Examples and counter-examples of log-symplectic manifolds

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Abstract

We study topological properties of log-symplectic structures and produce examples of compact manifolds with such structures. Notably we show that several symplectic manifolds do not admit log-symplectic structures and several log-symplectic manifolds do not admit symplectic structures, for example $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ has log-symplectic structures if and only if $m, n > 0$ while they only have symplectic structures for $m = 1$. We introduce surgeries that produce log-symplectic manifolds out of symplectic manifolds and show that any compact oriented log-symplectic four-manifold can be transformed into a collection of symplectic manifolds by reversing these surgeries. Finally we relate log-symplectic structures to achiral Lefschetz fibrations and use this relation to conclude that if M is a simply connected 4-manifold then $M\#(S^2 \times S^2)$ and $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ have log-symplectic structures

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1 Introduction

A log-symplectic structure on a manifold M^{2n} is a Poisson structure $\pi \in \mathfrak{X}^2(M)$ for which π^n has only nondegenerate zeros. This condition is weaker than asking that M is outright symplectic (in which case π^n would not vanish) and yet it is only a little less so, since it still requires that π is generically symplectic and that its failure to be so everywhere is as well behaved as one could ask. If we want to rule out log-symplectic structures which are in fact symplectic, we refer to them as *bona fide* or nonsymplectic log-symplectic structures.

These structures have been classified on surfaces by Radko [12] and already in dimension two there is a marked contrast with the symplectic case, namely, every surface (orientable or not) has a log-symplectic structure. Recently log-symplectic structures received renewed attention: Guillemin, Miranda and Pires [7] proved a local form for the Poisson structure in a neighbourhood of the zeros of π^n and Gualtieri and Li [5] managed to give a clear geometrical description of symplectic groupoids integrating log-symplectic structures.

Despite these recent advances in the theory of log-sympletic manifolds, the area still lacks examples and even topological obstructions to the existence of these structures are unknown. So given a manifold, the question "does it have a log-symplectic structure?" is a little hard to answer.

We tackle these shortcomings in this paper. Indeed, Marcut and Osorno-Torres's paper [9, 11] and the present one are the first to provide topological obstructions to the existence of log-symplectic structures. While Marcut and Osorno-Torres prove that a log-symplectic manifold must have a cohomology class $a \in H^2(M)$ such that $a^{n-1} \neq 0$, we prove a different property which is more contrastive with symplectic geometry:

Theorem. *A compact orientable manifold with a bona fide log-symplectic structure has a nontrivial cohomology class $b \in H^2(M)$ such that $b^2 = 0$.*

Different from Marcut and Osorno-Torres's topological constraint, this property is not necessarily shared by symplectic manifolds and, in effect, shows that there are several symplectic manifolds for which the only log-symplectic structures are outright symplectic while other manifolds do not admit log-symplectic structures at all.

We then move on to produce examples of manifolds admitting such structures. The first aproach consists simply of deforming a symplectic structure into a log-symplectic. We show:

Theorem. *Let (M^{2n}, ω) be a symplectic manifold and $k > 0$ be an integer. If M has a compact symplectic submanifold $F^{2n-2} \subset M$ with trivial normal bundle, then M has a log-symplectic structure for which the zero locus of π^n has k components, all diffeomorphic to $F \times S^1$.*

Using symplectic blow-up we can then construct log-symplectic structures on $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ for $m, n > 0$. Therefore coupling the two theorems we have a complete classification of which manifolds in the family $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ for $m, n \geq 0$ admit log-symplectic structures (see Figure 1).

Further we introduce two surgeries which produce log-symplectic manifolds out of log-symplectic manifolds and which increase the number of components of the singular locus of the Poisson structure, hence even if the starting manifolds are symplectic, the resulting manifolds will only be log-symplectic.

Following the lines of Gompf's theorem relating symplectic structures to Lefschetz fibrations, we prove an analogue result for log-symplectic manifolds

Theorem. *Let M^4 and Σ^2 be compact connected manifolds and $p : M \longrightarrow \Sigma$ be an achiral Lefschetz fibration with generic fiber F. If F is orientable and $[F] \neq 0 \in H_2(M; \mathbb{R})$, then M has a log-symplectic structure whose singular locus has one component and for which the fibers are symplectic submanifolds of the symplectic leaves of the Poisson structure.*

Using results of Etnyre and Fuller on such fibrations [2] we obtain a general existence result

$\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$	symplectic	*bona fide* log-symplectic
$m>1$, $n>0$	✗	✓
$m>1$, $n=0$	✗	✗
$m=1$, $n>0$	✓	✓
$m=1$, $n=0$	✓	✗
$m=0$, $n>0$	✗	✗

Figure 1: Table showing the values of m and n for which $m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ has symplectic or *bona fide* log-symplectic structures. In the symplectic case, we require that the orientation determined by the symplectic structure agrees with the orientation of the manifold.

Theorem. *Let M be a simply connected compact four-manifold. Then both $M\#(S^2 \times S^2)$ and $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admit bona fide log-symplectic structures.*

We finish showing that in four dimensions any compact orientable log-symplectic manifold is obtained out of a symplectic manifold using our surgeries. Expressed another way:

Theorem. *Let (M^4, π) be a compact, orientable, log-symplectic manifold with singular locus Z. Then each unoriented component of $M\backslash Z$ can be compactified as a symplectic manifold.*

This paper is organized as follows. Section 2 reviews the basics of Poisson geometry relevant for our study and Section 3 reviews Guillemin–Miranda–Pires normal form theorem [7]. Section 4 introduces a simple topological invariant that allows us to show that there are many symplectic manifolds which do not admit *bona fide* log-symplectic structures. Section 5 shows that under general assumptions one can deform a symplectic structure into a log-symplectic structure and Section 6 introduces the surgeries and gives the existence result for log-symplectic structures on achiral Lefschetz fibrations. Finally, Section 7 shows that in four dimensions the surgeries can be reversed and any compact, orientable, log-symplectic four manifold can be transformed into a symplectic manifold by surgeries.

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While carrying out this research, the author was made aware that Frejlich, Martinez-Torres and Miranda were carrying out a project [3] which overlaps with the results in this paper. Notably, they had independently produced our "Construction 1" and our Theorem 7.1.

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2 Poisson structures

This section we give a short account of the basic material on Poisson and log-symplectic structures. For more details we refer the reader to [5, 6, 7].

2.1 Log-symplectic manifolds

A Poisson structure on a manifold M^m is a bivector $\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM)$ for which

$$[\pi, \pi] = 0,$$

where the bracket used is the Schouten–Nijenhuis bracket of multivector fields. Assuming that M^m is even dimensional, say, $m = 2n$, a generic bivector (at a point) would give an isomorphism

$\pi : T_p^*M \xrightarrow{\cong} T_pM$. In this case π^n is a non zero element in $\wedge^{2n}T_pM$. If a Poisson bivector π is *everywhere generic* (i.e., everywhere invertible) then the 2-form $\omega = \pi^{-1}$ is a symplectic structure on M.

Definition 2.1. The locus where $\pi : T^*M \longrightarrow TM$ is an isomorphism is the *symplectic locus of* π and its complement is the *singular locus* of the Poisson structure.

Moving to a move general situation which is still modeled on a "generic" case, one can require that π^n only has nondegenerate zeros:

Definition 2.2. A *log-symplectic structure* on M^{2n} is a Poisson structure π for which the zeros of π^n are nondegenerate.

Since in a log-symplectic manifold the singular locus is given by the nondegenerate zero locus of a section of a line bundle, we have that it is a smooth submanifold of codimension one. Further, as the rank of the Poisson structure does not change along each of its symplectic leaves, we see that each connected component of the singular locus is itself a Poisson submanifold of M, i.e., a union of symplectic leaves.

Finally, any Poisson manifold comes equipped with two diferential operators which give rise to cohomology theories. The first is the Poisson differential on multivector fields:

$$d_\pi : \mathfrak{X}^\bullet(M) \longrightarrow \mathfrak{X}^{\bullet+1}(M); \qquad d_\pi(\xi) = [\pi, \xi],$$

The Poisson condition and the Jacobi identity for the Schouten–Nijenhuis bracket imply that $d_\pi^2 = 0$ and its cohomology is known as the *Poisson cohomology* of (M, π).

The second is the Koszul differential on forms:

$$\delta : \Omega^\bullet(M) \longrightarrow \Omega^{\bullet-1}(M); \qquad \delta\rho = \{\pi, d\}\rho,$$

where $\{\pi, d\} = \pi d - d\pi$ is the graded commutator of operators and π acts on forms by interior product. Again the Jacobi identity for $\{\cdot, \cdot\}$ and the Poisson condition imply that $\delta^2 = 0$ and its cohomology is known as the *canonical cohomology* of (M, π).

These operators are related:

Lemma 2.3. Let (M, π) be a Poisson manifold, $\xi \in \mathfrak{X}^\bullet(M)$ and $\rho \in \Omega^\bullet(M)$. Then

$$\{\delta, \xi\}\rho = (d_\pi\xi) \cdot \rho,$$

where ξ and $d_\pi\xi$ act on forms by inner product.

Proof. This follows automatically from the description of the Schouten–Nijnehuis bracket as a derived bracket:

$$\{\delta, \xi\}\rho = \{\{\pi, d\}, \xi\}\rho = [\pi, \xi] \cdot \rho = (d_\pi\xi)\rho.$$

□

2.2 The canonical bundle and the modular vector field

Given a Poisson manifold (M^m, π), the determinant bundle $K = \wedge^m T^*M$ is also known as the *canonical bundle* of M. Given any nonvanishing local section $\rho \in \Gamma(K)$ there is a unique vector field X such that

$$\delta\rho = \iota_X\rho.$$

The vector field X is called the *modular vector field*. Notice that changing the trivialization ρ by a nonvanishing function, say g, changes the modular vector field from X to $X + \pi(d\log|g|) = X + d_\pi \log|g|$. In particular, changing ρ to $-\rho$ does not change X and a modular vector field is determined by a section of the quotient sheaf $K/\mathbb{Z}_2$. If M is nonorientable, there is no global nonvanishing section of K, yet, $K/\mathbb{Z}_2$, the sheaf of densities, always has a nonvanishing section, so one can always find globally defined modular vector fields.

Notice that any modular vector field X is an infinitesimal symmetry of the Poisson structure, i.e., $[\pi, X] = 0$ since for a local section $\rho \in \Gamma(K_\pi)$ we have

$$0 = \delta^2\rho = \delta(X \cdot \rho) = (d_\pi X) \cdot \rho - X \cdot (X \cdot \rho) = (d_\pi X) \cdot \rho,$$

which implies $[\pi, X] = 0$. Since a different choice of section of $K_\pi/\mathbb{Z}_2$ changes X to $X + d_\pi log|g|$ we see that the modular vector field gives a well defined degree one Poisson cohomology class. A Poisson structure is *unimodular* if this class is trivial, which is therefore equivalent to the existence of a globally defined δ-closed section of $K_\pi/\mathbb{Z}_2$.

2.3 Representations

A representation of a Poisson structure is a vector bundle $E \longrightarrow (M, \pi)$ together with a flat Poisson connection $\nabla : \Gamma(E) \longrightarrow \Gamma(TM \otimes E)$.

Example 2.4. The canonical bundle of a Poisson manifold is a representation. Indeed, the operator $\delta : K \longrightarrow \Omega^{m-1}(M) \cong \Gamma(TM \otimes K)$ satisfies the properties required for a connection and $\delta^2 = 0$ is the flatness condition. Note that if M is orientable, a Poisson structure is unimodular if and only if its canonical bundle is the trivial representation. ■

Example 2.5. Let $Z^d \subset M^m$ be a Poisson submanifold, then $\mathcal{N}_Z^*$, the conormal bundle of Z, is a representation for (Z, π). Indeed we define

$$\nabla : \Gamma(T^*Z \otimes \mathcal{N}_Z^*) \longrightarrow \mathcal{N}_Z^*; \qquad \nabla_\eta \xi = \iota_\pi(\eta \wedge d\xi),$$

where above one must extend the section $\xi \in \Gamma(\mathcal{N}_Z^*)$ to a form on a neighbourhood of Z, but the result does not depend on the chosen extension.

In particular we also get that $\wedge^{m-d}\mathcal{N}_Z^*$ is a representation. ■

If $Z \subset M$ is a Poisson manifold of M, one has clearly the isomorphism of vector bundles.

$$\wedge^m T^*M|_Z \cong \wedge^d T^*Z \otimes \wedge^{m-d}\mathcal{N}_Z^*.$$

Naturality of the representations discussed above, means that this is also an isomorphism of representations, similar to the *adjunction formula* from complex geometry

Lemma 2.6 (Adjunction formula)**.** Let M^m be a Poisson manifold and $Z \subset M$ be a d-dimensional embedded Poisson submanifold and $\mathcal{N}_Z^*$ be its conormal bundle. Then

$$K_M|_Z \cong K_Z \otimes \wedge^{m-p}\mathcal{N}_Z^*.$$

as Poisson modules.

The following proposition adds up the basic facts about the singular locus of a log-symplectic structure.

Proposition 2.7. *Let M be a log-symplectic manifold, Z its singular locus, $\mathcal{N}_Z^*$ the conormal bundle of Z and K_M and K_Z the canonical bundles of M and Z respectively. Then*

1. *Z is an orientable Poisson submanifold of M;*
2. *K_Z is the trivial representation;*
3. *$\mathcal{N}_Z^* = \wedge^{2n}T^*M|_Z$ as Poisson modules.*

In particular if M is orientable each component of Z has trivial normal bundle.

Proof. Since Z is the nondegenerate zero locus of $\pi^n \in \Gamma(\wedge^{2n}TM)$, we have that, over Z, $d\pi^n$ gives an isomorphism of Poisson representations

$$d\pi^n : \wedge^{2n}TM|_Z \otimes \mathcal{N}_Z^* \xrightarrow{\cong} \mathbb{R}$$

that is $\wedge^{2n}T^*M|_Z$ is isomorphic to $\mathcal{N}_Z^*$. Finally, the adjunction formula gives that

$$\wedge^{2n-1}T^*Z \otimes \mathcal{N}_Z^* \cong \wedge^{2n}T^*M|_Z \cong \mathcal{N}_Z^*,$$

that is $\wedge^{2n-1}T^*Z$ is the trivial representation and Z is orientable. □

3 Invariants and local forms

While Proposition 2.7 gives a list of simple invariants associated to a log-symplectic structure in [7] Guillemin, Miranda and Pires showed that these are in fact all invariants associated to a neighbourhood of the singular locus. Indeed, the following is a direct consequence of the results in [7]:

Theorem 3.1. *Let (M, π) be a log-symplectic manifold and let Z be a compact connected component of the singular locus. Then a neighbourhood of Z is determined by the Poisson structure induced on Z and its representation on the conormal bundle of Z.*

Taking the inverse of the Poisson structure, one can translate this information into differential forms (c.f. [7]):

Theorem 3.2. *Let (M, π) be a log-symplectic manifold, let Z be a connected component of the singular locus and X a modular vector field of π. Then the pair (π, X) determines the following structure on Z:*

1. *The normal bundle of Z as a vector bundle, i.e., a class $w_1 \in H^1(Z, \mathbb{Z}_2)$.*
2. *A closed 1-form $\theta \in \Omega^1(Z)$ such that $\theta(X) = -1$.*
3. *A closed 2-form $\sigma \in \Omega^2(Z)$ such that $\iota_X\sigma = 0$ and*

$$\theta \wedge \sigma^{n-1} \neq 0. \tag{3.1}$$

Changing the modular vector field by $d_\pi f$ does not change θ and changes σ to $\sigma + df \wedge \theta$.

Further, if Z is compact, any log-symplectic structure inducing the data above on Z is equivalent to a neighbourhood of the zero section of the normal bundle of Z endowed with the following structure

$$d \log |x| \wedge \theta + \sigma, \tag{3.2}$$

where $|x|$ is the distance to the zero section measured with respect to a fixed fiberwise linear metric on $\mathcal{N}_Z$.

Under the conditions of the theorem, the annihilator of the form θ corresponds to the distribution in Z determined by the Poisson structure and σ agrees with the leafwise symplectic form on Z.

Definition 3.3. A *cosymplectic structure* on a manifold Z^{2n-1} is a pair of closed forms $\theta \in \Omega^1(Z)$ and $\sigma \in \Omega^2(Z)$ satisfying (3.1).

For special types of log-symplectic structure one can rephrase the data *1. — 3.* above as a more workable set.

Definition 3.4. A connected cosymplectic manifold (Z, σ, θ) is *proper* if it is compact and the distribution given by the annihilator of θ has a compact leaf. A component Z of the singular locus of a log-symplectic manifold is *proper* if the cosymplectic structure induced on Z is proper. A log-symplectic manifold is *proper* if all components of the singular locus are proper.

Given a cosymplectic manifold (Z, σ, θ), if we let X be a vector field such that $\theta(X) = -1$ and $\iota_X \sigma = 0$, we have that $\mathcal{L}_X \theta = 0$ and hence the flow of X preserves the leaves of the distribution determined by θ, hence, if Z is proper with compact (symplectic) leaf $(F, \sigma) \subset Z$, the flow of F by the vector field X will provide further leaves of π. Since X is transverse to F and Z is compact, we see that after finite time, say $\lambda > 0$, the flow of X brings F back to itself:

$$\varphi_\lambda : F \longrightarrow F,$$

Since $\mathcal{L}_X \sigma = 0$, the flow is a symplectomorphism of F and hence Z is a symplectic fiber bundle with fiber (F, σ) over the circle:

$$Z = \mathbb{R} \times F/\mathbb{Z},$$

where the quotient is taken with respect to the $\mathbb{Z}$-action generated by $(y, p) \mapsto (y+\lambda, \varphi_\lambda(p))$. Further, the modular vector field is $-\partial_y$ hence $\theta = dy$.

Different choices of nonvanishing sections of $K_\pi/\mathbb{Z}_2$, change the modular vector field over Z by adding Hamitonian vector fields of F so the symplectomorphism φ_λ is only determined up to Hamiltonian symmetries, i.e., the relevant data is only its class in $\mathrm{Symp}(F)/\mathrm{Ham}(F)$.

Finally, the normal bundle of Z is determined by its first Stiefel–Whitney class $w_1 \in H^1(Z, \mathbb{Z}_2) = H^1(F; \mathbb{Z}_2)^{\varphi_\lambda} \times H^1(S^1; \mathbb{Z}_2)$. So, in the proper case, Theorem 3.2 becomes (c.f. [7, 5])

Theorem 3.5. *Let Z be a proper component of the singular locus of a log-symplectic structure π and $F \subset Z$ be a compact symplectic leaf of π. Then π determines the following data:*

1. *The normal bundle of Z, i.e., a class $w_1 \in H^1(Z, \mathbb{Z}_2) = H^1(F; \mathbb{Z}_2)^{\varphi} \times H^1(S^1; \mathbb{Z}_2)$.*
2. *The symplectic structure σ of F;*
3. *A class $[\varphi] \in \mathrm{Symp}(F)/\mathrm{Ham}(F)$;*
4. *A period $\lambda > 0$;*

Further, any two log-symplectic structures inducing the same set of data are equivalent and given a set of data 1. — 4. there is a proper log-symplectic structure which realises it.

Notice that given a nonorientable Poisson manifold, M, one can always pass to the oriented double cover $\widetilde{M}$ of M which inherits a Poisson structure from M. For the log-symplectic case, this allows us to get a simpler local model for the singular locus as now its neighbourhood depends on one less parameter, since according to Proposition 2.7 $w_1 = 0$ in $\widetilde{M}$.

The following theorem, communicated to the author by Ioan Marcut (see also [11]), shows one can always deform a log-symplectic structure into a proper one.

Theorem 3.6. *If the components of the singular locus of a log-symplectic structure are compact, then the structure can be deformed into a proper one.*

Proof. Let Z be a connected component of the singular locus. The proof consist of two steps. Firstly we notice that one can deform the cosymplectic structure (θ, σ) of Z into $(\widetilde{\theta}, \sigma)$ so that the kernel of $\widetilde{\theta}$ gives a fibration structure to Z. The second step is to show that this deformation can be realised as a deformation of the log-symplectic structure.

For the first step, let $\widetilde{\theta}$ be a closed 1-form representing a class in $H^1(Z, \mathbb{Q})$ which is close enough to θ so that we still have

$$\widetilde{\theta} \wedge \sigma^{n-1} \neq 0.$$

Since $[\widetilde{\theta}]$ represents a rational class, $[\widetilde{\theta}](H_1(Z; \mathbb{Z}))$ is a lattice Λ in $\mathbb{R}$. Then we define the projection map

$$p : Z \longrightarrow \mathbb{R}/\Lambda; \qquad p(z) = \int_{z_0}^{z} \widetilde{\theta},$$

where $z_0 \in Z$ is a fixed reference point and the value of the integral modulo Λ does not depend of choice of path connecting z_0 to z. By construction $dp = \widetilde{\theta}$ is nowhere vanishing and hence $p : Z \longrightarrow S^1$ is a fibration.

For the second step, according to Theorem 3.2 there is $\delta > 0$ such that the log-symplectic structure in a neighbourhood of Z is equivalent to (3.2) for $|x| < \delta$. If we let ψ be a smooth function such that

$$\psi : [0,1] \longrightarrow [0,1]; \qquad \begin{cases} \psi(x) = 1 & \text{if } x < \delta/3 \\ \psi(x) = 0 & \text{if } x > 2\delta/3 \end{cases}$$

then the log-symplectic form

$$d\log|x| \wedge ((1 - \psi(|x|)\theta + \psi(|x|)\widetilde{\theta}) + \sigma,$$

induces the cosymplectic structure $(\widetilde{\theta}, \sigma)$ on Z and agrees with the original log-symplectic structure if $|x| > 2\delta/3$ hence can be extended to the rest of M by the original log-symplectic structure. □

4 A simple topological invariant

One of the simplest and yet restrictive topological property of compact symplectic manifolds is the existence of a class $a \in H^2(M)$ whose top power is nonzero. Of course, this does not hold on all log-symplectic manifolds, yet log-symplectic manifolds are just a little shy of satisfying this property as shown by Marcut and Osorno-Torres:

Theorem 4.1 (Marcut–Osorno-Torres [9, 11])**.** *Let M^{2n} be a log-symplectic manifold whose singular locus has a compact component. Then there is a cohomology class $a \in H^2(M;\mathbb{R})$ such that $a^{n-1} \neq 0$. Further, if $Z \subset M$ is a proper component of the singular locus and has (F, σ) as a symplectic leaf, a can be chosen so that $[a]|_F = [\sigma]$.*

Here we use a little more of the log-symplectic structure in the orientable case to find another topological property of these manifolds.

Theorem 4.2. *A compact oriented manifold with a bona fide log-symplectic structure has a cohomology class $b \in H^2(M;\mathbb{R})\backslash\{0\}$ such that $b^2 = 0$.*

Proof. Assume that M has a log-symplectic structure with singular locus $Z \neq \emptyset$. Then, due to Theorem 3.6, we may assume that the structure is proper, hence Z is a symplectic fibration over the circle with fiber a symplectic manifold F. On the one hand, due to Marcut–Osorno-Torres's Theorem there is a globally defined closed 2-form $\widetilde{\omega} \in \Omega^2(M)$ which restricts to the symplectic form on F, i.e., the homology class of F pairs nonzero with a cohomology class on M, so we have that $[F] \neq 0 \in H_{2n-2}(M;\mathbb{R})$. On the other hand, since, even within Z, F appears as a fiber of a fibration, we conclude that the Poincaré dual of F, $b \in H^2(M;\mathbb{R})\backslash\{0\}$, must satisfy $b^2 = 0$. □

A few immediate corollaries:

Corollary 4.3. An orientable, compact, *bona fide* log-symplectic manifold M of dimension greater than two has $b_2(M) \geq 2$.

Proof. If M is four dimensional the class b from Theorem 4.2 is non zero and squares to zero, hence nondegeneracy of the intersection form implies the existence of a second class $\beta \in H^2(M;\mathbb{R})$ which has nonvanishing cup product with b. If the dimension of M is greater than four the classes from Marcut–Osorno-Torres's theorem and from Theorem 4.2 are necessarily different. Either way, $H^2(M;\mathbb{R})$ must be at least 2-dimensional. □

Corollary 4.4. For $n > 1$, $\mathbb{C}P^n$ has no *bona fide* log-symplectic structure and, for $n > 2$, the blow-up of $\mathbb{C}P^n$ along a symplectic submanifold of real codimension greater than 4 also does not carry *bona fide* log-symplectic structures.

Corollary 4.5. A smooth orientable compact four-manifold with definite intersection form does not admit *bona fide* log-symplectic structures. In particular, for $n > 0$, $\#n\mathbb{C}P^2$ and $\#n\overline{\mathbb{C}P^2}$ do not admit *bona fide* log-symplectic structures

Proof. Indeed, under the hypothesis of both corollaries there is no element in second cohomology whose square is zero. □

Notice that due to Taubes result on Seiberg–Witten invariants of symplectic manifolds [13], $\#n\mathbb{C}P^2$ does not admit symplectic structures for $n > 1$, i.e., $\#n\mathbb{C}P^2$ simply does not admit log-symplectic structures *bona fide* or not.

5 Birth of singular loci

This section we show that under mild assumptions one can transform a symplectic structure into a log-symplectic structure with non-empty singular locus. As a consequence of this seemingly inoffensive fact we conclude that $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ has a log-symplectic structure with non-empty singular locus as long as $m > 0$ and $n > 0$.

Theorem 5.1. *Let (M^{2n}, ω) be a symplectic manifold and $k > 0$ be an integer. If M has a compact symplectic submanifold $F^{2n-2} \subset M$ with trivial normal bundle, then M has a log-symplectic structure for which the zero locus of π^n has k components all diffeomorphic to $F \times S^1$.*

Proof. Due to the symplectic neighbourhood theorem, F has a tubular neighbourhood diffeomorphic to $D^2 \times F$ endowed with the product symplectic structure, where D^2 is the 2-disc of radius $\varepsilon > 0$. To prove the theorem it is enough to endow D^2 with a log-symplectic structure whose singular locus has k components and which agrees with the standard symplectic structure near the boundary of the disc. Indeed, in this case we can consider $D^2 \times F$ with the product of the log-symplectic structure on D^2 and the symplectic structure on F. Since this new structure agrees with the original symplectic structure on the boundary of the disc, we can extend it to M using the original symplectic structure.

To produce the desired log-symplectic structure on D^2 we observe that in two dimensions every bivector is automatically Poisson, hence all we need to do is find a bivector in D^2 with the desired number of nondegenerate zeros. To achieve this we let $\pi \in \Gamma(\wedge^2 TD)$ be the inverse of the standard symplectic structure on D^2 and consider the bivector $f\pi$ where f is a smooth function which in polar coordinates has the graph shown in Figure 2. Then $f\pi$ is a log-symplectic structure of the desired type on D^2.

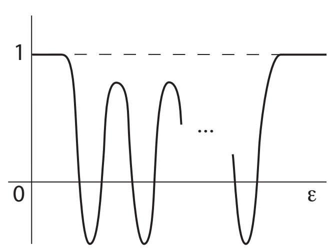

Figure 2: Graph of a possible scaling function that can be used to create a singular locus with an even number of components.

□

Corollary 5.2. For any positive integers m, n the manifolds $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ have a log-symplectic structure whose singular locus is diffeomorphic to $S^1 \times S^2$.

Proof. The blow-up of $\mathbb{C}P^2$ at a point, i.e., $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, has the structure of a symplectic $\mathbb{C}P^1$ fibration over $\mathbb{C}P^1$. In particular, the fibers satisfy the properties of Theorem 5.1 and hence we can endow $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ with a log-symplectic structure with non-empty singular locus, say, with one component diffeomorphic to $S^1 \times S^2$. Therefore, the top power of the log-symplectic form on the symplectic locus agrees with the orientation of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ at some points and disagrees in other points. By the Symplectic Blow-up Theorem [10], we can blow up points in the symplectic locus and the result still has a log-symplectic structure. If we blow up points in the symplectic locus where the orientation of the log-symplectic form agrees with the orientation of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, we are performing a connected sum with $\overline{\mathbb{C}P^2}$, while if we blow up points in the symplectic locus where the log-symplectic form gives the opposite orientation we are performing a connected sum with $\mathbb{C}P^2$. □

Notice that the manifolds obtained in Corollary 5.2 have vanishing Seiberg–Witten invariants and, for m and n even, those manifolds do not admit almost complex structures for either choice of orientation. These are contrasts between log-symplectic and symplectic geometries since symplectic manifolds have nonzero Seiberg–Witten invariants [13] and admit almost complex structures.

As for higher dimensions, Donaldson proved that every symplectic manifold admits a Lefschetz pencil [1] and hence is related to a Lefschetz fibration via the blow-up of the base locus of the pencil. Due to Theorem 5.1 any such fibration has log-symplectic structures.

6 Surgeries for log-symplectic manifolds

This section we introduce surgeries which produce new log-symplectic structures out of old ones. A main feature is that in these surgeries we create new components of the singular locus hence even if the starting manifolds are symplectic the results will be only log-symplectic.

6.1 Construction 1

This first construction produces (possibly) orientable log-symplectic manifolds out of pairs with matching data.

Building block: Using the language of Theorem 3.2, the local model that gives rise to the construction corresponds to the case when Z has trivial normal bundle. Given a cosymplectic manifold (Z, σ, θ), we let

$$\mathcal{N} = (-2, 2) \times Z$$

and endow $\mathcal{N}$ with a log-symplectic structure for which $\{0\} \times Z$ is the singular locus, namely, we consider the 2-form

$$\Omega = d\log|x| \wedge \theta + \sigma, \tag{6.1}$$

where $|x|$ denotes the absolute value of the real number x.

Ingredients: To perform this surgery we will need a (not necessarily connected) log-symplectic manifold (M^{2n}, π) together with two embeddings of a compact, connected, cosymplectic manifold (Z, σ, θ), $\iota_i : Z^{2n-1} \hookrightarrow M$, such that

1. Each $\iota_i(Z)$ is a separating submanifold of M lying in the symplectic locus of π and $\iota_1(Z) \cap \iota_2(Z) = \emptyset$;

2. There is $f \in \mathbb{C}^\infty(Z)$ such that $\iota_1^*\omega = \iota_2^*\omega - df \wedge \theta = \sigma$, where ω is the symplectic form on the symplectic locus of M

The surgery: Since each $\iota_i(Z)$ is in the symplectic locus of π, the underlying symplectic structure gives rise to an orientation of a neighbourhood of $\iota_i(Z)$. Since Z is cosymplectic, it has a natural orientation as defined by the volume form $\theta \wedge \sigma^{n-1}$. Since $\iota_i(Z)$ are separating submanifolds, each of them defines an interior and an exterior region: a vector $n \in T_{\iota_i(p)}M$ is outward pointing if for any positive basis $\{v_1, \cdots, v_{2n-1}\} \in T_pZ$ the set $\{n, \iota_{i*}v_1, \cdots, \iota_{i*}v_{2n+1}\}$ is a positive basis for T_pM. We let M^+ be the closures of the intersection of the exterior regions determined by $\iota_1(Z)$ and $\iota_2(Z)$, then M^+ has two boundary components: $\iota_1(Z)$ and $\iota_2(Z)$.

Theorem 6.1. *Let (M, π), (Z, θ, σ) and $\iota_1, \iota_2 : Z \longrightarrow M$ be the ingredients for the surgery and let M^+ be the closure of the intersection of the exterior regions defined by $\iota_1(Z)$ and $\iota_2(Z)$. Then the manifold*

$$M^+/\sim$$

obtained by identifying the boundary components of M^+ via the map $\iota_2 \circ \iota_1^{-1}$ has a log-symplectic structure which agrees with the original structure on M^+ outside a neighbourhood of $Z = \partial M^+/\sim$ and for which Z is part of the singular locus.

Proof. Notice that we have an embedding $Z \hookrightarrow \mathcal{N}$, as $\{-1\} \times Z$. For this embedding, Z lies in the symplectic locus of the log-symplectic structure and the restriction of the symplectic form (6.1) to Z is just σ. Hence, by Weinstein's coisotropic neighbourhood theorem, a neighbourhood of $\{-1\} \times Z \subset \mathcal{N}$ is symplectomorphic to a neighbourhood of $\iota_1(Z)$. Using this symplectomorphism, we can glue the exterior of $\iota_1(Z)$ in M to $(-1, 2) \times Z \subset N$, the interior of $\{-1\} \times Z$ in $\mathcal{N}$, and the resulting manifold has a log-symplectic structure.

Similarly, given a function $f \in C^\infty(Z)$, if we take $\varepsilon > 0$ small enough we have an embedding

$$\varphi : Z \hookrightarrow \mathcal{N}, \qquad \varphi(z) = (\varepsilon e^{f(z)}, z)$$

and the restriction of the symplectic form from $\mathcal{N}$ to this embedding is $df \wedge \theta + \sigma$, that is, this symplectic form agrees with $\iota_2^{-1}\omega$ and hence we can repeat the previous argument, but now to the embedding ι_2 and this second copy of Z inside $\mathcal{N}$.

It is clear that the manifold obtained by this gluing procedure is diffeomorphic to $M^+/\sim$ and that the structure on the complement on $\mathcal{N}$ is the original structure on M^+. $\square$

Observe that even if M is a symplectic manifold, and hence has a preferred orientation, the diffeomorphism used to glue the two boundaries together does not respect these orientations hence $\widetilde{M}$ does not have a preferred orientation.

As far as examples go, Theorem 6.1 leaves us with the question of how to find suitable submanifolds Z to which it can be applied. Next we identify two situations in which manifolds with the desired structure appear naturally. We start with the simplest setting:

Corollary 6.2. Let (M^{2n}, ω) be a log-symplectic manifold and $\iota_i : (F^{2n-2}, \sigma) \longrightarrow (M, \omega)$, $i = 1, 2$ be embeddings of a compact symplectic manifold F in the symplectic locus of M for which the images have trivial normal bundle. Let

$$\widetilde{M} = M \backslash (\mathcal{N}_1 \cup \mathcal{N}_2)/\sim,$$

where $\sim$ indicates the natural identification of the boundaries $\partial\mathcal{N}_1 \cong \partial\mathcal{N}_2$. Then $\widetilde{M}$ has a log-symplectic structure which agrees with the original structure outside a tubular neighbourhood of $\partial\mathcal{N}_i \subset \widetilde{M}$. The Euler characteristic of $\widetilde{M}$ is

$$\chi_{\widetilde{M}} = \chi_M - 2\chi_F.$$

Proof. Weinstein's symplectic neighbourhood theorem implies that $\iota_i(F)$ have neighbourhoods symplectomorphic to $D^2 \times F$ with the product symplectic structure. In particular the boundary of such neighbourhoods are diffeomorphic to $Z = S^1 \times F$ with the cosymplectic structure given by the volume from of S^1,θ, and $\sigma = \omega|_F$ and the restriction of the symplectic form to Z is simply the symplectic form of F pulled back to the product. Hence we can use the theorem to conclude that $\widetilde{M}$ has a log-symplectic structure.

The last claim follows from the additive properties of the Euler characteristic. $\square$

Next we present a setting which is a little more elaborate:

Corollary 6.3. Let $p_i : M_i \longrightarrow \Sigma_i$, $i = 1, 2$, be symplectic Lefschetz fibrations with the same generic fiber (F, σ). Let $\gamma_i : \mathbb{R}/\lambda\mathbb{Z} \longrightarrow \Sigma_i$ be separating loops which avoid the critical values of p_i, let M_i^+ be the exterior region of $p_i^{-1}(\gamma_i)$ and let $\varphi_i : F \longrightarrow F$ be the symplectomorphism of $F = p_i^{-1}(\gamma_i(0))$ obtained from symplectic parallel transport along γ_i. If there is a symplectomorphism $\vartheta : F \longrightarrow F$ such that $\varphi_2 = \vartheta \circ \varphi_1 \circ \vartheta^{-1}$, then

$$M_1^+ \cup M_2^+ / \sim$$

has a log-symplectic structure with singular locus $\partial M_1^+ \cong \partial M_2^+$.

Proof. Under the hypothesis, $Z_1 = p_1^{-1}(\gamma_1)$ is given by the quotient of $\mathbb{R} \times F$ by the $\mathbb{Z}$-action generated by

$$(x, z) \cong (x + \lambda, \varphi_1(z)).$$

The form $\theta = dx$ together with the symplectic form $\sigma \in \Omega^2(F)$ make Z_1 into a cosymplectic manifold.

Similarly, $Z_2 = p_2^{-1}(\gamma_2)$ is a cosymplectic manifold and the map

$$Z_1 \longrightarrow Z_2, \qquad (x, z) = (z, \vartheta(z))$$

is an isomorphism of cosymplectic structures as long as $\varphi_2 = \vartheta \circ \varphi_1 \circ \vartheta^{-1}$. The result now follows from Theorem 6.1. $\square$

Example 6.4 (Log-symplectic structures on $\#nS^2 \times S^2$)**.** Next we provide an explicit log-symplectic structure on $\#nS^2 \times S^2$. Our starting point is the elliptic surface $E(2k)$: the fiber sum of $2k$ copies of $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$. This is a Lefschetz fibration $p : E(2k) \longrightarrow \mathbb{C}P^1$ which, after appropriate identifications, has $24k$ singular fibers for which the vanishing cycles correspond to two basis elements $\{a, b\} \in H_1(F)$ appearing in an alternating fashion, as depicted, for $E(2)$, in Figure 3 (c.f. [4], Example 8.2.11).

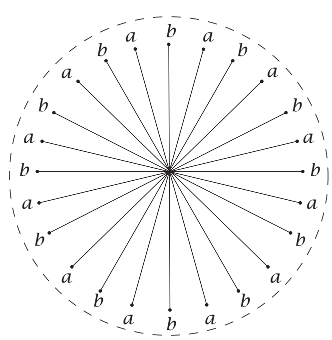

Figure 3: Graphic representation of the base of $E(2)$, showing the singular values of the projection map, a regular value of the map for reference and paths connecting the regular value to the singular values to determine the homology class of the vanishing cycles. In this case, a and b, the vanishing cycles which form an integral basis for $H_1(F; \mathbb{Z})$.

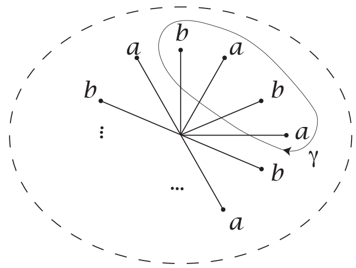

Figure 4: Graphic representation of the base of $E(2k)$ and a path whose exterior contains four consecutive singular values of the projection map.

In order to use Construction 1 we consider two copies of $E(2k)$ and in both of them consider the same path, namely one whose exterior contains $n+1$ consecutive singular fibers (see Figure **??**).

Since we are starting with two copies of the same data we can use Corollary 6.3 to introduce a log-symplectic structure

$$\widetilde{M} = M^+ \cup_\partial M^+,$$

where $M^+ = p^{-1}(\overline{\Sigma^+})$ is the inverse image of the closure of the exterior points of γ. Hence we see that Construction 1 consists of taking two identical copies of $M^+ \subset E(2k)$ and then glue them along the boundaries using the identity map. The resulting manifold is also known as the *double* of M^+. To precisely determine M^+ we observe two simple facts:

- Firstly, M^+ admits a handlebody decomposition which contains a zero handle, n 2-handles and no handles of other indices (that is M is a 2-handlebody). Indeed, according to [4], Example 8.2.8, a Kirby diagram for M has $n+2$ 2-handles and two 1-handles and one zero handle, but we can cancel two of 2-handles against the two 1-handles to obtain the desired handlebody decomposition);
- Secondly, since the interior of M^+ is an open subset of $E(2k)$ and $E(2k)$ is spin, the intersection form on M^+ is even.

These facts, together with the following Proposition determine the result of the surgery:

Proposition 6.5 ([4], Corollary 5.1.6)**.** *Let M^4 be a 2-handlebody with n 2-handles. Then the double of M is diffeomorphic to $\#nS^2 \times S^2$ if the intersection form of M is even and to $\#n\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ if the intersection form of M is odd.*

In our case, we conclude that $\widetilde{M}$ is diffeomorphic to $\#nS^2 \times S^2$. ∎

6.2 Achiral Lefschetz fibrations

Related to the construction of Corollary 6.3 is the notion of an achiral Lefschetz fibration:

Definition 6.6. Let M^{2n} be a manifold. An *achiral Lefschetz fibration* on M is a proper, smooth map $p : M \longrightarrow \Sigma^2$ such that the pre-image of any critical value has only one critical point and for any such pair of critical value, y, and critical point, x, there are complex coordinate systems centered at x and y for which p takes the following form

$$p(z_1, \cdots, z_n) = z_1^2 + \cdots + z_n^2.$$

Notice that in this definition we do not require M or Σ to be orientable. If they are, one can assign a sign to each critical point x: we demand that the complex structure on Σ is compatible with the orientation and then we say that x is *positive* if the complex structure on M used in the definition is compatible with the orientation of M and *negative* otherwise.

Given the construction of Corollary 6.3 and the issuing example, one might expect that achiral Lefschetz fibrations are related to log-symplectic structures in the same way that Lefschetz fibrations are related to symplectic structures. This is indeed the case, as we show next:

Theorem 6.7. *Let M^4 and Σ^2 be compact connected manifolds and $p : M \longrightarrow \Sigma$ be an achiral Lefschetz fibration with generic fiber F. If F is orientable and $[F] \neq 0 \in H_2(M;\mathbb{R})$, then M has a log-symplectic structure whose singular locus has one component and for which the fibers are symplectic submanifolds of the symplectic leaves of the Poisson structure.*

Proof. The proof follows closely that of Gompf's Theorem relating Lefschetz fibrations and symplectic structures ([4], Theorem 10.2.18). Before we delve into the proof we will fix some notation. We let

- $F_y = p^{-1}(y)$ for $y \in \Sigma$;
- Δ be the set of singular points of p;
- Δ' be the set of the singular values of p;
- $\Sigma_0 = \Sigma \backslash \Delta'$ and $M_0 = p^{-1}(\Sigma\backslash\Delta')$, so that $p : M_0 \longrightarrow \Sigma_0$ is a proper fibration.

Firstly, we observe that we can assume that p has connected fibers. Indeed, for achiral Lefschetz fibrations we have a short exact sequence of homotopy groups:

$$\pi_1(F) \longrightarrow \pi_1(M) \xrightarrow{p_*} \pi_1(\Sigma) \longrightarrow \pi_0(F) \longrightarrow \{0\}. \tag{6.2}$$

Since M is compact, $\pi_0(F)$ is finite and hence (6.2) implies that $p_*(\pi_1(M))$has finite index in $\pi_1(\Sigma)$. Let $\widetilde{\Sigma}$ be the cover of Σ corresponding to the subgroup $p_*(\pi_1(M)) \subset \pi_1(\Sigma)$. Then $\widetilde{\Sigma}$ is compact and the map $p : M \longrightarrow \Sigma$ lifts to a map $\widetilde{p} : M \longrightarrow \widetilde{\Sigma}$. The projection $\widetilde{p}$ is still an achiral Lefschetz fibration and, by construction, $\widetilde{p}_*(\pi_1(M)) = \pi_1(\widetilde{\Sigma})$, hence (6.2) implies that the fibers of $\widetilde{p}$ are connected. From now on we assume that the fibers of $p : M \longrightarrow \Sigma$ are connected.

Next we deal with the general lack of orientation of the manifolds involved. Firstly, since F is orientable, we fix an orientation for F for the remaining of this proof. If $p : M_0 \longrightarrow \Sigma_0$ was a nonorientable fibration, there would be a loop $\alpha : (I, \partial I) \longrightarrow M_0$ based at some point $y \in \Sigma$, where I is the unit interval, for which parallel transport (after a choice of connection) provided an orientation reversing diffeomorphism of F_y. In this case $p^{-1}(\alpha(I))$ would provide a chain whose boundary is $2[F_y]$, contradicting the condition $[F] \neq 0 \in H_2(M;\mathbb{R})$. Therefore $p : M_0 \longrightarrow \Sigma_0$ is an orientable fibration. Further this orientation induces orientations on the singular fibers of p and hence we can also integrate forms over the (components of the singular) fibers. It also follows that M is orientable if and only if Σ is.

If Σ and M are orientable, after choosing orientations, we can split the critical points of p into positive and negative ones. If there are no positive or no negative points, Gompf's Theorem ([4], Theorem 10.2.18) implies that M admits a symplectic structure and due to Theorem 5.1 it admits a log-symplectic structure with the desired properties. If there are positive and negative critical points, we can choose a separating loop $\Gamma \subset \Sigma_0$ whose interior locus contains all the negative points and whose exterior locus contains all the positive points. If Σ is nonorientable, we can choose a loop $\Gamma \subset \Sigma_0$ such that $\Sigma\backslash\Gamma$ is orientable and hence so is $M\backslash p^{-1}(\Gamma)$. After choosing orientations on both $\Sigma\backslash\Gamma$ and $M\backslash p^{-1}(\Gamma)$ we may homotope the loop Γ through the negative critical values of p so that all singular points of $p : M\backslash p^{-1}(\Gamma) \longrightarrow \Sigma\backslash\Gamma$ are positive. In either case, $M\backslash p^{-1}(\Gamma)$ is an oriented manifold and $p : M\backslash p^{-1}(\Gamma) \longrightarrow \Sigma\backslash\Gamma$ is a proper Lefschetz fibration possibly after changing the orientation of one of the components of $M\backslash\Gamma$ and $\Sigma\backslash\Gamma$. From now on we orient both $M\backslash\Gamma$ and $\Sigma\backslash\Gamma$ so that p is a Lefschetz fibration there.

The next steps aim to construct a closed 2-form σ on M which restricts to a symplectic form in every fiber.

Lemma 6.8. Under the hyphothesis of Theorem 6.7, there is a closed 2-form $\zeta \in \Omega^2(M)$ such that

1. $\int_S \zeta = 1$ over any fiber and
2. if a singular fiber F_y is a plumbing of two surfaces S_1 and S_2, and we are given $s_y \in (0,1)$ then ζ can be chosen so that $\int_{S_1} \zeta = s_y$.

Proof. Since $[F] \neq 0 \in H_2(M;\mathbb{R})$, there is a closed form $\xi \in \Omega^2(M)$ which integrates to 1 over the generic fibers and hence over all fibers. Next we need to argue that one can change ξ so that the property 2. holds. In this case, with the orientations chosen before on $M\backslash\Gamma$, the intersection number of S_1 and S_2 is 1. Since $S_1 \cup S_2$ is homologous to a regular fiber, $\int_{S_1} \xi + \int_{S_2} \xi = 1$. If $\int_{S_1} \xi = r$, let ψ be a form with support in a neighbourhood of $S_2 \subset M\backslash p^{-1}(\Gamma)$ which represents the Poincaré dual of S_2 and consider $\xi' = \xi + (-r + s_y)\psi$. Then for any closed surface S' inside another fiber $F_{y'}$, $\int_{F_{y'}} \psi = 0$ as $F_{y'}$ does not intersect S_2 hence $\int_{S'} \xi' = \int_{S'} \xi$. On the other hand

$$\int_{S_1} \xi' = \int_{S_1} \xi \pm (-r + s_y) \int_{S_1} \psi = s_y.$$

That is, after a change in ξ, we found another closed form for which the claim holds at a specific singular fiber. Since there are only finitely many such fibers, we can repeat the process for each of them to obtain the desired ζ. □

Lemma 6.9. Under the hyphothesis of Theorem 6.7, there is a finite good open cover $\mathcal{U}$ of Σ such that for each $U_\alpha \in \mathcal{U}$, there is a closed form $\eta_\alpha \in \Omega^2(p^{-1}(U_\alpha))$ which is symplectic on the fibers of p.

Proof. Since $p : M\backslash p^{-1}(\Gamma) \longrightarrow \Sigma\backslash\Gamma$ is a proper Lefschetz fibration for which $[F] \neq 0$, it follows from Gompf's Theorem ([4], Theorem 10.2.18) that each component of $M\backslash p^{-1}(\Gamma)$ has a symplectic form for which the fibers are symplectic of area 1. Letting $\mathcal{N}$ be a small tubular neighbourhood of Γ without singular values of p, it follows that, $p : p^{-1}(\mathcal{N}) \longrightarrow \mathcal{N}$ is a proper Lefschetz fibration for which $[F] \neq 0$ hence we can also apply Gompf's result here to conclude that there is a symplectic form ω_0 on $p^{-1}(\mathcal{N})$ for which the fibers are symplectic of area 1. Finally, let $\mathcal{U}$ be a finite good refinement of the cover $\{\Sigma\backslash\Gamma, \mathcal{N}\}$ and for each $U_\alpha \in \mathcal{U}$ let η_α be the restriction of one of the symplectic forms above to $p^{-1}(U_\alpha)$. □

Lemma 6.10. Under the hyphothesis of Theorem 6.7, there is a closed 2-form $\sigma \in \Omega^2(M)$ such that $\sigma|_{F_y}$ is a symplectic form for every fiber F_y.

Proof. The forms η_α and ζ are cohomologous on $p^{-1}(U_\alpha)$ (in the case of singular fibers, one must choose the value s_y so that the integrals of these forms over each cycle agree). Therefore there are $\theta_i \in \Omega^1(p^{-1}(U_i))$ such that $\eta_i = \zeta + d\theta_i$. Let $\{\kappa_\alpha\}$ be a partitition of unity subbordinate to the cover $\mathcal{U}$ and consider the form

$$\sigma = \zeta + d\sum_i p^*(\kappa_i)\theta_i.$$

Since $p^*\kappa|_{F_y}$ is a constant, we have that

$$\sigma|_{F_y} = \zeta|_{F_y} + \sum_i p^*(\kappa_i) d\theta_i|_{F_y} = \sum_i p^*(\kappa_i)(\zeta|_{F_y} + d\theta_i|_{F_y}) = \sum_i p^*(\kappa_i)\eta_i|_{F_y}.$$

Since each $\eta_i|_{F_y}$ is a symplectic form and all of them determine the same orientation $\sum_i p^*(\kappa_i)\eta_i|_{F_y}$ is a symplectic form on F_y. □

End of proof of Theorem 6.7. Finally, to obtain the log-symplectic structure on M, observe that since $\Sigma\backslash\Gamma$ is oriented, Γ is a real divisor on Σ representing $\wedge^2 T\Sigma$, i.e., there is a section $\pi \in \Gamma(\wedge^2(T\Sigma))$ which has Γ as its (transverse) zero locus. Since Σ is two-dimensional, π is a log-symplectic structure whose singular locus is Γ. We further choose π so that it agrees with the orientation of $\Sigma\backslash\Gamma$. Inverting π we obtain $\omega_\Sigma \in \Omega^2(\Sigma)$ with a log-singularity at Γ. Then the standard argument shows that $\omega = p^*\omega_\Sigma + \varepsilon\sigma$ is a log-symplectic structure for ε small enough. Indeed, away from critical points of p, ω_Σ dominates σ and hence determines a log-symplectic structure on the complement of a small neighbourhood of Δ. In particular, ω determines an orientation on its symplectic locus. With respect to this orientation on $M\backslash p^{-1}(\Gamma)$ and the orientation determined by π on $\Sigma\backslash\Gamma$, all singular points are positive and the argument from [4], Exercise 10.2.21, shows that ω is symplectic on $M\backslash p^{-1}(\Gamma)$. $\square$

Remarks (*The condition* $[F] \neq 0$).

- If the genus of the fiber is different from 1, then $\ker(p)$ defines a line bundle over $M\backslash\Delta$ and this line bundle extends to the singular locus. Letting c_1 be the first Chern class of this bundle, naturality implies that $c_1|_F$ is just the Euler class of the fiber and hence, if the genus of the fiber is not 1, c_1 evaluates nonzero on $[F]$, showing that $[F] \neq 0$;
- If an achiral Lefschetz fibration over an oriented surface has a section, then any fiber represents a nontrivial class since it has nontrivial intersection with the section.
- S^4 admits an achiral Lefschetz fibration. Since $H^2(S^4) = \{0\}$, the fibers are homologically trivial and hence are tori and there is no section. Further, S^4 also does not admit log-symplectic structures due to Theorem 4.1. Therefore the condition $[F] \neq 0$ can not be removed from Theorem 6.7.

Achiral Lefschetz fibrations have been studied by Etnyre and Fuller [2] and are present in several four-manifolds:

Theorem 6.11 (Etnyre–Fuller [2])**.** *Let M be a simply-connected compact four-manifold. Then both $M\#(S^2 \times S^2)$ and $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admit achiral Lefschetz fibrations with a section over S^2.*

Combining Theorem 6.7 with Etnyre–Fuller's Theorem we get:

Theorem 6.12. *Let M be a simply connected compact four-manifold. Then both $M\#(S^2 \times S^2)$ and $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admit bona fide log-symplectic structures.*

6.3 Construction 2

The second construction is a nonorientable version of the first which produces proper log-symplectic manifolds.

Building block: Given a symplectic manifold (F, σ) a symplectomorphism $\varphi : F \longrightarrow F$ and $\lambda > 0$ we form the quotient of the log-symplectic manifold

$$\mathcal{N} = (-2, 2) \times \mathbb{R} \times F; \qquad \Omega|_{(x,y,p)} = d\log|x| \wedge dy + \sigma$$

by the $\mathbb{Z}$-action generated by $(x, y, p) \sim (-x, y + \lambda, \varphi(p))$:

$$\mathcal{N}_\varphi = \mathcal{N}/\mathbb{Z}.$$

Then $\mathcal{N}_\varphi$ is a log-symplectic manifold with singular locus $Z = \{0\} \times \mathbb{R} \times F/\mathbb{Z}$. Notice that $\mathcal{N}_\varphi\backslash Z$ is a fiber bundle over $\mathbb{R}_+$, with projection map induced by the invariant map $\pi_1(x, y, p) = |x|$ defined on $(-2, 2) \times \mathbb{R} \times F$. Then $Z' = \pi_1^{-1}(1)$ is a coisotropic submanifold of the symplectic locus given by

$$Z' = \mathbb{R} \times F/\mathbb{Z} \qquad (y, p) \sim (y + 2\lambda, \varphi^2(p)). \tag{6.3}$$

Ingredients: We will need a proper cosymplectic manifold (Z', θ, σ) with symplectic fiber F for which the monodromy map is the square of a symplectomorphism $\varphi : F \longrightarrow F$, i.e., Z' is given by (6.3). We will need further a log-symplectic manifold (M, π) and a separating embedding $\iota : Z' \hookrightarrow M$ in the symplectic locus of M such that $\iota^*\omega = \sigma$, where ω is the induced symplectic structure on M.

The surgery: The surgery follows the same lines of Construction 1: Since $\iota(Z)$ is separating, it defines an exterior and an interior region of M. Let M^+ be the closure of the exterior. Then there is a $\mathbb{Z}_2$-action on $Z' = \partial M^+$, namely, in terms of (6.3), the action is generated by the map

$$(y, p) \mapsto (y + \lambda, \varphi(p))$$

and the orbits of this action form an equivalence relation on ∂M^+ which allows us to form the space

$$\widetilde{M} = M^+/\sim$$

obtained by taking the quotient of ∂M^+ by this equivalence relation.

Theorem 6.13. *Let (M, π), (Z', θ, σ) and $\iota : Z' \longrightarrow M$ be the ingredients for the surgery and let M^+ be the closure of the exterior region defined by $\iota(Z')$. Then the manifold*

$$\widetilde{M} = M^+/\sim$$

obtained by taking the quotient of ∂M^+ by the $\mathbb{Z}_2$-action has a log-symplectic structure which agrees with the original structure on M^+ outside a neighbourhood of $Z = \partial M^+/\sim$ and for which Z is part of the singular locus.

The proof of this theorem is completely analogous to that of Theorem 6.1. A particular case of this surgery has a geometric interpretation.

Corollary 6.14 (Real blow-up)**.** Let (M^{2n}, ω) be a log-symplectic manifold and let $F^{2n-2} \subset M$ be a symplectic submanifold which does not intersect the singular locus and has trivial normal bundle. Then the real blow-up of M along F has a log-symplectic structure for which the exceptional divisor is a component of the singular locus.

Proof. Just as in Corollary 6.2, the requirement that F has trivial normal bundle implies that a neighbourhood of F is symplectomorphic to $D^2 \times F$ and hence we obtain an embedding of the proper cosymplectic manifold $S^1 \times F$ into M. The monodromy of this cosymplectic manifold is the identity map which is obviously the square of a symplectomorphism. Now, the local model is based on using $\varphi = \mathrm{Id}$, that is $\mathcal{N}_F = \mathbb{M} \times F$, where $\mathbb{M}$ is the Möbius band and the effect of the surgery is that we remove a neighbourhood of F (which is diffeomorphic to $D^2 \times F$) and glue back $\mathbb{M} \times F$. This is precisely the underlying surgery of the real blow-up of F. $\square$

7 Reversing the surgeries

Last section we managed to produce several examples of log-symplectic manifolds out of symplectic manifolds. One might rightfully expect that there are more examples of such structures: for one thing the Stiefel–Whitney class either vanished (first construction) or corresponded to the generator of $H^1(S^1; \mathbb{Z}_2)$ (second construction), therefore leaving out a number of possibilities. On the other hand, if we assume that M is orientable or, in the nonorientable case, take the orientable double cover, then any singular locus automatically is associated to the zero Stiefel-Whitney class and hence it has neighboorhood diffeomorphic, as a Poisson, manifold to the building blocks used in Construction 1. Next we show that in four dimensions any log-symplectic structure is created out of our surgeries and hence can be cut up and filled into a collection of compact symplectic manifolds.

Theorem 7.1. *Let (M^4, π) be a compact, orientable, log-symplectic manifold with singular locus Z. Then each unoriented component of $M \backslash Z$ can be compactified as a symplectic manifold.*

Proof. According to Theorem 3.2, a neighbourhood of each connected component of Z is equivalent to the building block of Construction 1 and hence we have two copies of Z in such neighbourhood (one on either side of the singular locus) as a coisotropic submanifold. To reverse the surgery, one would need to prove that such coisotropic submanifold appears as the boundary of the (interior of) a symplectic manifold. But in four dimensions any cosymplectic manifold is automatically a taut foliation and hence the conclusion follows from the following theorem:

Theorem 7.2 ([8], Theorem 41.3.1)**.** *Let Z be a closed 3-manifold and $\mathcal{F} \subset TZ$ be a smooth taut foliation. Let $\sigma \in \Omega^2(Z)$ be the closed form which is positive on the leaves of $\mathcal{F}$. Then there is a closed symplectic manifold (X, ω) containing Z as a separating submanifold such that $\omega|Z = \sigma$.*

□

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