tag{2.7}$$

Given a generalized metric $\mathbb{G}$ and an orientation on M, following [9], one can define a generalized Hodge star operator on $\wedge^\bullet T^*M$: Since V_+ is isomorphic to TM, the orientation on M induces one on V_+. Then we let $\{e_1, e_2, \cdots, e_m\}$ be a positive orthonormal basis of V_+, let $\star = -e_m \cdots e_2 \cdot e_1 \in \mathrm{Clif}(\mathbb{T}M)$ and define *the (generalized) Hodge star* as the Clifford action of $\star$ on spinors:

$$\star : \wedge^\bullet T^*M \longrightarrow \wedge^\bullet T^*M \qquad \star\,\alpha = \star \cdot \alpha. \tag{2.8}$$

Notice that $\star^2 = (-1)^{\frac{m(m-1)}{2}}$, so if m is a multiple of four, $\star$ decomposes the space of forms into its $\pm 1-$eigenspaces.

Definition 2.2. In a four-dimensional manifold, we say that a form is *self-dual* if it lies in the $+1$-eigenspace of the generalized Hodge star and is *anti-self-dual* if it lies in its -1-eigenspace.

Using the Chevalley pairing (2.6), the operator (2.8) induces a positive definite metric on spinors via

$$(\varphi, \psi) \mapsto (\varphi, \star\psi)_{Ch}.$$

In the metric splitting of $\mathbb{T}M$, the generalized Hodge star relates to the classical Hodge star, denoted by $\star_{Hod}$, via the Chevalley pairing:

$$(\varphi, \star\psi)_{Ch} = (\varphi \wedge \star_{Hod}\psi)_{top}.$$

This means that, in the metric splitting, the genereralized Hodge star agrees with its classical counterpart, up to signs: if ψ has degree j, we have

$$\star\,\psi = (-1)^{\frac{(m-j)(m-j-1)}{2}+1} \star_{Hod} \psi. \tag{2.9}$$

Remark. In this paper we will be interested in the case $m = 4$ and, in particular, on the behaviour of $\star$ on even forms. The relation above shows that, in the metric splitting, $\star$ agrees with $\star_{Hod}$ on 2-forms and is minus the classic Hodge star on 0 and 4-forms. ■

Definition 2.3. A *generalized complex structure* on a manifold M equipped with a closed 3-form $H \in \Omega^3(M)$ is a bundle automorphism $\mathbb{J}$ of $\mathbb{T}M$ such that $\mathbb{J}^2 = -\mathrm{Id}$, $\mathbb{J}$ is orthogonal with respect to (2.1) and integrable, i.e., its $+i$-eigenspace, L, is involutive with respect to the Courant bracket (2.2).

The existence of a generalized complex structure forces the dimension of M to be even, so we let $m = 2n$. Since $\mathbb{J}^2 = -\mathrm{Id}$ and $\mathbb{J}$ is orthogonal, $\mathbb{J}$ is also an element in $\mathfrak{so}(\mathbb{T}M)$, and hence it acts on spinors accordingly, giving rise to a decomposition of $\wedge^\bullet T^*_{\mathbb{C}}M$ into its eigenspaces. We define $U^k \subset \wedge^\bullet T^*_{\mathbb{C}}M$ to be the ik-eigenspace of $\mathbb{J}$. These spaces are nonempty for $-n \leq k \leq n$, are related by conjugation, i.e., $U^{-k} = \overline{U^k}$, and U^n is a line subbundle of $\wedge^\bullet T^*_{\mathbb{C}}M$, referred to as the *canonical bundle* of $\mathbb{J}$. The line U^n is generated by either an even or an odd form and the *parity* of $\mathbb{J}$ is the parity of one such generator. Further, Clifford action of elements of L maps U^k to U^{k+1} and action by elements of $\overline{L}$ maps U^k to U^{k-1}.

Letting $\mathcal{U}^k$ denote the sheaf of sections of the bundle U^k, integrability of $\mathbb{J}$ is equivalent to the condition

$$d^H : \mathcal{U}^k \longrightarrow \mathcal{U}^{k+1} \oplus \mathcal{U}^{k-1}, \tag{2.10}$$

where $d^H = d + H\wedge$.

The decomposition of forms into subspaces U^k is compatible with the $\mathbb{Z}_2$ grading of spinors. Further, since the Chevalley pairing is spin invariant and $\mathbb{J}$ acts on spinors as an element of $\mathfrak{spin}(\mathbb{T}M)$, the space U^k is orthogonal to U^l unless $k = -l$, in which case the pairing in nondegenerate.

In what follows we will frequently use the exponential of the action of $\mathbb{J}$ on forms, namely, the action of $\mathcal{J} = e^{\frac{\pi \mathbb{J}}{2}}$ which, restricted to U^p, is multiplication by i^p.

A generalized complex structure on M also naturally induces an orientation: if $\rho \in U^n \backslash \{0\}$ then $(-1)^{\deg(\rho)+1} i^{-n} (\rho, \overline{\rho})_{Ch}$ is a nonzero real volume form, and any other choice of trivialization of the line U^n changes this form by a positive number.

Definition 2.4. A *generalized Hermitian* structure on M is pair $(\mathbb{J}_1, \mathbb{G})$ consisting of a generalized complex structure and generalized metric which commute.

Given a generalized Hermitian structure, the orthogonal automorphism $\mathbb{J}_2 = \mathbb{J}_1 \mathbb{G}$ also squares to $-\mathrm{Id}$, but is not necessarily integrable. If $\mathbb{J}_2$ is integrable, we have a generalized Kähler structure:

Definition 2.5. A *generalized Kähler structure* on a manifold M is a pair $(\mathbb{J}_1, \mathbb{J}_2)$ of generalized complex structures which commute and for which $\mathbb{G} = -\mathbb{J}_1 \mathbb{J}_2$ is a generalized metric.

Since $\mathbb{J}_1$ and $\mathbb{J}_2$ commute in a generalized Hermitian manifold, $\mathbb{T}_{\mathbb{C}}M$ splits as the intersections of their eigenspaces. Letting L_i be the $+i$-eigenspace of $\mathbb{J}_i$, we define

$$V_+^{1,0} = L_1 \cap L_2; \quad V_-^{1,0} = L_1 \cap \overline{L_2}; \qquad V_+^{0,1} = \overline{L_1} \cap \overline{L_2}; \qquad V_-^{0,1} = \overline{L_1} \cap L_2.$$

and then we have

$$\mathbb{T}_{\mathbb{C}}M = V_+^{1,0} \oplus V_+^{0,1} \oplus V_-^{1,0} \oplus V_-^{0,1}. \tag{2.11}$$

These subspaces are related to the eigenspaces of the generalized metric:

$$V_+ \otimes \mathbb{C} = V_+^{1,0} \oplus V_+^{0,1}; \quad V_- \otimes \mathbb{C} = V_-^{1,0} \oplus V_-^{0,1}.$$

Further, $\wedge^\bullet T^*_{\mathbb{C}}M$ also acquires a bi-grading as the intersection of the eigenspaces of $\mathbb{J}_1$ and $\mathbb{J}_2$:

$$U^{p,q} = U^p_{\mathbb{J}_1} \cap U^q_{\mathbb{J}_2}. \tag{2.12}$$

$$
\begin{array}{ccccc}
 & & U^{0,2} & & \\
 & U^{-1,1} & & U^{1,1} & \\
U^{-2,0} & & U^{0,0} & & U^{2,0} \\
 & U^{-1,-1} & & U^{1,-1} & \\
 & & U^{0,-2} & &
\end{array}
$$

Figure 1: Nontrivial spaces in the decomposition of forms of a generalized Hermitian four-manifold

$$
\begin{tikzcd}
U^{p-1,q+1} & & U^{p+1,q+1} \\
 & U^{p,q} \arrow[ul, "V_-^{0,1}"'] \arrow[ur, "V_+^{1,0}"] \arrow[dl, "V_+^{0,1}"] \arrow[dr, "V_-^{1,0}"'] & \\
U^{p-1,q-1} & & U^{p+1,q-1}
\end{tikzcd}
$$

Figure 2: Clifford action of $V_\pm^{1,0}$ and $V_\pm^{0,1}$ on $U^{p,q}$.

We can represent the spaces $U^{p,q}$ as points in a lattice. If M is four-dimensional, the only nontrivial entries appear in Figure 1. The Clifford action of elements in $V_\pm^{1,0}$ and $V_\pm^{0,1}$ maps $U^{p,q}$ into adjacent spaces in this splitting, as depicted in Figure 2. The decomposition of forms by parity of degree may be deduced from the decomposition into spaces $U^{p,q}$, once the parity of $\mathbb{J}_1$ is given. For example, if M is four-dimensional and $\mathbb{J}_1$ is of even type, then

$$
\begin{aligned}
\wedge^{ev} T^*_{\mathbb{C}} M &= U^{2,0} \oplus U^{0,2} \oplus U^{-2,0} \oplus U^{0,-2} \oplus U^{0,0} \\
\wedge^{od} T^*_{\mathbb{C}} M &= U^{1,1} \oplus U^{1,-1} \oplus U^{-1,1} \oplus U^{-1,-1}
\end{aligned}
$$

Due to (2.10), we have that for a generalized Kähler structure, d^H has total degree 1, that is,

$$
d^H : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p+1,q+1} \oplus \mathcal{U}^{p-1,q+1} \oplus \mathcal{U}^{p+1,q-1} \oplus \mathcal{U}^{p-1,q-1}, \tag{2.13}
$$

where $\mathcal{U}^{p,q}$ denotes the sheaf of sections of $U^{p,q}$.

Finally, since $\mathbb{G} = -\mathbb{J}_1\mathbb{J}_2$, it follows that the action of the generalized Hodge star on forms can be expressed in terms of the exponentials of the actions of $\mathbb{J}_1$ and $\mathbb{J}_2$.

Lemma 2.6 ([9])**.** In a generalized Hermitian manifold, $\star = -\mathcal{J}_1\mathcal{J}_2$, where $\mathcal{J}_k = \exp(\frac{\pi}{2}\mathbb{J}_k)$.

Proof. This lemma is obtained simply by lifting the identity $\mathbb{G} = -\mathbb{J}_1\mathbb{J}_2$ to the spin group. We include an alternative proof for concreteness.

Let V_+ be the $+1$-eigenspace of $\mathbb{G}$. Since $\mathbb{J}_1$ and $\mathbb{J}_2$ commute, they preserve V_+ and since $\mathbb{G} = -\mathbb{J}_1\mathbb{J}_2$, they agree on V_+, hence V_+ has a complex structure. The anchor map gives an isomorphism between V_+ and TM, and the orientations induced by $\mathbb{J}_1$ and $\mathbb{J}_2$ on TM agree with the orientation determined by the complex structure on V_+, so the star operator is well defined.

We will prove the result by induction on p, starting with $p = n$, i.e., at $U^{n,0}$. Let $e_1, \mathbb{J}_1 e_1, \cdots, e_n, \mathbb{J}_1 e_n$ be a positive orthonormal basis of V_+, and let $\alpha \in U^{n,0}$. We have

$$0 = (e_k + i\mathbb{J}_1 e_k)(e_k - i\mathbb{J}_1 e_k) \cdot \alpha = 2\alpha + 2i\mathbb{J}_1 e_k \cdot e_k \cdot \alpha.$$

Hence $\mathbb{J}_1 e_k \cdot e_k \cdot \alpha = i\alpha$, and it follows from the definition of $\star$ that $\star\alpha = -i^n\alpha$.

Now we assume that $\star = -i^{p+q}$ on $U^{p,q}$ and prove that $\star = -i^{p+q}$ on $U^{p-1,q+1}$ and $-i^{p+q-2}$ on $U^{p-1,q-1}$. Indeed, $U^{p-1,q-1}$ is generated by elements of the form $(e_k + i\mathbb{J}_1 e_k) \cdot \alpha$ with $\alpha \in U^{p,q}$, and for such elements we have

$$\begin{aligned}
\star(e_k + i\mathbb{J}_1 e_k) \cdot \alpha &= -\mathbb{J}_1 e_n \cdot e_n \cdots \mathbb{J}_1 e_k \cdot e_k \cdots \mathbb{J}_1 e_1 \cdot e_1 \cdot (e_k + i\mathbb{J}_1 e_k) \cdot \alpha \\
&= (e_k + i\mathbb{J}_1 e_k) \cdot \mathbb{J}_1 e_n \cdot e_n \cdots \mathbb{J}_1 e_k \cdot e_k \cdots \mathbb{J}_1 e_1 \cdot e_1 \alpha \\
&= -(e_k + i\mathbb{J}_1 e_k) \cdot \star\alpha \\
&= -i^{p+q+2}(e_k + i\mathbb{J}_1 e_k) \cdot \alpha
\end{aligned}$$

Similarly, $U^{p-1,q+1}$ is generated by elements of the form $v \cdot \alpha$ with $v \in \overline{L_1} \cap L_2 \subset V_- \otimes \mathbb{C}$ and $\alpha \in U^{p,q}$. Since elements of V_- are orthogonal to elements of V_+, we see that Clifford multiplication by v (graded) commutes with $\star$. Since $\star$ is multiplication by an even element in the Clifford algebra, we have

$$\star(v \cdot \alpha) = v \cdot \star\alpha = -i^{p+q} v \cdot \alpha.$$

□

According to Lemma 2.6, one can also read the spaces of self-dual and anti self-dual forms off from the generalized Kähler decomposition.

Proposition 2.7. *Let $\wedge^\bullet_+ T^*M$ and $\wedge^\bullet_- T^*M$ denote self-dual and anti-self-dual forms, respectively, for the generalized Hodge star operator. On an even generalized Kähler four-manifold, we have the following identities for their complexifications:*

$$\begin{aligned}
&\wedge^{ev}_+ T^*_{\mathbb{C}} M = U^{2,0} \oplus U^{0,2} \oplus U^{-2,0} \oplus U^{0,-2}; \quad && \wedge^{od}_+ T^*_{\mathbb{C}} M = U^{1,1} \oplus U^{-1,-1}; \\
&\wedge^{ev}_- T^*_{\mathbb{C}} M = U^{0,0}; && \wedge^{od}_- T^*_{\mathbb{C}} M = U^{1,-1} \oplus U^{-1,1}.
\end{aligned}$$

3 Generalized reduction

We now summarize the results which we require from the generalized reduction theory developed in [5, 6, 7].

3.1 Courant reduction

Let M be a smooth manifold equipped with a closed 3-form $H \in \Omega^3(M)$. Reducing the Courant algebroid structure on $\mathbb{T}M$ (defined by (2.1), (2.2) and (2.3)) is the first step for the reduction of generalized geometric structures on M. One can carry out Courant reduction with the following ingredients:

(1) An action of a connected Lie group G on M, generated infinitesimally by a map of Lie algebras
$$\psi : \mathfrak{g} \longrightarrow \Gamma(TM);$$

(2) A *lift* of this action, i.e., a map $\widetilde{\psi} : \mathfrak{g} \longrightarrow \Gamma(\mathbb{T}M)$ making the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\widetilde{\psi}"] \arrow[d, "\mathrm{Id}"] & \Gamma(\mathbb{T}M) \arrow[d, "\pi_T"] \\ \mathfrak{g} \arrow[r, "\psi"] & \Gamma(TM) \end{tikzcd}$$

commute, and satisfying the following compatibility conditions: the image of $\widetilde{\psi}$ in $\mathbb{T}M$ is isotropic with respect to (2.1), the map $\widetilde{\psi}$ preserves brackets, and the condition

$$i_{X_\gamma} H = -d\xi_\gamma \tag{3.1}$$

holds for every $\gamma \in \mathfrak{g}$, where $\widetilde{\psi}(\gamma) = X_\gamma + \xi_\gamma$, $X_\gamma \in \Gamma(TM)$ and $\xi_\gamma \in \Gamma(T^*M)$.

(3) An equivariant map $\mu : M \longrightarrow \mathfrak{h}^*$, where $\mathfrak{h}^*$ is a G-module. We say that μ is the *moment map* for the action.

Remark. The G-action on M in (1) induces a canonical G-action on $\mathbb{T}M$, and we regard $\mathbb{T}M$ as a G-equivariant bundle in this way. The lift in (2) also defines a $\mathfrak{g}$-action on $\mathbb{T}M$ via $\gamma \mapsto [\widetilde{\psi}(\gamma), \cdot]_H$, and condition (3.1) guarantees that these actions coincide, see [6, Sec. 2.3]. ■

We will assume that $0 \in \mathfrak{h}^*$ is a regular value for μ, and that the induced G-action on the submanifold $P := \mu^{-1}(0) \hookrightarrow M$ is free and proper, so that $P \to P/G$ is a principal bundle. Following [6], we refer to the set $(\widetilde{\psi}, \mathfrak{h}, \mu)$ as in (1)–(3), satisfying these extra regularity conditions, as *reduction data.*

It will be convenient to consider the direct sum $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ and pack all reduction data into a single map:
$$\begin{aligned} &\Psi : \mathfrak{a} \longrightarrow \Gamma(\mathbb{T}M), \\ &\Psi(\gamma, \lambda) = \widetilde{\psi}(\gamma) + d\langle \mu, \lambda \rangle, \qquad \gamma \in \mathfrak{g}, \lambda \in \mathfrak{h}. \end{aligned} \tag{3.2}$$

Remark. As shown in [6], $\mathfrak{a}$ can be equipped with a bracket making it into a *Courant algebra*, in the sense of [5], so that Ψ is bracket preserving; i.e., Ψ is an example of an *extended action* [5, Sec. 2.2]. ■

Starting with reduction data $(\widetilde{\psi}, \mathfrak{h}, \mu)$ and $P = \mu^{-1}(0)$ as above, the quotient

$$M_{red} := P/G$$

is a smooth manifold, called the *reduced manifold.* Using that 0 is a regular value and the freeness of the G-action on P, one checks that the distribution

$$\mathbb{K} := \Psi(\mathfrak{a}) \subseteq \mathbb{T}M$$

is a vector bundle over P. Since the lift $\widetilde{\psi}$ is isotropic, $\mathbb{K}|_P$ is an isotropic subbundle of $\mathbb{T}M|_P$. Also, letting $\mathbb{K}^\perp$ be the orthogonal complement of $\mathbb{K}$ with respect to the pairing (2.1), one can consider the bracket of G-invariant sections $v_1, v_2 \in \Gamma(\mathbb{K}^\perp|_P)^G$ by (locally) extending them to $\widetilde{v_1}, \widetilde{v_2} \in \Gamma(\mathbb{T}M)$, sections defined on a neighbourhood of P, taking their bracket, and restricting the result back to P:

$$[v_1, v_2] := [\widetilde{v_1}, \widetilde{v_2}]|_P.$$

This bracket on $\Gamma(\mathbb{K}^\perp|_P)^G$ always gives back an element in $\Gamma(\mathbb{K}^\perp|_P)^G$, but it is not well defined, as different choices of extensions can change the final result by an element of $\Gamma(\mathbb{K}|_P)^G$. Since $\Gamma(\mathbb{K}|_P)^G$ is an ideal of the G-invariant sections of $\mathbb{K}^\perp|_P$, the vector-bundle quotient

$$\mathcal{E}_{red} := \frac{\mathbb{K}^\perp|_P}{\mathbb{K}|_P} \Big/ G \to M_{red} \tag{3.3}$$

inherits a bracket from the Courant bracket on $\mathbb{T}M$; it also inherits a nondegenerate pairing, as well as a projection map $\pi : \mathcal{E}_{red} \longrightarrow TM_{red}$, obtained as the composition $\mathbb{K}^\perp \xrightarrow{\pi_T} TP \xrightarrow{p_*} TM_{red}$, where $p : P \longrightarrow M_{red}$ is the quotient map. These make $\mathcal{E}_{red}$ into an exact Courant algebroid over M_{red} [16]; i.e., $\mathcal{E}_{red}$, equipped with its bracket, pairing and projection, is locally isomorphic to $\mathbb{T}M_{red}$ with the Courant bracket, natural pairing and anchor map, see e.g. [5, Sec. 2.1] for details.

Example 3.1 (Tangent action)**.** Let G act on M freely and properly with infinitesimal action $\psi : \mathfrak{g} \longrightarrow \Gamma(TM)$. Let us consider the trivial lift for this action:

$$\widetilde{\psi} : \mathfrak{g} \longrightarrow \Gamma(\mathbb{T}M), \qquad \widetilde{\psi}(\gamma) = \psi(\gamma).$$

Then, condition (3.1) holds if and only if H is a basic form, i.e., it is the pull back of a 3-form H_{red} on M/G, which we assume to be the case. Finally, choose $\mathfrak{h} = \{0\}$, so that the moment map is trivial, and $\Psi = \widetilde{\psi}$ and $M_{red} = M/G$.

In this case $\mathbb{K} = \Psi(\mathfrak{g}) \subset TM$ corresponds to the tangent space to the G-orbits, hence $\mathbb{K}^\perp = TM \oplus \mathrm{Ann}(\psi(\mathfrak{g}))$ and

$$\mathcal{E}_{red} = \frac{\mathbb{K}^\perp}{\mathbb{K}} \Big/ G = \frac{TM}{\psi(\mathfrak{g})} \oplus \mathrm{Ann}(\psi(\mathfrak{g})) \Big/ G = TM_{red} \oplus T^*M_{red};$$

the Courant bracket on $\mathbb{T}M_{red}$ is the one determined by the 3-form H_{red} via (2.2).

Example 3.2 (Cotangent action)**.** Courant reduction can be also carried out for the action of the trivial group $G = \{e\}$ on M. In this case, any map $\mu : M \longrightarrow \mathfrak{h}^*$, where $\mathfrak{h}^*$ is a vector space, can be taken as a moment map. The reduced space is simply $M_{red} = \mu^{-1}(0)$, since there is no group action, and $\mathbb{K} = d\langle \mu, \mathfrak{h} \rangle = \mathrm{Ann}(TM_{red})$ and $\mathbb{K}^\perp = TM_{red} \oplus T^*M$. The reduced Courant algebroid over M_{red} is given by

$$\mathcal{E}_{red} = TM_{red} \oplus \frac{T^*M}{\mathrm{Ann}(TM_{red})} = TM_{red} \oplus T^*M_{red}.$$

The Courant bracket on $\mathcal{E}_{red}$ is the one determined by the pull-back of H to $M_{red} = \mu^{-1}(0)$.

Example 3.3. Given general reduction data $(\widetilde{\psi}, \mathfrak{h}, \mu)$, the Courant reduction can be described in two steps. First, we consider the cotangent action determined by the moment map μ, as in Example 3.2, and take the corresponding reduction. The result is the Courant algebroid $\mathbb{T}P$, with 3-form H_P given by the pull-back of H to $P = \mu^{-1}(0)$. One verifies that the lifted action $\widetilde{\psi} : \mathfrak{g} \to \Gamma(\mathbb{T}M)$ restricts to a lifted action

$$\widetilde{\psi}_P : \mathfrak{g} \to \Gamma(\mathbb{T}P).$$

Splitting $\widetilde{\psi}_P$ into its tangent and cotangent parts, we write $\widetilde{\psi}_P = X + \xi$, with $X \in \Gamma(TP \otimes \mathfrak{g}^*)$ and $\xi \in \Omega^1(P, \mathfrak{g}^*)$. Let $\theta \in \Omega^1(P, \mathfrak{g})$ be a connection on P, viewed as a principal G-bundle. We will use the following notation: for $\alpha \in \Omega^1(P, V)$ and $\Omega^1(P, V^*)$, where V is a vector bundle over P, we denote by $\langle \alpha, \beta \rangle \in \Omega^2(P)$ the 2-form given by $\langle \alpha, \beta \rangle(Y, Z) = \beta(Z)(\alpha(Y)) - \beta(Y)(\alpha(Z))$. We consider the invariant 2-form $B_\theta \in \Omega^2(P)$,

$$B_\theta := \langle \theta, \xi \rangle + \frac{1}{2} \langle X \circ \theta, \xi \circ \theta \rangle, \tag{3.4}$$

where we define $X \circ \theta \in \Omega^1(P, TP)$, $\xi \circ \theta \in \Omega^1(P, T^*P)$ by viewing $X : P \times \mathfrak{g} \to TP$, $\xi : P \times \mathfrak{g} \to T^*P$, and $\theta : TP \to P \times \mathfrak{g}$. This 2-form satisfies

$$i_{X_\gamma} B_\theta = \xi_\gamma, \qquad \forall \gamma \in \mathfrak{g}.$$

Indeed, any $Y \in TP$ can be written as $Y = X_{\widetilde{\gamma}} + Y_h$, for some $\widetilde{\gamma} \in \mathfrak{g}$ and $\theta(Y_h) = 0$, so

$$\begin{aligned} i_Y i_{X_\gamma} B_\theta &= \xi(Y)(\theta(X_\gamma)) - \xi(X_\gamma)(\theta(X_{\widetilde{\gamma}})) + \frac{1}{2}(\xi_{\widetilde{\gamma}}(X_\gamma) - \xi_\gamma(X_{\widetilde{\gamma}})) \\ &= \xi_\gamma(Y) - \xi_{\widetilde{\gamma}}(X_\gamma) + \xi_{\widetilde{\gamma}}(X_\gamma) = \xi_\gamma(Y), \end{aligned}$$

since $\theta(X_\gamma) = \gamma$ and $\xi_\gamma(X_{\widetilde{\gamma}}) = -\xi_{\widetilde{\gamma}}(X_\gamma)$, which follows from the lifted action $\widetilde{\psi}_P$ having isotropic image. We use the 2-form B_θ to change the splitting of $\mathbb{T}P$, and in this new splitting the lifted action is given by

$$e^{B_\theta}(X_\gamma + \xi_\gamma) = X_\gamma + \xi_\gamma - i_{X_\gamma} B_\theta = X_\gamma.$$

Therefore, after the change of splitting by B_θ, the lifted action is given purely by tangent vectors. We hence complete the reduction procedure as in Example 3.1. Note that changing the splitting by B_θ modifies the 3-form on $\mathbb{T}P$ to $H_P - dB_\theta$, see (2.5). This 3-form is invariant and satisfies

$$i_{X_\gamma} H_P - i_{X_\gamma} dB_\theta = -d\xi_\gamma + d i_{X_\gamma} B_\theta = 0, \qquad \forall \gamma \in \mathfrak{g};$$

hence $H_P - dB_\theta$ is basic, and it determines the 3-form on the reduced Courant algebroid $\mathbb{T}M_{red}$ over $M_{red} = P/G$.

3.2 Reduction of generalized geometries

Once Courant reduction is in place, one may reduce generalized geometric structures on M. We will be interested in reducing generalized metrics and generalized Kähler structures. For the following theorems, we assume that we are given reduction data $(\widetilde{\psi}, \mathfrak{h}, \mu)$ as in (1), (2), (3), so that 0 is a regular value of μ and the G-action on $P = \mu^{-1}(0)$ is free and proper. We consider

$\Psi : \mathfrak{a} \longrightarrow \Gamma(\mathbb{T}M)$ as in (3.2), $\mathbb{K} = \Psi(\mathfrak{a}) \subseteq \mathbb{T}M$, and let $\mathbb{K}^\perp$ be its orthogonal complement with respect to (2.1). We let $\mathcal{E}_{red}$ be the associated reduced Courant algebroid (3.3).

A distribution of $\mathbb{T}M$ of particular importance when considering the reduction of structures which involve a generalized metric $\mathbb{G}$ is $\mathbb{K}^\mathbb{G}$, the orthogonal complement of $\mathbb{K}$ inside $\mathbb{K}^\perp$ with respect to $\mathbb{G}$, i.e.,

$$\mathbb{K}^\mathbb{G} := \mathbb{K}^\perp \cap \mathbb{G}(\mathbb{K}^\perp).$$

The relevance of this distribution stems from the fact that at every point in M, the projection $\mathbb{K}^\perp \to \mathbb{K}^\perp/\mathbb{K}$ restricts to an isomorphism $\mathbb{K}^\mathbb{G} \xrightarrow{\sim} \mathbb{K}^\perp/\mathbb{K}$. So if $\mathbb{G}$ is G-invariant, then we have a natural identification

$$\mathbb{K}^\mathbb{G}|_P/G \cong \mathcal{E}_{red}, \tag{3.5}$$

showing that $\mathcal{E}_{red}$ inherits a generalized metric.

Theorem 3.4 (Metric reduction [7])**.**

(a) If $\mathbb{G}$ is a generalized metric on M that is G-invariant, then it reduces to a generalized metric $\mathbb{G}_{red}$ on $\mathcal{E}_{red}$ via (3.5).

(b) Let us consider $\mathbb{T}M$ with the metric splitting and, in this splitting, suppose that the lifted action over $\mu^{-1}(0)$ has infinitesimal generators $X + \xi$, with $X \in \Gamma(TM \otimes \mathfrak{g}^*)$ and $\xi \in \Omega^1(M, \mathfrak{g}^*)$. Then the metric induced by $\mathbb{G}_{red}$ on M_{red} is the restriction of $\mathbb{G}$ to the distribution transversal to the G-orbits in $P = \mu^{-1}(0)$ given by

$$\tau_+ = \{Y \in TP : \langle \mathbb{G}(X) + \xi, Y\rangle = 0\}. \tag{3.6}$$

(c) Let θ be the connection on $P = \mu^{-1}(0)$, seen as a principal G-bundle, for which τ_+ is the horizontal distribution. The 3-form associated to the metric splitting of $\mathcal{E}_{red}$ is given by $H - dB_\theta$, where B_θ is given by (3.4) for this choice of connection.

We call $\mathbb{G}_{red}$ the reduced metric.

The importance of the distribution $\mathbb{K}^\mathbb{G}$ goes beyond metric reduction. Any G-invariant metric structure on $\mathbb{T}M$ is usually defined by two types of condition: a linear algebraic condition, which determines the pointwise behaviour of the structure, and a differential condition which regards integrability of the structure and is phrased in terms of the Courant bracket. Reduction involves checking that both the linear algebraic and the differential conditions hold on $\mathcal{E}_{red}$. As a rule of thumb, checking the linear algebraic conditions boils down to proving that they hold on $\mathbb{K}^\mathbb{G}$, since (3.5) then implies that they hold on $\mathcal{E}_{red}$. As for the differential conditions, since the Courant bracket on $\mathcal{E}_{red}$ is determined by the Courant bracket on $\mathcal{E}$, integrability of the reduced structures usually follows from integrability of the structures on $\mathcal{E}$.

In the case of a generalized Kähler structure, this translates to:

Theorem 3.5 (Generalized Kähler reduction [6])**.** *Let $(\mathbb{J}_1, \mathbb{J}_2)$ be a G-invariant generalized Kähler structure on M. If $\mathbb{J}_1\mathbb{K}^\mathbb{G} = \mathbb{K}^\mathbb{G}$ over $P = \mu^{-1}(0)$, then the generalized Kähler structure on M reduces to a generalized Kähler structure on M_{red}.*

Indeed, under the hypothesis of the theorem, $\mathbb{K}^\mathbb{G}$ is invariant under $\mathbb{J}_1$ and $\mathbb{G}$, so it is also invariant under $\mathbb{J}_2$. In $\mathbb{K}^\mathbb{G}$, $\mathbb{J}_1$ and $\mathbb{J}_2$ commute and give rise to a metric $(\mathbb{G}|_{\mathbb{K}^\mathbb{G}})$. By (3.5), the structure on $\mathbb{K}^\mathbb{G}$ gives pointwise a generalized Kähler structure on $\mathcal{E}_{red}$. Integrability follows from integrability of the structures in M.

4 The moduli space of instantons

Let M be a compact, oriented four-manifold, equipped with a closed 3-form H and a generalized metric $\mathbb{G}$. After an appropriate change in the splitting of $\mathbb{T}M$, we may assume $\mathbb{G}$ has the form (2.7), for a Riemannian metric g on M.

Fix a principal G-bundle E over M, for G a compact, connected, semi-simple Lie group equipped with an Ad-invariant inner product κ on its Lie algebra $\mathfrak{g}$. We denote by $\mathfrak{g}_E \to M$ the vector bundle associated to the adjoint representation of G. The space $\mathcal{A}$ of connections on E is an affine space modeled on $\Omega^1(M, \mathfrak{g}_E)$, so for each $A \in \mathcal{A}$ we have a natural identification $T_A\mathcal{A} = \Omega^1(M, \mathfrak{g}_E)$. Let $\mathscr{G}$ be the group of gauge transformations, i.e. automorphisms of E.

A connection $A \in \mathcal{A}$ is anti-self-dual, and called an instanton, when its curvature has vanishing self-dual part:

$$F_+^A = 0.$$

This gauge-invariant condition gives rise to an elliptic complex

$$0 \longrightarrow \Omega^0(M, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(M, \mathfrak{g}_E) \xrightarrow{d_{A+}} \Omega^2_+(M, \mathfrak{g}_E) \longrightarrow 0, \tag{4.1}$$

where d_A is the covariant exterior derivative and d_{A+} is its self-dual projection. Let $H^i(M, \mathfrak{g}_E)$, $i = 0, 1, 2$, be the cohomology groups of the above complex, and let h^0, h^1, h^2 be their dimensions.

We now restrict our attention to the open set $\mathcal{A}^*$ of connections satisfying $h^0 = 0$ (meaning that (E, A) is irreducible) and $h^2 = 0$. By the theorem of Atiyah, Hitchin, and Singer [1], the quotient space

$$\mathcal{M} = \{A \in \mathcal{A}^* : F_+^A = 0\}/\mathscr{G} \tag{4.2}$$

is a smooth, finite-dimensional manifold of dimension $h^1 = p_1(\mathfrak{g}_E) - \frac{1}{2}\dim G(\chi + \tau)$, where χ, τ are the Euler characteristic and signature, respectively, of M. We refer to this space as the moduli space of instantons.

In the remainder of this section, we shall apply the reduction procedure of §3 to the passage from the space of connections $\mathcal{A}^*$ to the moduli space of instantons $\mathcal{M}$. This method explains how structures defined on M, such as a Courant algebroid, generalized metric, or generalized Kähler structure, induce similar structures on $\mathcal{M}$.

The moduli space is described in (4.2) as an infinite-dimensional quotient, and so we shall proceed only formally with the computations required by the reduction procedure in §3; the natural setting for this technique is that of Banach manifold quotients.

4.1 Extending the gauge action

Recall that for $A \in \mathcal{A}$ the tangent space is $T_A\mathcal{A} = \Omega^1(M, \mathfrak{g}_E)$. We may also identify the cotangent space to $A \in \mathcal{A}$ with $\Omega^3(M, \mathfrak{g}_E)$, using the pairing

$$\xi(X) := 2\int_M \kappa(X \wedge \xi), \tag{4.3}$$

for $\xi \in \Omega^3(M, \mathfrak{g}_E)$ and $X \in \Omega^1(M, \mathfrak{g}_E)$. Therefore, the fiber of the generalized tangent bundle $\mathbb{T}\mathcal{A} = T\mathcal{A} \oplus T^*\mathcal{A}$ over a connection $A \in \mathcal{A}$ is

$$\mathbb{T}_A\mathcal{A} = \Omega^{od}(M, \mathfrak{g}_E) = \Omega^1(M, \mathfrak{g}_E) \oplus \Omega^3(M, \mathfrak{g}_E).$$

The form of the duality pairing (4.3) implies that the natural inner product on $\mathbb{T}\mathcal{A}$ can be expressed in terms of the Chevalley pairing (2.6): for $v_1, v_2 \in \Omega^{od}(M, \mathfrak{g}_E)$, we have

$$\langle v_1, v_2 \rangle = \int_M \kappa(v_1, v_2)_{Ch}. \tag{4.4}$$

Remark. Since the expression on the right hand side of (4.4) is defined for any pair of forms, odd or not, we will use it to extend the definition of $\langle \cdot, \cdot \rangle$ to a bilinear form on $\Omega^\bullet(M, \mathfrak{g}_E)$. ■

Now consider the action of the gauge group $\mathscr{G}$ on $\mathcal{A}$. The Lie algebra is

$$\mathrm{Lie}(\mathscr{G}) = \Omega^0(M, \mathfrak{g}_E),$$

and the infinitesimal action of $\mathscr{G}$ on $\mathcal{A}$ is given by

$$\psi : \mathrm{Lie}(\mathscr{G}) \longrightarrow \Gamma(T\mathcal{A}), \qquad \psi(\gamma)|_A = d_A\gamma, \tag{4.5}$$

where we abuse notation by using A for the connection in $\mathcal{A}$ as well as the induced connection on the adjoint bundle $\mathfrak{g}_E$.

We now describe a lift of this gauge action to $\mathbb{T}\mathcal{A}$, as well as a moment map for the action.

4.1.1 Lifting the gauge action

The lift of the gauge action (4.5) to $\mathbb{T}\mathcal{A}$ uses the closed 3-form $H \in \Omega^3(M)$; we define it by

$$\widetilde{\psi} : \mathrm{Lie}(\mathscr{G}) \longrightarrow \Gamma(\mathbb{T}\mathcal{A}), \qquad \widetilde{\psi}(\gamma)|_A = d_A^H\gamma, \tag{4.6}$$

where

$$d_A^H := d_A + H\wedge : \Omega^0(M, \mathfrak{g}_E) \to \Omega^1(M, \mathfrak{g}_E) \oplus \Omega^3(M, \mathfrak{g}_E).$$

Proposition 4.2 below shows that this is indeed a lift for the gauge action in the sense of (2), Section 3.1, where we equip $\mathcal{A}$ with the *zero 3-form*.

Lemma 4.1 (Integration by parts)**.** Let $\alpha_j \in \Omega^\bullet(M, \mathfrak{g}_E)$, $j = 1, 2$. Then

$$\langle d_A^H\alpha_1, \alpha_2 \rangle = (-1)^{\dim(M)} \langle \alpha_1, d_A^H\alpha_2 \rangle.$$

Since in our case $\dim(M) = 4$, we have $\langle d_A^H\alpha_1, \alpha_2 \rangle = \langle \alpha_1, d_A^H\alpha_2 \rangle$.

Proof. It is enough to prove the result in a local trivialization assuming that α_1, α_2 have compact support. It suffices to assume that α_j is either an even or an odd form; we denote its parity by $|\alpha_j|$. Also, if $|\alpha_1|+|\alpha_2|+1 \neq \dim(M) \bmod 2$, then neither $\kappa(d_A^H\alpha_1, \alpha_2^t)$ nor $\kappa(\alpha_1, d_A^H\alpha_2^t)$ has a top degree component, hence the identity holds trivially. So we may assume that $|\alpha_1| + |\alpha_2| + 1 = \dim(M)$ mod 2. Locally, we write $d_A^H = d + A + H$, for $A \in \Omega^1(M, \mathfrak{g}_E)$. Integrating by parts, we obtain

$$\begin{aligned}
\int \kappa(d_A^H\alpha_1, \alpha_2)_{Ch} &= -\int \kappa(d\alpha_1 + [A, \alpha_1] + H \wedge \alpha_1, \alpha_2^t) \\
&= -(-1)^{|\alpha_1|+1} \int \kappa(\alpha_1, d\alpha_2^t + [A, \alpha_2^t] - H \wedge \alpha_2^t) \\
&= -(-1)^{|\alpha_1|+1+|\alpha_2|} \int \kappa(\alpha_1, (d\alpha_2 + [A, \alpha_2] + H \wedge \alpha_2)^t) \\
&= (-1)^{\dim(M)} \int \kappa(\alpha_1, d_A^H\alpha_2)_{Ch},
\end{aligned}$$

where in the second equality we used integration by parts for d, Ad-invariance of κ for A and the commutation rule for the 3-form H, and in the third equality we commuted Clifford transposition with each operator d, A and $H\wedge$. $\square$

Proposition 4.2. *Consider the map $\widetilde{\psi}$ in (4.6). Then*

(a) The image of $\widetilde{\psi}$ is isotropic in $\mathbb{T}\mathcal{A}$.

(b) For every $\gamma \in \Omega^0(M, \mathfrak{g}_E)$, $\xi_\gamma = H\gamma \in \Gamma(T^*\mathcal{A})$ is a closed 1-form on $\mathcal{A}$ (hence (3.1) holds for $\widetilde{\psi}$, since $\mathcal{A}$ is equipped with the zero 3-form).

(c) The map $\widetilde{\psi}$ is bracket preserving.

So $\widetilde{\psi}$ is a lift of the gauge action in the sense of Section 3.1.

Proof. To prove (a), take $\gamma \in \Omega^0(M, \mathfrak{g}_E)$, $A \in \mathcal{A}$, and note that

$$\langle \widetilde{\psi}(\gamma)|_A, \widetilde{\psi}(\gamma)|_A \rangle = \langle d_A^H\gamma, d_A^H\gamma \rangle = \langle \gamma, (d_A^H)^2\gamma \rangle = \int_M \kappa(\gamma, [F^A, \gamma])_{Ch} = 0,$$

where we have used integration by parts in the second equality, that the $(d_A^H)^2$ is the curvature of the connection A in the third equality, and that the Chevalley pairing of γ with $[F^A, \gamma]$ vanishes identically since γ has degree 0 and $[F^A, \gamma]$ has degree 2.

For (b), just note that, for each $\gamma \in \mathrm{Lie}(\mathscr{G})$, $H\gamma$, viewed as a 1-form on $\mathcal{A}$, is independent of the point $A \in \mathcal{A}$, that is, it is a constant 1-form and hence it is closed.

Finally, we check that (c) holds, i.e., $\widetilde{\psi}$ preserves brackets: we have

$$\begin{aligned}
[\widetilde{\psi}(\gamma_1), \widetilde{\psi}(\gamma_2)] &= [d_A\gamma_1, d_A\gamma_2] + \mathcal{L}_{d_A\gamma_1}H\gamma_2 - i_{d_A\gamma_2}d(H\gamma_1) \\
&= [\psi(\gamma_1), \psi(\gamma_2)] + \mathcal{L}_{d_A\gamma_1}H\gamma_2,
\end{aligned}$$

where the first equality is just the definition of the Courant bracket on $\mathcal{A}$, and in the second we used the definition of ψ and the fact that $H\gamma_1 \in \Omega^1(\mathcal{A})$ is closed. Since ψ is a map of Lie algebras, the first summand on the right hand side is $\psi([\gamma_1, \gamma_2])$. So, for $\widetilde{\psi}$ to be a bracket-preserving map, we must show that $\mathcal{L}_{d_A\gamma_1}H\gamma_2 = H[\gamma_1, \gamma_2]$. We verify that by fixing a connection A and taking $X \in \Omega^1(M, \mathfrak{g}_E) = T_A\mathcal{A}$. We compute the contraction of $\mathcal{L}_{d_A\gamma_1}H\gamma_2 \in \Omega^1(\mathcal{A})$ with the vector X:

$$\begin{aligned}
i_X\mathcal{L}_{d_A\gamma_1}H\gamma_2 = i_X d i_{d_A\gamma_1}H\gamma_2 &= 2i_X d \int_M \kappa(H\gamma_2, d_A\gamma_1)_{Ch} \\
&= 2\frac{d}{dt}\int_M \kappa(H\gamma_2, d_A\gamma_1 + t[X, \gamma_1])_{Ch}\Big|_{t=0} \\
&= 2\int_M \kappa(H\gamma_2, [X, \gamma_1])_{Ch} \\
&= 2\int_M \kappa(H[\gamma_1, \gamma_2], X)_{Ch} \\
&= 2\langle H[\gamma_1, \gamma_2], X \rangle.
\end{aligned}$$

$\square$

4.1.2 The moment map

Following the procedure outlined in Section 3.1, we now define a moment map for $\widetilde{\psi}$, using the Riemannian structure on M.

Let $\mathfrak{h} = \Omega^2_+(M, \mathfrak{g}_E)$ be the $\mathscr{G}$-module of self-dual 2-forms with coefficients in the adjoint bundle. Using κ and integration over M, we formally identify $\mathfrak{h}^*$ with $\mathfrak{h}$ and define the moment map to be the equivariant map

$$\mu : \mathcal{A} \longrightarrow \mathfrak{h}^* \qquad \mu(A) = F^A_+, \tag{4.7}$$

where F^A_+ denotes the self-dual part of the curvature of the connection A.

We now combine the lifted action and the moment map as in (3.2): we let $\mathfrak{a} := \mathrm{Lie}(\mathscr{G}) \oplus \mathfrak{h} = \Omega^0(M, \mathfrak{g}_E) \oplus \Omega^2_+(M, \mathfrak{g}_E)$ and consider the map

$$\Psi : \mathfrak{a} \to \Gamma(\mathbb{T}\mathcal{A}), \quad \Psi(\gamma, \lambda) = \widetilde{\psi}(\gamma) + d\langle \mu, \lambda \rangle, \quad \text{ for } \gamma \in \mathrm{Lie}(\mathscr{G}), \lambda \in \mathfrak{h}. \tag{4.8}$$

Lemma 4.3. For $\alpha \in \mathfrak{a}$, $\Psi(\alpha)|_A = d^H_A \alpha$.

Proof. It is enough to check that, for $\lambda \in \Omega^2_+(M, \mathfrak{g}_E)$,

$$\Psi(0, \lambda)|_A = d\langle \mu, \lambda \rangle|_A = d^H_A \lambda.$$

To determine the value of $\Psi(0, \lambda) \in \Omega^1(\mathcal{A})$ at a point $A \in \mathcal{A}$, we let $X \in T_A\mathcal{A}$ and compute

$$i_X \Psi(0, \lambda) = i_X d\langle \mu, \lambda \rangle = \mathcal{L}_X \langle \mu, \lambda \rangle.$$

Using the fact that $\mathcal{L}_X F^A|_A = d_A X$ and denoting by $d_{A\pm}$ the operator d_A composed with the projection onto the self-dual/anti self-dual forms, we have

$$\mathcal{L}_X \langle \mu, \lambda \rangle = \mathcal{L}_X \int_M \kappa(F^A_+, \lambda)_{Ch} = \int_M \kappa(d_{A+}X, \lambda)_{Ch} = \int_M \kappa(d_A X, \lambda)_{Ch} = \int_M \kappa(X, d_A \lambda)_{Ch},$$

where in the third equality we used the fact that $\lambda \in \Omega^2_+(M, \mathfrak{g}_E)$, hence it is orthogonal to $d_{A-}X$ and its pairing with $d_{A+}X$ is the same as its pairing with $d_A X$. The equation above shows that $\Psi(0, \lambda) = d_A \lambda$. Since $H \wedge \lambda = 0$, we conclude that $\Psi(0, \lambda) = d^H_A \lambda$. ☐

It is convenient to describe the space $\mathfrak{a} = \Omega^0(M, \mathfrak{g}_E) \oplus \Omega^2_+(M, \mathfrak{g}_E)$ using the natural extension of the Hodge star operator $\star$ described in (2.9) to $\mathfrak{g}_E$-valued forms. Indeed, $\mathfrak{a}$ is naturally isomorphic to the space $\Omega^{ev}_+(M; \mathfrak{g})$ of self-dual even forms via the map

$$\begin{aligned} \Omega^0(M, \mathfrak{g}_E) \oplus \Omega^2_+(M, \mathfrak{g}_E) &\longrightarrow \Omega^{ev}_+(M, \mathfrak{g}_E) \\ \gamma + \lambda &\mapsto \gamma + \lambda + \star\gamma. \end{aligned}$$

Since the operator d^H_A is trivial on elements in $\Omega^4(M, \mathfrak{g}_E)$, we use the identification

$$\mathfrak{a} = \Omega^{ev}_+(M, \mathfrak{g}_E),$$

and, by Lemma 4.3, we may write the map (4.8) as

$$\begin{aligned} &\Psi : \mathfrak{a} \longrightarrow \Gamma(\mathbb{T}\mathcal{A}) \\ &\Psi(\alpha)|_A = d^H_A \alpha. \end{aligned} \tag{4.9}$$

4.2 The reduced Courant algebroid

In this section we describe the reduced Courant algebroid associated with the lifted action $\widetilde{\psi}$ (4.6) and moment map μ (4.7) on the space $\mathcal{A}^*$. Since the tangent part of $\widetilde{\psi}$ is the classical gauge action and the zero set of μ consists of the anti-self-dual connections, the reduced space

$$\mathcal{A}^*_{red} := \{A \in \mathcal{A}^* : \mu(A) = 0\}/\mathscr{G}$$

coincides with $\mathcal{M}$ from (4.2). According to (3.3), the reduced Courant algebroid $\mathcal{E}_{red} \to \mathcal{M}$ is given by

$$\mathcal{E}_{red} = \frac{\mathbb{K}^\perp|_{\mu^{-1}(0)}}{\mathbb{K}|_{\mu^{-1}(0)}} \Big/ \mathscr{G}, \tag{4.10}$$

where $\mathbb{K} \subseteq \mathbb{T}\mathcal{A}$ is defined by the image of Ψ (4.9):

$$\mathbb{K}|_A = \{d^H_A \alpha : \ \alpha \in \Omega^{ev}_+(M, \mathfrak{g}_E)\}. \tag{4.11}$$

4.2.1 Cohomological description

We now give a cohomological description of the reduced Courant algebroid (4.10) as a bundle of cohomology groups over the moduli space. For an anti-self-dual connection A, consider the complex

$$0 \longrightarrow \Omega^{ev}_+(M, \mathfrak{g}_E) \xrightarrow{d^H_A} \Omega^{od}(M, \mathfrak{g}_E) \xrightarrow{d^H_{A+}} \Omega^{ev}_+(M, \mathfrak{g}_E) \longrightarrow 0, \tag{4.12}$$

and the cohomology group

$$H^{od}_{d^H_A}(M, \mathfrak{g}_E) := \frac{\ker d^H_{A+}}{\operatorname{Im} d^H_A}. \tag{4.13}$$

Proposition 4.4. *Let $A \in \mathcal{A}$ be anti-self-dual. Then*

$$\left.\frac{\mathbb{K}^\perp}{\mathbb{K}}\right|_A = H^{od}_{d^H_A}(M, \mathfrak{g}_E).$$

Proof. Note that $v \in \mathbb{K}^\perp$ if and only if

$$0 = \int_M \kappa(v, d^H_A \alpha)_{Ch} = \int_M \kappa(d^H_A v, \alpha)_{Ch}, \qquad \forall \alpha \in \Omega^2_+(M, \mathfrak{g}_E),$$

i.e., the self-dual part of $d^H_A v$ must vanish: $d^H_{A+} v = 0$. So we conclude that

$$\mathbb{K}^\perp|_A = \ker d^H_{A+} \subseteq \Omega^{od}(M, \mathfrak{g}_E). \tag{4.14}$$

It immediately follows that $\left.\frac{\mathbb{K}^\perp}{\mathbb{K}}\right|_A = \frac{\ker d^H_{A+}}{\operatorname{Im} d^H_A} = H^{od}_{d^H_A}(M, \mathfrak{g}_E)$. $\square$

Therefore, from (4.10), we conclude that $H^{od}_{d^H_A}(M, \mathfrak{g}_E)$ may be seen as the fibre of $\mathcal{E}_{red}$ over $[A] \in \mathcal{M}$. In fact, we may extend this cohomological description to obtain the structure of $\mathcal{E}_{red}$ as an extension of $T\mathcal{M}$ by $T^*\mathcal{M}$, as follows.

Recall that $T\mathcal{M}$ is given by $H^1(M, \mathfrak{g}_E)$, the middle cohomology of the sequence (4.1). Dualizing this sequence, we obtain

$$0 \longrightarrow \Omega^2_+(M, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^3(M, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^4(M, \mathfrak{g}_E) \longrightarrow 0. \tag{4.15}$$

We denote the cohomology of (4.15) by $H_k(M, \mathfrak{g}_E)$. Poincaré duality then provides a nondegenerate pairing

$$H^k(M, \mathfrak{g}_E) \times H_{2-k}(M, \mathfrak{g}_E) \longrightarrow \mathbb{R}.$$

Proposition 4.5. *Let $A \in \mathcal{A}$ be anti-self-dual and let $H^\bullet_{d_A^H}(M, \mathfrak{g}_E)$ denote the cohomology of (4.12). If $H^0(M, \mathfrak{g}_E) = H^2(M, \mathfrak{g}_E) = \{0\}$, then $H^0_{d_A^H}(M, \mathfrak{g}_E) = H^2_{d_A^H}(M, \mathfrak{g}_E) = \{0\}$, and the following sequence is exact:*

$$0 \longrightarrow H_1(M, \mathfrak{g}_E) \xrightarrow{\iota^*} H^{od}_{d_A^H}(M, \mathfrak{g}_E) \xrightarrow{\pi^*} H^1(M, \mathfrak{g}_E) \longrightarrow 0, \tag{4.16}$$

where ι is the inclusion of 3-forms into the odd forms and π is the projection of odd forms onto 1-forms.

Proof. If A is anti-self-dual, then the complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^2_+(M, \mathfrak{g}_E) \arrow[r, "\iota"] \arrow[d, "d_A"] & \Omega^{ev}_+(M, \mathfrak{g}_E) \arrow[r, "\pi"] \arrow[d, "d_A^H"] & \Omega^0(M, \mathfrak{g}_E) \arrow[r] \arrow[d, "d_A"] & 0 \\
0 \arrow[r] & \Omega^3(M, \mathfrak{g}_E) \arrow[r, "\iota"] \arrow[d, "d_A"] & \Omega^{od}(M, \mathfrak{g}_E) \arrow[r, "\pi"] \arrow[d, "d_{A+}^H"] & \Omega^1(M, \mathfrak{g}_E) \arrow[r] \arrow[d, "d_{A+}"] & 0 \\
0 \arrow[r] & \Omega^4(M, \mathfrak{g}_E) \arrow[r, "\iota"] \arrow[d] & \Omega^{ev}_+(M, \mathfrak{g}_E) \arrow[r, "\pi"] \arrow[d] & \Omega^2_+(M, \mathfrak{g}_E) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4.17}$$

is a short exact sequence of differential complexes. Since $H^0(M, \mathfrak{g}_E)$ and $H^2(M, \mathfrak{g}_E)$ vanish, the long exact sequence obtained from (4.17) implies that $H^0_{d_A^H}(M, \mathfrak{g}_E)$ and $H^2_{d_A^H}(M, \mathfrak{g}_E)$ vanish, and furthermore that (4.16) is exact, as required. □

In conclusion, the cohomology exact sequence (4.16) exhibits $\mathcal{E}_{red}$ as an extension of $T\mathcal{M}$ by $T^*\mathcal{M}$, with anchor map $\mathcal{E}_{red} \to T\mathcal{M}$ given by the projection of odd forms to 1-forms.

4.2.2 Harmonic forms and the reduced metric

The generalized Hodge star $\star$ (2.9) has a natural extension to $\mathfrak{g}_E$-valued forms. This operator preserves parity, in particular:

$$\star : \Omega^{od}(M, \mathfrak{g}_E) \to \Omega^{od}(M, \mathfrak{g}_E). \tag{4.18}$$

Using the identification with the generalized tangent space to the space of connections $\mathbb{T}_A\mathcal{A} = \Omega^{od}(M, \mathfrak{g}_E)$, we obtain an automorphism

$$\mathcal{G} : \mathbb{T}\mathcal{A} \to \mathbb{T}\mathcal{A}, \tag{4.19}$$

which is orthogonal and self-adjoint. The associated bilinear form

$$\langle v, \mathcal{G}w\rangle = \int_M \kappa(v, \star w)_{Ch} \tag{4.20}$$

is positive definite, therefore $\mathcal{G}$ defines a generalized metric on $\mathcal{A}$.

Following Section 3.2, we would like to use the metric orthogonal of $\mathbb{K}$ in $\mathbb{K}^\perp$,

$$\mathbb{K}^{\mathcal{G}} = \mathbb{K}^\perp \cap \mathcal{G}(\mathbb{K}^\perp),$$

to model the reduced Courant algebroid $\mathcal{E}_{red}$. Viewing $\mathcal{E}_{red}$ as the cohomology of the elliptic complex (4.12), we will see below that its identification with $\mathbb{K}^{\mathcal{G}}$ corresponds to using harmonic forms as specific representatives for elements in $\mathcal{E}_{red}$. For clarity, let us state the harmonic condition. The pairing (4.20) can be extended, using the same expression, to the space of $\mathfrak{g}_E$-valued forms and hence we can compute the adjoints of the operators in the elliptic complex (4.12). A form is d_A^H*-harmonic* if it is closed and co-closed with respect to the appropriate operators.

Theorem 4.6 (The reduced metric)**.** *Let A be an anti-self-dual connection.*

(a) The space $\mathbb{K}^{\mathcal{G}}|_A$ consists of the d_A^H-harmonic odd forms, and the reduced metric corresponds to the L^2-inner product $(v, w) \mapsto \int_M \kappa(v, \star w)_{Ch}$.

(b) The $+1$-eigenspace V_+^{red} of the reduced metric is the space of self-dual d_A^H-harmonic odd forms,

$$V_+^{red} = \{X + \star X : d_{A+}^H(X + \star X) = 0 \text{ and } X \in \Omega^1(M, \mathfrak{g}_E)\},$$

and the norm of $\hat{X} \in T\mathcal{M} = H^1(M, \mathfrak{g}_E)$ is given by the L^2-norm of the unique self-dual, d_A^H-harmonic, odd form $X + \star X$ for which the d_A-cohomology class of X is $\hat{X}$. Equivalently, the induced metric on $T\mathcal{M}$ is given by the L^2-norm of 1-forms satisfying

$$\begin{cases} d_{A+}X = 0, \\ d_A \star X + H \wedge X = 0. \end{cases} \tag{4.21}$$

Proof. Recall that $\mathbb{K}^\perp = \ker(d_{A+}^H)$ (see (4.14)), while v is in the metric orthogonal of $\mathbb{K}$ if and only if, for all $\alpha \in \Omega_+^{ev}(M, \mathfrak{g}_E)$,

$$\langle d_A^H \alpha, \star v\rangle = \langle \alpha, d_A^H \star v\rangle = 0,$$

i.e., $d_{A+}^H \star v = 0$. So $v \in \mathbb{K}^{\mathcal{G}}$ if and only if it is closed and co-closed, hence harmonic. The induced generalized metric is the restriction of the pairing (4.20) to $\mathbb{K}^{\mathcal{G}}$, as required.

To prove (b), notice that $+1$-eigenspace of $\star$ on $\mathbb{K}^{\mathcal{G}}$ is precisely the space of self-dual d_A^H-harmonic odd forms, and the reduced metric on $T\mathcal{M}$ is induced by the natural pairing (4.4) on V_+ and the isomorphism given by projection from V_+ onto $T\mathcal{M}$; that is, for $\hat{X} \in H^1(M, \mathfrak{g}_E)$, there is a unique $X \in \Omega^1(M, \mathfrak{g}_E)$ representing this class such that $X + \star X \in V_+$, and the norm of $\hat{X}$ is $\int_M \kappa(X, \star X)_{Ch}$. Finally, the condition $X + \star X \in V_+$ is equivalent to (4.21), and the norm of X is precisely the norm of $X + \star X$ with respect to the natural pairing. $\square$

Remark. Theorem 4.6 shows that the usual isomorphism between $\mathbb{K}^{\mathcal{G}}/\mathscr{G}$ and $\mathcal{E}_{red}$, familiar from the finite-dimensional setting (3.5), continues to hold here. In this case, $\mathcal{E}_{red}$ has a cohomological description (4.13) as $H^{od}(M, \mathfrak{g}_E)$, while $\mathbb{K}^{\mathcal{G}}$ consists of the d_A^H-harmonic odd forms. The isomorphism between these spaces is provided by the usual argument in Hodge theory. ∎

4.2.3 The Ševera class and Donaldson's μ-map

We now consider the closed 3-form H_{red} on $\mathcal{M}$ arising from the metric splitting of the reduced Courant algebroid $\mathcal{E}_{red} \to \mathcal{M}$. In Theorem 4.8, we express H_{red} in terms of Donaldson's μ-map (not to be confused with the moment map).

To find an explicit expression for H_{red}, we follow part (c) of Theorem 3.4. The space $\mathcal{A}_{\mathrm{asd}}$ of anti-self-dual connections admits a horizontal distribution τ_+, transverse to the action of the gauge group, given by the 1-forms $X \in \Omega^1(M, \mathfrak{g}_E)$ satisfying (4.21); we denote the associated connection 1-form on the $\mathscr{G}$-bundle $\mathcal{A}_{\mathrm{asd}} \to \mathcal{M}$ by θ, and its curvature 2-form by $\Theta \in \Omega^2(\mathcal{A}_{\mathrm{asd}}, \mathrm{Lie}(\mathscr{G}))$.

According to Theorem 3.4, part (c), $H_{red} = -dB_\theta$ (since we take the zero 3-form on $\mathcal{A}$), where B_θ is given by (3.4). One can equally describe H_{red} by specifying the restriction of $-dB_\theta$ to the horizonal distribution τ_+; since $\theta|_{\tau_+} = 0$, we have that

$$H_{red} = -dB_\theta|_{\tau_+} = -(d\langle\theta, \xi\rangle)|_{\tau_+} = -\langle d\theta, \xi\rangle|_{\tau_+} = -\langle\Theta, \xi\rangle|_{\tau_+}.$$

Since the cotangent part of the lifted action is $\xi = H\wedge$, see (4.6), we obtain, for $\hat{X}, \hat{Y}, \hat{Z} \in H^1(M, \mathfrak{g}_E)$,

$$H_{red}(\hat{X}, \hat{Y}, \hat{Z}) = \int_M \kappa(\Theta(X, Y), Z) \wedge H + c.p., \tag{4.22}$$

where X, Y, Z are the representatives of the classes $\hat{X}, \hat{Y}, \hat{Z}$ which satisfy (4.21). This is the same expression obtained by Hitchin (c.f. [12, Eq. (31)]), under the assumption that M is generalized Kähler and the cohomology class of H is trivial.

The computation above can be rephrased as follows: given a closed 3-form H on a compact oriented Riemannian 4-manifold, we get a corresponding closed 3-form on $\mathcal{M}$. One can also use expression (4.22) to show that if H is exact, then so is H_{red}, hence, in fact, we have a map in cohomology

$$H^3(M; \mathbb{R}) \longrightarrow H^3(\mathcal{M}; \mathbb{R}).$$

We now argue that this map coincides with Donaldson's μ-map, which is normally used to obtain degree two cohomology classes on the instanton moduli space. We use the description of the μ-map in terms of differential forms from [8].

Fix a principal G-bundle E, and let ∇ be the universal connection on $\pi_2^*\mathfrak{g}_E$, the pull-back of the adjoint bundle to $\mathcal{A}^* \times M$ via the second projection. Recall that ∇ is tautological in the M direction and trivial in the $\mathcal{A}^*$ direction. We then view $\mathcal{A}^* \times M$ as a principal $\mathscr{G}$-bundle over $\mathcal{B}^* \times M$, where $\mathcal{B}^* = \mathcal{A}^*/\mathscr{G}$ is the moduli space of irreducible connections on E. We then use Theorem 3.4 to endow $\mathcal{A}^*$ with the principal connection θ with horizontal spaces

$$\tau_+ = \{X \in T\mathcal{A}^* : d_A \star X + H \wedge X = 0\}. \tag{4.23}$$

Together, ∇ and θ give rise to a connection $\hat{\nabla}$ on the quotient bundle $\hat{\mathfrak{g}}_E = \pi_2^*\mathfrak{g}_E/\mathscr{G}$ over $\mathcal{B}^* \times M$, namely, given $\hat{v} \in \Gamma(T(\mathcal{B}^* \times M))$ and $\hat{s} \in \Gamma(\hat{\mathfrak{g}}_E)$ we let $v \in \Gamma(T(\mathcal{A}^* \times M))$ be the horizontal lift of $\hat{v}$ with respect to θ and let $s \in \Gamma(\mathfrak{g}_E)$ be pull back of $\hat{s}$, that is, the $\mathscr{G}$-invariant section of $\mathfrak{g}_E$ which projects to $\hat{s}$. Then we define

$$\hat{\nabla}_{\hat{v}}\hat{s}|_{([A],x)} = (\nabla_v s|_{(A,x)}) \in \mathfrak{g}_E|_x \cong \hat{\mathfrak{g}}_E|_{[A,x]}.$$

The curvatures $F^\nabla, F^{\hat{\nabla}}$, of $\nabla, \hat{\nabla}$ have three components corresponding to the decomposition

$$\wedge^2 T^*(\mathcal{A}^* \times M) = \wedge^2 T^*\mathcal{A}^* \oplus (T^*\mathcal{A}^* \otimes T^*M) \oplus \wedge^2 T^*M,$$

and its analogue for $\mathcal{B}^* \times M$. At a point $(A,x) \in \mathcal{A}^* \times M$, we obtain

$$\begin{aligned} F^\nabla(u,v) &= F^A(u,v); \\ F^\nabla(X,v) &= \langle X, v\rangle; \\ F^\nabla(X,Y) &= 0, \end{aligned}$$

where $u,v \in T_xM$ and $X,Y \in T_A\mathcal{A}^* \cong \Omega^1(M;\mathfrak{g}_E)$ and the pairing in the second expression is simply evaluation of a 1-form in a tangent vector.

Since $\hat{\nabla}$ is determined by ∇ and θ, one can also compute its curvature (cf. Proposition 5.2.17 in [8]):

Lemma 4.7. At a point $([A],x) \in \mathcal{B}^* \times M$ we have

$$F^{\hat{\nabla}}(u,v) = F^A(u,v); \tag{i}$$
$$F^{\hat{\nabla}}(\hat{X},v) = \langle X, v\rangle; \tag{ii}$$
$$F^{\hat{\nabla}}(\hat{X},\hat{Y}) = \Theta(X,Y)|_x; \tag{iii}$$

Where $u,v \in T_xM,\ X,Y \in T_A\mathcal{A}^* \cong \Omega^1(M;\mathfrak{g}_E)$ are horizontal representatives of $\hat{X},\hat{Y} \in T_A\mathcal{B}^*$ and $\Theta \in \Omega^2(\mathcal{A}^*;\Omega^0(M;\mathfrak{g}_E))$ is the curvature of the connection θ.

The μ-map involves the choice of a characteristic class of the bundle E, which in this case will be a multiple of the first Pontryagin class, represented by the form $\frac{1}{2}\kappa(F^\nabla, F^\nabla)$. A representative for $\mu([H]) \in H^3(\mathcal{M},\mathbb{R})$ is then given by the restriction of the 3-form

$$\Omega = \frac{1}{2}\int_M \kappa(F^\nabla, F^\nabla) \wedge H$$

to $\mathcal{M} \subset \mathcal{B}^*$. Since H is a 3-form on M, the only component of $F^\nabla \wedge F^\nabla$ which contributes to this integral is the section of $\wedge^3 T^*\mathcal{B}^* \otimes T^*M$ which is obtained from parts (ii) and (iii) of Lemma 4.7. So we have

$$\Omega_{[A]}(\hat{X},\hat{Y},\hat{Z}) = \int_M \kappa(\Theta(X,Y),Z) \wedge H + c.p..$$

Combining this result with Equation 4.22, we obtain:

Theorem 4.8. *The Ševera class of the reduced Courant algebroid over $\mathcal{M}$ coincides with the result of Donaldson's μ-map applied to $[H] \in H^3(M,\mathbb{R})$.*

4.3 Generalized Kähler structure

Let $(\mathbb{J}_1, \mathbb{J}_2)$ define a generalized Kähler structure on M, integrable with respect to the 3-form H, and with generalized metric $\mathbb{G} = -\mathbb{J}_1\mathbb{J}_2$. As above, we work in the metric splitting of $\mathbb{T}M$, and we study the moduli space of instantons associated to the underlying Riemannian metric of $\mathbb{G}$ and the orientation induced by the generalized complex structures.

The operators

$$\mathcal{J}_k = \exp(\tfrac{\pi}{2}\mathbb{J}_k) \in Spin(\mathbb{T}M)$$

act on differential forms, and we extend this action to $\mathfrak{g}_E$-valued forms in the natural way; the corresponding (p, q)-spaces (2.12) of $\mathfrak{g}_E$-valued forms are denoted by $U^{p,q}_{\mathfrak{g}}$ and their sheaf of sections by $\mathcal{U}^{p,q}_{\mathfrak{g}}$.

Finally, we assume that the generalized Kähler structure on M is *even*. It then follows from the (p, q)-decomposition of $\mathfrak{g}_E$-valued forms that, when acting on $\Omega^{od}(M, \mathfrak{g}_E) = \mathbb{T}\mathcal{A}$, both $\mathcal{J}_1$ and $\mathcal{J}_2$ square to $-\mathrm{Id}$. Since the Chevalley pairing is Spin-invariant, $\mathcal{J}_1$ and $\mathcal{J}_2$ are orthogonal operators with respect to the natural pairing (4.4) on $\mathbb{T}\mathcal{A}$, and since $\mathcal{J}_1$ and $\mathcal{J}_2$ are constant (i.e., they do not depend on the particular $A \in \mathcal{A}$), they are automatically integrable with respect to the Courant bracket on $\mathcal{A}$ (for the zero 3-form on $\mathcal{A}$). Hence $\mathcal{J}_1$ and $\mathcal{J}_2$ are generalized complex structures on $\mathcal{A}$. By Lemma 2.6, we know that $\star = -\mathcal{J}_1\mathcal{J}_2$, and hence $\mathcal{J}_1$ and $\mathcal{J}_2$ define a generalized Kähler structure on $\mathcal{A}$, with generalized metric give by the Hodge star operator (see (4.19)).

Lemma 4.9. The generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ on $\mathcal{A}$ is invariant under the the action of the gauge group.

Proof. In a local trivialization, an element of the gauge group is given by a map $g : U \subset M \longrightarrow G$, a connection $A \in \mathcal{A}$ can be written as $A = d + a$ with $a \in \Omega^1(M; \mathfrak{g}_E)$ and the action of g on D is $g \cdot (d + a) = d + gag^{-1} + g^{-1}dg$. So, the action of the gauge group on $\mathbb{T}\mathcal{A} \cong \mathcal{A} \times \Omega^{od}(M, \mathfrak{g}_E)$ is the adjoint action on $\mathfrak{g}_E$, tensored by the trivial action on forms. The distributions $V^{1,0}_{\pm}, V^{0,1}_{\pm} \subset \mathbb{T}\mathcal{A}$ defining the generalized Kähler structure on $\mathcal{A}$ are given by the decomposition of forms into $\mathcal{U}^{p,q}_{\mathfrak{g}} = \mathcal{U}^{p,q} \otimes \mathfrak{g}_E$, for $p = \pm 1$, $q = \pm 1$, and these subspaces are individually preserved by the gauge action, yielding the result. □

In view of the previous lemma, it is natural to ask whether this generalized Kähler structure descends to a generalized Kähler structure on the moduli space of instantons, along the lines of Theorem 3.5. That is indeed the case.

Theorem 4.10. *The generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ on $\mathcal{A}$ satisfies*

$$\mathcal{J}_1\mathbb{K}^{\mathcal{G}}|_A = \mathbb{K}^{\mathcal{G}}|_A \tag{4.24}$$

for all anti-self-dual connections $A \in \mathcal{A}$. Hence the moduli space $\mathcal{M}$ of instantons over an even generalized Kähler compact four-manifold inherits a generalized Kähler structure by the reduction procedure (c.f. Theorem 3.5).

According to Proposition 4.6, $\mathbb{K}^{\mathcal{G}}$ at an anti-self-dual connection A is given by the odd d^H_A-harmonic forms in the complex (4.12), so (4.24) amounts to proving that these forms are invariant under the action of each generalized complex structure. We verify this fact in the remainder of this section, ending with the proof of Theorem 4.10.

Lemma 4.11. Let M^{2n} be a generalized Kähler manifold with respect to a closed 3-form H, and let $E \to M$ be a principal G-bundle with a connection A. Then

$$d_A^H(\mathcal{U}_{\mathfrak{g}}^{p,q}) \subset \mathcal{U}_{\mathfrak{g}}^{p+1,q+1} \oplus \mathcal{U}_{\mathfrak{g}}^{p+1,q-1} \oplus \mathcal{U}_{\mathfrak{g}}^{p-1,q+1} \oplus \mathcal{U}_{\mathfrak{g}}^{p-1,q-1},$$

so that d_A^H defines four operators

$$\begin{array}{ll} \delta_+ : \mathcal{U}_{\mathfrak{g}}^{p,q} \longrightarrow \mathcal{U}_{\mathfrak{g}}^{p+1,q+1} & \qquad \delta_- : \mathcal{U}_{\mathfrak{g}}^{p,q} \longrightarrow \mathcal{U}_{\mathfrak{g}}^{p+1,q-1} \\ \overline{\delta_+} : \mathcal{U}_{\mathfrak{g}}^{p,q} \longrightarrow \mathcal{U}_{\mathfrak{g}}^{p-1,q-1} & \qquad \overline{\delta_-} : \mathcal{U}_{\mathfrak{g}}^{p,q} \longrightarrow \mathcal{U}_{\mathfrak{g}}^{p-1,q+1} \end{array}$$

such that $d_A^H = \delta_+ + \delta_- + \overline{\delta_+} + \overline{\delta_-}$.

Proof. In a local trivialization, $d_A^H = d^H + a$, for some $a \in \Omega^1(M, \mathfrak{g}_E) \subset \Gamma(\mathbb{T}_{\mathbb{C}}M \otimes \mathfrak{g}_E)$. Since in a generalized Kähler manifold d^H decomposes as a sum of four operators mapping $\mathcal{U}_{\mathfrak{g}}^{p,q}$ into the desired spaces due to (2.13) and the same is true for the Clifford action of $T^*M \subset \mathbb{T}M$ (see Figure 2), we see that d_A^H decomposes into four operators as described above. $\square$

$$\begin{tikzcd} \mathcal{U}_{\mathfrak{g}}^{p-1,q+1} & & \mathcal{U}_{\mathfrak{g}}^{p+1,q+1} \\ & \mathcal{U}_{\mathfrak{g}}^{p,q} \arrow[ul, "\overline{\delta_-}"'] \arrow[ur, "\delta_+"] \arrow[dl, "\overline{\delta_+}"] \arrow[dr, "\delta_-"'] & \\ \mathcal{U}_{\mathfrak{g}}^{p-1,q-1} & & \mathcal{U}_{\mathfrak{g}}^{p+1,q-1} \end{tikzcd}$$

Figure 3: Decomposition of d_A^H for a generalized Kähler manifold.

Lemma 4.12 (Integration by parts)**.** If δ is one of the operators δ_+, δ_-, $\overline{\delta_+}$ or $\overline{\delta_-}$, we have

$$\langle \delta\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle.$$

Proof. We prove the result for δ_+. For $\alpha \in \mathcal{U}_{\mathfrak{g}}^{p,q}$ and $\beta \in \mathcal{U}_{\mathfrak{g}}^{-p-1,-q-1}$ we have

$$\int_M \kappa(\delta_+\alpha, \beta)_{Ch} = \int_M \kappa(d_A^H\alpha, \beta)_{Ch} = \int_M \kappa(\alpha, d_A^H\beta)_{Ch} = \int_M \kappa(\alpha, \delta_+\beta)_{Ch},$$

where we have used in the first and last equalities the fact that the only component of $d_A^H\alpha$ (resp. $d_A^H\beta$) which pair nontrivially with β (resp. α) is the one given by δ_+. $\square$

Lemma 4.13. With the same notation as Lemma 4.11, and using the Hermitian inner product induced by the Hodge star:

$$(\alpha, \beta) \mapsto \langle \alpha, \star\overline{\beta} \rangle,$$

the adjoints of the operators δ_+, δ_- are

$$\delta_+^* = -\overline{\delta_+} \qquad \delta_-^* = \overline{\delta_-},$$

and

$$d_A^{H*} = -\delta_+ - \overline{\delta_+} + \delta_- + \overline{\delta_-}.$$

Proof. For $\alpha \in \mathcal{U}_{\mathfrak{g}}^{p,q}$ and $\beta \in \mathcal{U}_{\mathfrak{g}}^{p+1,q+1}$ we have

$$\begin{aligned}\langle \delta_+\alpha, \star\overline{\beta}\rangle &= i^{-p-q-2}\langle \delta_+\alpha, \overline{\beta}\rangle \\ &= i^{-p-q-2}\langle \alpha, \delta_+\overline{\beta}\rangle = i^{-p-q-2}\langle \alpha, \overline{\star}\,\overline{\star}\delta_+\overline{\beta}\rangle \\ &= i^{-2}\langle \alpha, \overline{\star}\overline{\delta_+}\beta\rangle = -\langle \alpha, \star\overline{\overline{\delta_+}\beta}\rangle,\end{aligned}$$

where in the first and fourth equalities we used that for a (p,q)-form φ, $\overline{\star}\varphi = \star\overline{\varphi} = -i^{-p-q}\overline{\varphi}$, and in the second we integrated by parts.

The proof for δ_- is totally analogous and the final claim follows from $d_A^{H*} = \delta_+^* + \overline{\delta_+}^* + \delta_-^* + \overline{\delta_-}^*$. □

Theorem 4.14. *Let $\triangle^H$ be the Laplacian corresponding to the sequence (4.12), and let $\triangle_\pm$ be the Laplacians corresponding to the sequences*

$$0 \longrightarrow \Omega_+^{ev}(M, \mathfrak{g}_E) \xrightarrow{\delta_\pm} \Omega^{od}(M, \mathfrak{g}_E) \xrightarrow{(\delta_\pm)_+} \Omega_+^{ev}(M, \mathfrak{g}_E). \tag{4.25}$$

Then

$$\begin{cases} \triangle^H = 2\triangle_{\delta_+} = 2\triangle_{\delta_-} & \text{on } \Omega_+^{ev}(M, \mathfrak{g}_E) \\ \triangle^H = 2\triangle_{\delta_+} \text{ and } \triangle_- = 0 & \text{on } \mathcal{U}_{\mathfrak{g}}^{\mp 1, \pm 1} \\ \triangle^H = 2\triangle_{\delta_-} \text{ and } \triangle_+ = 0 & \text{on } \mathcal{U}_{\mathfrak{g}}^{\pm 1, \pm 1}, \end{cases}$$

and all the Laplacians preserve the (p,q)-decomposition. In particular, if a form is $\triangle^H$-harmonic, so are its (p,q)-components.

Proof. We study the sequences in question term by term, the first being

$$\Omega_+^{ev}(M, \mathfrak{g}_E) = \mathcal{U}_{\mathfrak{g}}^{2,0} \oplus \mathcal{U}_{\mathfrak{g}}^{0,2} \oplus \mathcal{U}_{\mathfrak{g}}^{-2,0} \oplus \mathcal{U}_{\mathfrak{g}}^{0,-2}.$$

Since $d_{A+}^H d_A^H = 0$, this operator must vanish when applied to each individual summand above. Applied to $\mathcal{U}_{\mathfrak{g}}^{2,0}$, this translates to

$$\overline{\delta_+}^2 = \overline{\delta_-}^2 = 0 \qquad \textit{and} \qquad \triangle_{\delta_+} = \triangle_{\delta_-}. \tag{4.26}$$

Also, for $\alpha \in \mathcal{U}_{\mathfrak{g}}^{2,0}$, we have

$$\triangle^H\alpha = d_{A+}^{H*} d_A^H \alpha = (-\delta_+ - \overline{\delta_+} + \delta_- + \overline{\delta_-})_-(\overline{\delta_+} + \overline{\delta_-})\alpha = (-\overline{\delta_+}^2 + \overline{\delta_-}^2 + \triangle_{\delta_+} + \triangle_{\delta_-})\alpha.$$

Therefore, due to (4.26), we see that $\triangle^H = 2\triangle_{\delta_+} = 2\triangle_{\delta_-}$ on $\mathcal{U}^{2,0}$. By the same argument, this also holds for the remaining summands of $\Omega_+^{ev}(M, \mathfrak{g}_E)$.

To prove $\triangle^H = 2\triangle_{\delta_-}$ on $\mathcal{U}_{\mathfrak{g}}^{1,1}$, let $\alpha \in \mathcal{U}^{1,1}$ and compute

$$\begin{aligned}\triangle^H\alpha &= (d_A^H d_{A+}^{H*} + d_A^{H*} d_{A+}^H)\alpha = (d_A^H(\delta_- + \overline{\delta}_-) + d_A^{H*}(\delta_- + \overline{\delta}_-))\alpha \\ &= (d_A^H + d_A^{H*})(\delta_- + \overline{\delta}_-)\alpha = 2(\delta_- + \overline{\delta_-})^2\alpha \\ &= 2\triangle_{\delta_-}\alpha.\end{aligned}$$

Finally, at $\mathcal{U}_{\mathfrak{g}}^{1,1}$, δ_+ vanishes and $\overline{\delta_+}$ has codomain $\mathcal{U}_{\mathfrak{g}}^{0,0}$, which lies in the anti self-dual forms. Hence the projections of δ_+ and δ_+^* to the self-dual forms vanish on $\mathcal{U}_{\mathfrak{g}}^{1,1}$ and $\triangle_{\delta_+} = 0$. The same argument applies to the other summands of $\Omega^{od}(M, \mathfrak{g}_E)$. □

Proof of Theorem 4.10. From Theorem 4.14, we know that if an odd form is d_A^H-harmonic, so are its (p,q)-components. Now if $\alpha \in \mathcal{U}_{\mathfrak{g}}^{p,q}$ is harmonic, then $\mathcal{J}_i\alpha = \pm i\alpha$ is also harmonic. □

$$
\begin{tikzcd}
 & & \mathcal{U}_{\mathfrak{g}}^{0,2} & & \\
 & \mathcal{U}_{\mathfrak{g}}^{-1,1} & & \mathcal{U}_{\mathfrak{g}}^{1,1} \arrow[ul, "\overline{\delta_-}"] \arrow[dr, shift left, "\delta_-"] & \\
\mathcal{U}_{\mathfrak{g}}^{-2,0} & & \mathcal{U}_{\mathfrak{g}}^{0,0} & & \mathcal{U}_{\mathfrak{g}}^{2,0} \arrow[ul, shift left, "\overline{\delta_-}"] \arrow[dl, shift left, "\overline{\delta_+}"] \\
 & \mathcal{U}_{\mathfrak{g}}^{-1,-1} & & \mathcal{U}_{\mathfrak{g}}^{1,-1} \arrow[ur, shift left, "\delta_+"] \arrow[dl, "\overline{\delta_+}"] & \\
 & & \mathcal{U}_{\mathfrak{g}}^{0,-2} & &
\end{tikzcd}
$$

Figure 4: Contributions to $(d_A^H)^2$ when applied to $\mathcal{U}_{\mathfrak{g}}^{2,0}$.

4.4 Bi-Hermitian structure and degeneracy loci

The generalized Kähler structure on the moduli space $\mathcal{M}$ described in §4.3 comprises a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of generalized complex structures, each of which has type which may vary throughout $\mathcal{M}$. Recall that the type of a generalized complex structure $\mathbb{J}$ is half the corank of its associated real Poisson structure $\pi_T \circ \mathbb{J}|_{T^*}$, so that a symplectic structure has type 0 while a complex structure has maximal type.

In this section we provide an effective method for computing the types of $(\mathcal{J}_1, \mathcal{J}_2)$ at a given equivalence class $[A] \in \mathcal{M}$ of connections on the principal G-bundle E. We express each type as the dimension of a certain holomorphic sheaf cohomology group of the restriction of E to a distinguished complex curve in the original generalized Kähler 4-manifold M. To make sense of this, we must first use the Hitchin–Kobayashi correspondence to interpret (E, A) as a stable holomorphic principal G^c-bundle $\mathbf{E}$ over the 4-manifold M, which itself is viewed as a complex surface using the equivalence [10] between generalized Kähler and bi-Hermitian geometry. To compute the types of $(\mathcal{J}_1, \mathcal{J}_2)$, we then restrict $\mathbf{E}$ to the complex curves (D_1, D_2) in M where the generalized Kähler structures $(\mathbb{J}_1, \mathbb{J}_2)$ undergo type change in the 4-manifold.

Theorem 4.15. *Let $(M, \mathbb{J}_1, \mathbb{J}_2)$ be an even generalized Kähler four-manifold with corresponding bi-Hermitian structure (M, I_+, I_-, g), and let X denote the complex surface (M, I_+). Let $D_1, D_2 \subset X$ be the divisors where $\mathbb{J}_1, \mathbb{J}_2$ respectively have complex type. Finally, let $\mathcal{M}$ be the moduli space of instantons for the principal G-bundle E over M, and let $(\mathcal{J}_1, \mathcal{J}_2)$ be the induced generalized Kähler structure on $\mathcal{M}$.*

Then the type of $\mathcal{J}_i$, $i = 1, 2$ at $[A] \in \mathcal{M}$ is given by the dimension of the sheaf cohomology group

$$H^0(D_i, \mathfrak{g}_{\mathbf{E}}|_{D_i}), \tag{4.27}$$

where $\mathfrak{g}_{\mathbf{E}}$ denotes the adjoint bundle of the holomorphic G^c-bundle $\mathbf{E}$ which corresponds to (E, A) under the Hitchin–Kobayashi correspondence.

Remark. The canonical line bundles $K_1 = U^{2,0}$ and $K_2 = U^{0,2}$ of the generalized complex structures $\mathbb{J}_1, \mathbb{J}_2$ are both holomorphic line bundles over the complex surface X. The projection

$\Omega^\bullet \to \Omega^0$, upon restriction to K_i, yields maps

$$s_i : K_i \to \Omega^0, \qquad i = 1, 2, \tag{4.28}$$

defining holomorphic sections of K_1^* and K_2^*. The section s_i vanishes precisely when $\mathbb{J}_i$ has complex type (i.e. type 2), allowing us to define divisors D_1, D_2 via

$$D_i = (s_i).$$

Since the sum of the types of $\mathbb{J}_1$ and $\mathbb{J}_2$ is bounded above by 2, the zero loci $D_1 = s_1^{-1}(0), D_2 = s_2^{-1}(0)$ are disjoint curves in X. Furthermore, the natural factorization

$$K_1 \otimes K_2 = K_X \tag{4.29}$$

of the canonical line bundle of X indicates that $D_1 + D_2$ is an anticanonical divisor. In particular, if either of D_1 or D_2 is smooth, it must be a genus 1 curve, by adjunction. ■

As a corollary, we recover a generalization of Hitchin's computation of the rank of a certain canonical holomorphic Poisson structure σ on $\mathcal{M}$, where $\mathcal{M}$ is viewed as a complex manifold by the Hitchin–Kobayashi correspondence as above. It was shown in [12] that any generalized Kähler manifold has a canonical holomorphic Poisson structure relative to each of its underlying complex structures. As explained in [10], the symplectic leaves of this holomorphic Poisson structure are transverse intersections of the symplectic leaves of the constituent pair of generalized complex structures. This implies that the corank of σ coincides with the sum of the types of $\mathcal{J}_1, \mathcal{J}_2$, yielding the following result.

Corollary 4.16. The corank of the holomorphic Poisson structure at $[A] \in \mathcal{M}$ is

$$\operatorname{corank}(\sigma_{\mathcal{M}}) = \dim H^0(D_1, \mathfrak{g}_{\mathbf{E}}|_{D_1}) + \dim H^0(D_2, \mathfrak{g}_{\mathbf{E}}|_{D_2}). \tag{4.30}$$

The remainder of this section contains the proof of the above results.

4.4.1 Holomorphic Dirac geometry on the moduli of stable bundles

A generalized Kähler manifold has a natural holomorphic Courant algebroid over each of its pair of underlying complex manifolds. According to [10, §2.2], the holomorphic Courant algebroid $\mathscr{E}$ over $X = (M, I_+)$ may be described as the quotient $\mathscr{E} = (V_+^{1,0})^\perp / V_+^{1,0}$, where $V_+^{1,0}$ is the common $+i$ eigenspace of the generalized complex structures, as in the decomposition (2.11). This vector bundle inherits a holomorphic structure, and using the tangent projection, which identifies $V_+^{1,0}$ with $T_{0,1}X$, we obtain $\mathscr{E}$ as an extension of the holomorphic tangent by the holomorphic cotangent bundle:

$$0 \longrightarrow T_{1,0}^*X \longrightarrow \mathscr{E} \xrightarrow{\pi} T_{1,0}X \longrightarrow 0\ . \tag{4.31}$$

In the case of the moduli space of instantons $\mathcal{M}$, the decomposition (2.11) is given, at $[A] \in \mathcal{M}$, by the decomposition of the cohomology group $H^{od}_{d_A^H}(M, \mathfrak{g}_E)$ provided by Theorem 4.14. That is, corresponding to the decomposition of forms in Figure 1, we have

$$H^{od}_{d_A^H}(M, \mathfrak{g}_E) = \mathcal{H}_{\mathfrak{g}}^{-1,-1} \oplus \mathcal{H}_{\mathfrak{g}}^{-1,1} \oplus \mathcal{H}_{\mathfrak{g}}^{1,-1} \oplus \mathcal{H}_{\mathfrak{g}}^{1,1}. \tag{4.32}$$

The common $+i$ eigenspace of $\mathcal{J}_1, \mathcal{J}_2$ at $[A] \in \mathcal{M}$ is then given by $\mathcal{H}_{\mathfrak{g}}^{1,1}$. As a result, we obtain that the fiber over $[A]$ of the holomorphic Courant algebroid is given by

$$\mathscr{E}|_{[A]} = (\mathcal{H}_{\mathfrak{g}}^{1,1})^{\perp}/\mathcal{H}_{\mathfrak{g}}^{1,1} \cong \mathcal{H}_{\mathfrak{g}}^{-1,1} \oplus \mathcal{H}_{\mathfrak{g}}^{1,-1}. \tag{4.33}$$

Note that forms in $\mathcal{U}_{\mathfrak{g}}^{1,1}$ are annihilated by the Clifford action by $V_+^{1,0}$, implying that their 1-form components lie in $\Omega^{1,0}(\mathfrak{g}_{\mathbf{E}})$. As a result, under the projection map π^* given by Equation 4.16, $\mathscr{E}|_{[A]}$ is sent to the Dolbeault cohomology group

$$T_{1,0}\mathcal{M}|_{[A]} = H^{0,1}(X, \mathfrak{g}_{\mathbf{E}}),$$

where $\mathbf{E}$ is the holomorphic G^c-bundle over X defined by (E, A). Of course, this is nothing but the tangent space at $[\mathbf{E}]$ to the moduli space of stable holomorphic G^c-bundles over X, in agreement with the Hitchin–Kobayashi correspondence.

To complete our description of $\mathscr{E}$, we provide a purely holomorphic interpretation of the fibre (4.33) as follows. By the Hodge identities of Theorem 4.14, we may compute $\mathcal{H}_{\mathfrak{g}}^{-1,1}$ and $\mathcal{H}_{\mathfrak{g}}^{1,-1}$ using the complex defined by the $\overline{\delta_+}$ operator, shown below.

$$\begin{tikzcd}
 & & \mathcal{U}_{\mathfrak{g}}^{0,2} \arrow[dl, "\overline{\delta_+}"'] & & \\
 & \mathcal{U}_{\mathfrak{g}}^{-1,1} \arrow[dl, "\overline{\delta_+}"'] & & \mathcal{U}_{\mathfrak{g}}^{1,1} & \\
\mathcal{U}_{\mathfrak{g}}^{-2,0} & & \mathcal{U}_{\mathfrak{g}}^{0,0} & & \mathcal{U}_{\mathfrak{g}}^{2,0} \arrow[dl, "\overline{\delta_+}"] \\
 & \mathcal{U}_{\mathfrak{g}}^{-1,-1} & & \mathcal{U}_{\mathfrak{g}}^{1,-1} \arrow[dl, "\overline{\delta_+}"] & \\
 & & \mathcal{U}_{\mathfrak{g}}^{0,-2} & &
\end{tikzcd} \tag{4.34}$$

In view of the Clifford actions described in Figure 2, we see that the two complexes above coincide with Dolbeault resolutions of holomorphic vector bundles over X: the upper complex is the Dolbeault complex for $K_2 \otimes \mathfrak{g}_{\mathbf{E}}$, while the lower complex is the Dolbeault complex for $K_1 \otimes \mathfrak{g}_{\mathbf{E}}$, where $K_1 = U^{2,0}$ and $K_2 = U^{0,2}$ are the canonical line bundles of the generalized complex structures $\mathbb{J}_1$ and $\mathbb{J}_2$, respectively. This leads to the following description of $\mathscr{E}$ as a holomorphic vector bundle over the moduli space of stable bundles over X.

Proposition 4.17. *The holomorphic Courant algebroid $\mathscr{E}$ over the moduli space of stable holomorphic G^c-bundles over X has fibre above $[\mathbf{E}]$ given by*

$$\mathscr{E}|_{[\mathbf{E}]} = H^1(X, K_1 \otimes \mathfrak{g}_{\mathbf{E}}) \oplus H^1(X, K_2 \otimes \mathfrak{g}_{\mathbf{E}}). \tag{4.35}$$

The fact that the holomorphic Courant algebroid over $\mathcal{M}$ naturally decomposes into a direct sum (4.35) is a general phenomenon, explained in [10]. For any generalized Kähler manifold, the $+i$ eigenbundles L_1, L_2 of the generalized complex structures, since they satisfy $L_1 \cap L_2 = V_+^{1,0}$, induce a decomposition

$$\mathscr{E} = (V_+^{1,0})^{\perp}/V_+^{1,0} = \mathcal{D}_1 \oplus \mathcal{D}_2,$$

which is compatible with the Courant bracket in the sense that $\mathcal{D}_1, \mathcal{D}_2$ are transverse holomorphic Dirac structures.

Since $\mathcal{J}_1$ and $\mathcal{J}_2$ act on $\mathcal{U}_{\mathfrak{g}}^{p,q}$ by $\exp(\pi p/2)$ and $\exp(\pi q/2)$, respectively, we see that on the moduli space, the above holomorphic Dirac structures are given by

$$\mathcal{D}_1|_{[\mathbf{E}]} = H^1(X, K_1 \otimes \mathfrak{g}_{\mathbf{E}}) \qquad \mathcal{D}_2|_{[\mathbf{E}]} = H^1(X, K_2 \otimes \mathfrak{g}_{\mathbf{E}}).$$

The significance of the summand $\mathcal{D}_i$, $i = 1, 2$, is that it captures information about the generalized complex structure $\mathcal{J}_i$, but in a holomorphic fashion. Importantly for us, the type of $\mathcal{J}_i$ may be computed as the complex corank of the projection of $\mathcal{D}_i$ to the holomorphic tangent bundle deriving from sequence (4.31). Since $\mathcal{D}_i$ has the same rank as $T_{1,0}\mathcal{M}$, the corank and nullity of the projection coincide. So, to prove Theorem 4.15, it remains to compute the kernel of the "anchor" maps

$$\mathcal{D}_i \xrightarrow{\pi|_{\mathcal{D}_i}} T_{1,0}\mathcal{M} \; . \tag{4.36}$$

Lemma 4.18. At $[\mathbf{E}] \in \mathcal{M}$, the anchor map of $\mathcal{D}_i$ coincides with the homomorphism of cohomology groups

$$H^1(X, K_i \otimes \mathfrak{g}_{\mathbf{E}}) \to H^1(X, \mathfrak{g}_{\mathbf{E}})$$

induced by the anticanonical section $s_i \in H^0(X, K_i^*)$ defined by (4.28).

Proof. We argue in the case $i = 1$, but $i = 2$ works similarly. The complex computing $\mathcal{D}_1$ is

$$\mathcal{U}_{\mathfrak{g}}^{2,0} \xrightarrow{\overline{\delta_+}} \mathcal{U}_{\mathfrak{g}}^{1,-1} \xrightarrow{\overline{\delta_+}} \mathcal{U}_{\mathfrak{g}}^{0,-2} \; .$$

By definition, $\mathcal{U}_{\mathfrak{g}}^{2,0}$ is the space of sections of $K_1 \otimes \mathfrak{g}_{\mathbf{E}}$. The Clifford action by $V_+^{0,1}$ identifies $\mathcal{U}_{\mathfrak{g}}^{1,-1}$ with $\Omega^{0,1}(X, K_1 \otimes \mathfrak{g}_{\mathbf{E}})$, and similarly $\mathcal{U}_{\mathfrak{g}}^{0,-2}$ is identified with $\Omega^{0,2}(X, K_1 \otimes \mathfrak{g}_{\mathbf{E}})$.

The projection π to the tangent space of the moduli space is described at the level of differential forms as follows: we must project $\mathcal{U}_{\mathfrak{g}}^{2,0}$, $\mathcal{U}_{\mathfrak{g}}^{1,-1}$, and $\mathcal{U}_{\mathfrak{g}}^{0,-2}$ to Dolbeault forms of degree $(0, 0)$, forms of degree $(0, 1)$, and forms of degree $(0, 2)$, respectively, leaving the coefficients in $\mathfrak{g}_{\mathbf{E}}$ unaffected. In the case of $\mathcal{U}_{\mathfrak{g}}^{2,0}$, this is, by definition, the contraction with s_1.

For $\mathcal{U}_{\mathfrak{g}}^{1,-1}$, we argue as follows. Let $\rho = \rho^0 + \rho^2 + \rho^4$ be a local generator of $U^{2,0}$, so that a general section of $U^{1,-1}$ can be written $v \cdot \rho$, for $v \in V_+^{0,1}$. Recall that

$$V_+^{0,1} = \{V - i(i_V \omega_+) \;:\; V \in T_{1,0}X\},$$

where ω_+ is the canonical Hermitian 2-form which generates $\Omega^{1,1} \cap \Omega_+^2$. Then $v \cdot \rho$ has $(0, 1)$-form component given by the $(0, 1)$ part of

$$i_V(\rho_2 - i\rho_0\omega_+), \tag{4.37}$$

for some section V of $T_{1,0}X$. But recall that since ρ generates $U^{2,0}$, it is annihilated by $V_+^{1,0}$, and hence we have that

$$i_W(\rho_2 + i\rho_0\omega_+) = 0, \qquad W \in T_{0,1}X,$$

and hence $\rho_2 + i\rho_0\omega_+$ has type $(2, 0)$. This then implies that the $(0, 1)$ part of (4.37) is exactly $-2i\rho_0 i_V\omega_+$, which is identified with $-i(i_V\omega_+)s_1(\rho)$ in $\Omega^{0,1}$, meaning that $\pi|_{\mathcal{U}_{\mathfrak{g}}^{1,-1}} = s_1$.

For $\mathcal{U}_{\mathfrak{g}}^{0,-2}$, a similar argument yields the fact that the $(0, 2)$ component of

$$(V - i(i_V\omega_+))(W - i(i_W\omega_+)) \cdot \rho, \qquad V, W \in T_{1,0}X,$$

consists of four terms, all equal to $-\rho_0(i_V\omega_+) \wedge (i_W\omega_+)$, proving that $\pi|_{\mathcal{U}^{0,-2}_{\mathfrak{g}}} = s_1$.

Summarizing, the morphism of cochain complexes

$$\begin{tikzcd} \Omega^{0,0}(K_1 \otimes \mathfrak{g}_{\mathbf{E}}) \arrow[r] \arrow[d, "s_1"'] & \Omega^{0,1}(K_1 \otimes \mathfrak{g}_{\mathbf{E}}) \arrow[r] \arrow[d, "s_1"'] & \Omega^{0,2}(K_1 \otimes \mathfrak{g}_{\mathbf{E}}) \arrow[d, "s_1"'] \\ \Omega^{0,0}(\mathfrak{g}_{\mathbf{E}}) \arrow[r] & \Omega^{0,1}(\mathfrak{g}_{\mathbf{E}}) \arrow[r] & \Omega^{0,2}(\mathfrak{g}_{\mathbf{E}}) \end{tikzcd}$$

gives an induced map in degree one cohomology which is the required projection from $\mathcal{D}_1$ to the tangent space to the moduli space of stable bundles. □

Proof of Theorem 4.15. The type of $\mathcal{J}_i$ at a point $[\mathbf{E}]$ in the moduli space is given by the dimension of the kernel of the projection of $\mathcal{D}_i$ to $T_{1,0}\mathcal{M}$. Having identified this projection in Lemma 4.18 as a map on Dolbeault cohomology, we may now use sheaf cohomology on the complex surface X to localize the computation of its kernel.

The section $s_i \in H^0(X, K_i^*)$, $i = 1, 2$, defines a short exact sequence of sheaves

$$\mathcal{O}_X(K_i) \xrightarrow{s_i} \mathcal{O}_X \longrightarrow \mathcal{O}_{D_i} \ . \tag{4.38}$$

Tensoring with $\mathfrak{g}_{\mathbf{E}}$, the long exact sequence in cohomology yields the exact sequence

$$H^0_X(\mathfrak{g}_{\mathbf{E}}) \longrightarrow H^0_{D_i}(\mathfrak{g}_{\mathbf{E}}|_{D_i}) \longrightarrow H^1_X(\mathfrak{g}_{\mathbf{E}} \otimes K_i) \xrightarrow{s_{i*}} H^1_X(\mathfrak{g}_{\mathbf{E}}) \ . \tag{4.39}$$

Stability implies $H^0_X(\mathfrak{g}_{\mathbf{E}}) = 0$, and we conclude that the kernel of $(s_i)_*$ has the same dimension as the algebra of endomorphisms of the restriction to D_i:

$$\operatorname{type}(\mathcal{J}_i) = \dim\ker(s_i)_* = \dim H^0(D_i, \mathfrak{g}_{\mathbf{E}}). \tag{4.40}$$

□

Remark. If either D_1 or D_2 is empty, as in the case that Hitchin investigated, then the corresponding generalized complex structure has type zero, i.e. it defines a symplectic structure. If D_i is smooth, then it has genus 1, and a generic vector bundle has $\dim H^0(D_i, \operatorname{End}_0(E)) = \operatorname{rank}(E) - 1$, so we expect $\mathcal{J}_i$ to have type $\geq n-1$ on the moduli space of $SU(n)$ instantons.

■

4.5 Example: The Hopf surface

Let X be the Hopf surface given by the quotient of $\mathbb{C}^2 \setminus \{0\}$ by an infinite cyclic group of dilations. This is a principal elliptic fibration, via the projection $\pi : X \to \mathbb{C}P^1$. The Hopf surface is diffeomorphic to the Lie group $SU(2) \times U(1)$, and has a natural even generalized Kähler structure first described in the context of WZW models [15] (see also [10, Example 1.21]). This generalized Kähler structure has the property that $\mathbb{J}_1$ and $\mathbb{J}_2$ have generic type 0, jumping to type 2 along two divisors $D_1 = \pi^{-1}(0)$, $D_2 = \pi^{-1}(\infty)$, where $0, \infty \in \mathbb{C}P^1$.

We now make use of the work of Braam–Hurtubise [3] and Moraru [14] to describe the generalized complex structures on the moduli space $\mathcal{M}_k$ of stable holomorphic $SL(2,\mathbb{C})$ bundles over X with fixed second Chern class k. The moduli space $\mathcal{M}_k$ is a smooth, non-empty complex manifold of dimension $4k$. By the Hitchin–Kobayashi correspondence [4], $\mathcal{M}_k$ may be viewed as the moduli space of $SU(2)$ instantons of charge k over $SU(2) \times U(1)$.

Stable bundles over X are studied by restricting them to each elliptic curve $\pi^{-1}(p),\ p \in X$. For $k > 1$, the restriction of a stable bundle $\mathbf{E}$ to a fixed fiber $D_p = \pi^{-1}(p)$ has an endomorphism algebra with the following possible ranks:

$$\dim H^0(D_p, \mathrm{End}_0(\mathbf{E})|_{D_p}) = 1 + 2l,\ \ l \in \{0, 1, \ldots, k\}. \tag{4.41}$$

From this we can conclude that the type of $\mathcal{J}_1$ on $\mathcal{M}_k$ varies from the generic value of 1 to a maximum value of $2k + 1$.

In fact, using the constructions in [14], one can show that the pair of types for $\mathcal{J}_1$ and $\mathcal{J}_2$ takes on all possible values $1 \le \mathrm{type}(\mathcal{J}_1) \le 2k + 1$ and $1 \le \mathrm{type}(\mathcal{J}_2) \le 2k + 1$ such that $\mathrm{type}(\mathcal{J}_1) + \mathrm{type}(\mathcal{J}_2) \le 2(k + 1)$. The result of Corollary 4.16 is then consistent with Moraru's computation [14, Proposition 6.3] of the rank of the holomorphic Poisson structure, which equals

$$\mathrm{rk}\ \sigma = 4k - \dim H^0(D, \mathrm{End}_0(\mathbf{E}|_D)),$$

where $D = D_1 + D_2$ is the anticanonical divisor defined above.

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