On non-formality of a simply-connected symplectic 8-manifold

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Abstract. We show an alternative construction of the first example of a simply-connected compact symplectic non-formal 8-manifold given in [6]. We also give an alternative proof of its non-formality using higher order Massey products.

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INTRODUCTION

In [1, 2, 10] Babenko–Taimanov and Rudyak–Tralle give examples of non-formal simply-connected compact symplectic manifolds of any even dimension bigger than or equal to 10. Babenko and Taimanov raise the question of the existence of non-formal simply-connected compact symplectic manifolds of dimension 8, which cannot be constructed with their methods. In [6], it is constructed the first example of a simply-connected compact symplectic 8-dimensional manifold which is non-formal, thereby completing the solution to the question of existence of non-formal symplectic manifolds for all allowable dimensions. This example is constructed by starting with a suitable complex 8-dimensional compact nilmanifold M which has a symplectic form (but is not Kähler). Then one quotients by a suitable action of the finite group $\mathbb{Z}_3$ acting symplectically and freely except at finitely many fixed points. This gives a symplectic orbifold $\widehat{M} = M/\mathbb{Z}_3$, which is non-formal and simply-connected thanks to the choice of $\mathbb{Z}_3$-action. The last step is a process of symplectic resolution of singularities to get a smooth symplectic manifold. The symplectic resolution of isolated orbifold singularities has been described in detail in [4]. The non-formality of $\widehat{M}$ is checked via a newly defined product in cohomology. This is a product of Massey type, which is called a-product, and it is discussed at length in [4].

The purpose of the present note is to give a new description of the symplectic orbifold $\widehat{M}$ defined in [6]. The description presented here is in terms of real nilpotent Lie groups. Secondly, we prove the non-formality of $\widehat{M}$ by using higher order Massey products instead of a-products. It remains thus open the question of the existence of a smooth 8-manifold with non-zero a-products but trivial (higher order) Massey products.

A NILMANIFOLD OF DIMENSION 6

Let G be the simply connected nilpotent Lie group of dimension 6 defined by the structure equations

$$\begin{aligned} d\beta_i &= 0, \qquad i=1,2 \\ d\gamma_i &= 0, \qquad i=1,2 \\ d\eta_1 &= -\beta_1\wedge\gamma_1+\beta_2\wedge\gamma_1+\beta_1\wedge\gamma_2+2\beta_2\wedge\gamma_2, \\ d\eta_2 &= 2\beta_1\wedge\gamma_1+\beta_2\wedge\gamma_1+\beta_1\wedge\gamma_2-\beta_2\wedge\gamma_2, \end{aligned} \tag{1}$$

where $\{\beta_i,\gamma_i,\eta_i; 1\le i\le 2\}$ is a basis of the left invariant 1–forms on G. Because the structure constants are rational numbers, Mal'cev theorem [7] implies the existence of a discrete subgroup Γ of G such that the quotient space $N=\Gamma\backslash G$ is compact.

Using Nomizu's theorem [9] we can compute the real cohomology of N. We get

$$\begin{aligned} H^0(N) &= \langle 1\rangle, \\ H^1(N) &= \langle [\beta_1],[\beta_2],[\gamma_1],[\gamma_2]\rangle, \\ H^2(N) &= \langle [\beta_1\wedge\beta_2],[\beta_1\wedge\gamma_1],[\beta_1\wedge\gamma_2],[\gamma_1\wedge\gamma_2],[\beta_1\wedge\eta_2-\beta_2\wedge\eta_1],[\gamma_1\wedge\eta_2-\gamma_2\wedge\eta_1], \\ &\quad [\beta_1\wedge\eta_1+\beta_1\wedge\eta_2+\beta_2\wedge\eta_2],[\gamma_1\wedge\eta_1+\gamma_1\wedge\eta_2+\gamma_2\wedge\eta_2]\rangle, \\ H^3(N) &= \langle [\beta_1\wedge\beta_2\wedge\eta_1],[\beta_1\wedge\beta_2\wedge\eta_2],[\gamma_1\wedge\gamma_2\wedge\eta_1],[\gamma_1\wedge\gamma_2\wedge\eta_2],[\beta_1\wedge\gamma_1\wedge(\eta_1+2\eta_2)], \\ &\quad [\beta_1\wedge\gamma_1\wedge\eta_2-\beta_1\wedge\gamma_2\wedge\eta_1],[\beta_1\wedge\gamma_2\wedge\eta_1-\beta_1\wedge\gamma_2\wedge\eta_2],[\beta_2\wedge\gamma_2\wedge(\eta_2+2\eta_1)], \\ &\quad [\beta_2\wedge\gamma_2\wedge\eta_1-\beta_2\wedge\gamma_1\wedge\eta_2],[\beta_2\wedge\gamma_1\wedge\eta_2-\beta_2\wedge\gamma_1\wedge\eta_1]\rangle, \\ H^4(N) &= \langle [\beta_1\wedge\beta_2\wedge\gamma_1\wedge\eta_1],[\beta_1\wedge\beta_2\wedge\gamma_1\wedge\eta_2],[\beta_1\wedge\beta_2\wedge\eta_1\wedge\eta_2],[\beta_1\wedge\gamma_1\wedge\gamma_2\wedge\eta_2], \\ &\quad [\beta_2\wedge\gamma_1\wedge\gamma_2\wedge\eta_2],[\gamma_1\wedge\gamma_2\wedge\eta_1\wedge\eta_2],[\beta_1\wedge\gamma_2\wedge\eta_1\wedge\eta_2-\beta_2\wedge\gamma_1\wedge\eta_1\wedge\eta_2], \\ &\quad [\beta_1\wedge\gamma_2\wedge\eta_1\wedge\eta_2+\beta_1\wedge\gamma_1\wedge\eta_1\wedge\eta_2+\beta_2\wedge\gamma_2\wedge\eta_1\wedge\eta_2]\rangle, \\ H^5(N) &= \langle [\beta_1\wedge\beta_2\wedge\gamma_1\wedge\eta_1\wedge\eta_2],[\beta_1\wedge\beta_2\wedge\gamma_2\wedge\eta_1\wedge\eta_2],[\beta_1\wedge\gamma_1\wedge\gamma_2\wedge\eta_1\wedge\eta_2], \\ &\quad [\beta_2\wedge\gamma_1\wedge\gamma_2\wedge\eta_1\wedge\eta_2]\rangle, \\ H^6(N) &= \langle [\beta_1\wedge\beta_2\wedge\gamma_1\wedge\gamma_2\wedge\eta_1\wedge\eta_2]\rangle. \end{aligned}$$

We can give a more explicit description of the group G. As a differentiable manifold $G=\mathbb{R}^6$. The nilpotent Lie group structure of G is given by the multiplication law

$$\begin{aligned} m: \quad & G\times G \longrightarrow G \\ & ((y_1',y_2',z_1',z_2',v_1',v_2'),(y_1,y_2,z_1,z_2,v_1,v_2)) \mapsto \Big(y_1+y_1',y_2+y_2',z_1+z_1',z_2+z_2', \\ & \quad v_1+v_1'+(y_1'-y_2')z_1-(y_1'+2y_2')z_2, \\ & \quad v_2+v_2'-(2y_1'+y_2')z_1+(y_2'-y_1')z_2\Big). \end{aligned} \tag{2}$$

We also need a discrete subgroup, which it could be taken to be $\mathbb{Z}^6\subset G$. However, for later convenience, we shall take the subgroup

$$\Gamma=\{(y_1,y_2,z_1,z_2,v_1,v_2)\in\mathbb{Z}^6\,|\,v_1\equiv v_2 \pmod 3\}\subset G,$$

and define the nilmanifold

$$N=\Gamma\backslash G\,.$$

In terms of a (global) system of coordinates $(y_1,y_2,z_1,z_2,v_1,v_2)$ for G, the 1–forms β_i, γ_i and η_i, $1 \le i \le 2$, are given by

$$\begin{aligned} \beta_i &= dy_i, \quad 1 \le i \le 2, \\ \gamma_i &= dz_i, \quad 1 \le i \le 2, \\ \eta_1 &= dv_1 - y_1 dz_1 + y_2 dz_1 + y_1 dz_2 + 2y_2 dz_2, \\ \eta_2 &= dv_2 + 2y_1 dz_1 + y_2 dz_1 + y_1 dz_2 - y_2 dz_2. \end{aligned}$$

Note that N is a principal torus bundle

$$T^2 = \mathbb{Z}\langle(1,1),(3,0)\rangle\backslash\mathbb{R}^2 \hookrightarrow N \longrightarrow T^4 = \mathbb{Z}^4\backslash\mathbb{R}^4,$$

with the projection $(y_1,y_2,z_1,z_2,v_1,v_2) \mapsto (y_1,y_2,z_1,z_2)$.

The Lie group G can be also described as follows. Consider the basis $\{\mu_i, \nu_i, \theta_i; 1 \le i \le 2\}$ of the left invariant 1–forms on G given by

$$\begin{aligned} &\mu_1 = \beta_1 + \frac{1+\sqrt{3}}{2}\beta_2, \qquad \mu_2 = \beta_1 + \frac{1-\sqrt{3}}{2}\beta_2, \\ &\nu_1 = \gamma_1 + \frac{1+\sqrt{3}}{2}\gamma_2, \qquad \nu_2 = \gamma_1 + \frac{1-\sqrt{3}}{2}\gamma_2, \\ &\theta_1 = \frac{2}{\sqrt{3}}\eta_1 + \frac{1}{\sqrt{3}}\eta_2, \qquad \theta_2 = \eta_2. \end{aligned}$$

Hence, the structure equations can be rewritten as

$$\begin{aligned} &d\mu_i = 0, \quad 1 \le i \le 2, \\ &d\nu_i = 0, \quad 1 \le i \le 2, \\ &d\theta_1 = \mu_1 \wedge \nu_1 - \mu_2 \wedge \nu_2, \\ &d\theta_2 = \mu_1 \wedge \nu_2 + \mu_2 \wedge \nu_1. \end{aligned} \tag{3}$$

This means that G is the complex Heisenberg group $H_{\mathbb{C}}$, that is, the complex nilpotent Lie group of complex matrices of the form

$$\begin{pmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In fact, in terms of the natural (complex) coordinate functions (u_1,u_2,u_3) on $H_{\mathbb{C}}$, we have that the complex 1–forms

$$\mu = du_1, \ \nu = du_2, \ \theta = du_3 - u_2 du_1$$

are left invariant and $d\mu = d\nu = 0$, $d\theta = \mu \wedge \nu$. Now, it is enough to take $\mu_1 = \Re(\mu)$, $\mu_2 = \Im(\mu)$, $\nu_1 = \Re(\nu)$, $\nu_2 = \Im(\nu)$, $\theta_1 = \Re(\theta)$, $\theta_2 = \Im(\theta)$ to recover equations (3), where $\Re(\mu)$ and $\Im(\mu)$ denote the real and the imaginary parts of μ, respectively.

Lemma 1 *Let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi i/3}$, and consider the discrete subgroup $\Gamma_H \subset H_{\mathbb{C}}$ formed by the matrices in which $u_1,u_2,u_3 \in \Lambda$. Then there is a natural identification of $N = \Gamma\backslash G$ with the quotient $\Gamma_H\backslash H_{\mathbb{C}}$.*

Proof We have constructed above an isomorphism of Lie groups $G \to H_{\mathbb{C}}$, whose explicit equations are

$$(y_1, y_2, z_1, z_2, v_1, v_2) \mapsto (u_1, u_2, u_3),$$

where

$$\begin{aligned} u_1 &= \left(y_1 + \frac{1+\sqrt{3}}{2} y_2\right) + i\left(y_1 + \frac{1-\sqrt{3}}{2} y_2\right), \\ u_2 &= \left(z_1 + \frac{1+\sqrt{3}}{2} z_2\right) + i\left(z_1 + \frac{1-\sqrt{3}}{2} z_2\right), \\ u_3 &= \frac{1}{\sqrt{3}}\left(2v_1 + v_2 + 3z_1 y_2 + 3z_2 y_1 + 3z_2 y_2\right) + i\left(v_2 + 2z_1 y_1 + z_2 y_1 + z_1 y_2 - z_2 y_2\right). \end{aligned}$$

Note that the formula for u_3 can be deduced from

$$du_3 - u_2 du_1 = \theta = \left(\frac{2}{\sqrt{3}}\eta_1 + \frac{1}{\sqrt{3}}\eta_2\right) + i\eta_2 \ .$$

Now the group $\Gamma \subset G$ corresponds under this isomorphism to

$$\left\{(u_1, u_2, u_3) | u_1, u_2 \in \mathbb{Z}\left\langle 1+i, \frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2} i \right\rangle, u_3 \in \mathbb{Z}\left\langle 2\sqrt{3}, \sqrt{3}+i \right\rangle \right\}.$$

Using the isomorphism of Lie groups $H_{\mathbb{C}} \to H_{\mathbb{C}}$ given by

$$(u_1, u_2, u_3) \mapsto (u_1', u_2', u_3') = \left(\frac{u_1}{1+i}, \frac{u_2}{1+i}, \frac{u_3}{(1+i)^2}\right),$$

we get that $u_1', u_2', u_3' \in \Lambda = \mathbb{Z}\langle 1, \zeta \rangle$, which completes the proof. $\square$

Remark 2 *If we had considered the discrete subgroup $\mathbb{Z}^6 \subset G$ instead of $\Gamma \subset G$, then we would not have obtained the fact $u_3' \in \Lambda$ in the proof of Lemma 1. Note that $N = \Gamma \backslash G \twoheadrightarrow \mathbb{Z}^6 \backslash G$ is a $3:1$ covering.*

Under the identification $N = \Gamma \backslash G \cong \Gamma_H \backslash H_{\mathbb{C}}$, N becomes the principal torus bundle

$$T^2 = \Lambda \backslash \mathbb{C} \hookrightarrow N \longrightarrow T^4 = \Lambda^2 \backslash \mathbb{C}^2,$$

with the projection $(u_1, u_2, u_3) \mapsto (u_1, u_2)$.

A SYMPLECTIC ORBIFOLD OF DIMENSION 8

We define the 8–dimensional compact nilmanifold M as the product

$$M = T^2 \times N.$$

By Lemma 1 there is an isomorphism between M and the manifold $(\Gamma_H \backslash H_{\mathbb{C}}) \times (\Lambda \backslash \mathbb{C})$ studied in [6, Section 2] (we have to send the factor T^2 of M to the factor $\Lambda\backslash\mathbb{C}$). Clearly, M is a principal torus bundle

$$T^2 \hookrightarrow M \xrightarrow{\pi} T^6.$$

Let (x_1,x_2) be the Lie algebra coordinates for T^2, so that $(x_1,x_2,y_1,y_2,z_1,z_2,v_1,v_2)$ are coordinates for the Lie algebra $\mathbb{R}^2 \times G$ of M. Then $\pi(x_1,x_2,y_1,y_2,z_1,z_2,v_1,v_2) = (x_1,x_2,y_1,y_2,z_1,z_2)$. A basis for the left invariant (closed) 1–forms on T^2 is given as $\{\alpha_1,\alpha_2\}$, where $\alpha_1 = dx_1$ and $\alpha_2 = dx_2$. Then $\{\alpha_i,\beta_i,\gamma_i,\eta_i; 1 \le i \le 2\}$ constitutes a (global) basis for the left invariant 1–forms on M. Note that $\{\alpha_i,\beta_i,\gamma_i; 1 \le i \le 2\}$ is a basis for the left invariant closed 1–forms on the base T^6. (We use the same notation for the differential forms on T^6 and their pullbacks to M.) Using the computation of the cohomology of N, we get that the Betti numbers of M are: $b_0(M) = b_8(M) = 1$, $b_1(M) = b_7(M) = 6$, $b_2(M) = b_6(M) = 17$, $b_3(M) = b_5(M) = 30$, $b_4(M) = 36$. In particular, $\chi(M) = 0$, as for any nilmanifold.

Consider the action of the finite group $\mathbb{Z}_3$ on $\mathbb{R}^2$ given by

$$\rho(x_1,x_2) = (-x_1 - x_2, x_1),$$

for $(x_1,x_2) \in \mathbb{R}^2$, ρ being the generator of $\mathbb{Z}_3$. Clearly $\rho(\mathbb{Z}^2) = \mathbb{Z}^2$, and so ρ defines an action of $\mathbb{Z}_3$ on the 2-torus $T^2 = \mathbb{Z}^2\backslash\mathbb{R}^2$ with 3 fixed points: $(0,0)$, $(\frac{1}{3},\frac{1}{3})$ and $(\frac{2}{3},\frac{2}{3})$. The quotient space $T^2/\mathbb{Z}_3$ is the orbifold 2–sphere S^2 with 3 points of multiplicity 3. Let x_1, x_2 denote the natural coordinate functions on $\mathbb{R}^2$. Then the 1–forms dx_1, dx_2 satisfy $\rho^*(dx_1) = -dx_1 - dx_2$ and $\rho^*(dx_2) = dx_1$, hence $\rho^*(-dx_1 - dx_2) = dx_2$. Thus, we can take the 1–forms α_1 and α_2 on T^2 such that

$$\rho^*(\alpha_1) = -\alpha_1 - \alpha_2, \quad \rho^*(\alpha_2) = \alpha_1. \tag{4}$$

Define the following action of $\mathbb{Z}_3$ on M, given, at the level of Lie groups, by $\rho \colon \mathbb{R}^2 \times \mathbb{R}^6 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^6$,

$$\rho(x_1,x_2,y_1,y_2,z_1,z_2,v_1,v_2) = (-x_1-x_2,x_1,-y_1-y_2,y_1,-z_1-z_2,z_1,-v_1-v_2,v_1).$$

Note that $m(\rho(p'),\rho(p)) = \rho(m(p',p))$, for all $p,p' \in G$, where m is the multiplication map (2) for G. Also $\Gamma \subset G$ is stable by ρ since

$$v_1 \equiv v_2 \pmod 3 \Longrightarrow -v_1 - v_2 \equiv v_1 \pmod 3.$$

Therefore there is a induced map $\rho \colon M \to M$, and this covers the action $\rho : T^6 \to T^6$ on the 6–torus $T^6 = T^2 \times T^2 \times T^2$ (defined as the action ρ on each of the three factors simultaneously). The action of ρ on the fiber $T^2 = \mathbb{Z}\langle(1,1),(3,0)\rangle$ has also 3 fixed points: $(0,0)$, $(1,0)$ and $(2,0)$. Hence there are $3^4 = 81$ fixed points on M.

Remark 3 *Under the isomorphism $M \cong (\Gamma_H\backslash H_{\mathbb{C}}) \times (\Lambda\backslash\mathbb{C})$, we have that the action of ρ becomes $\rho(u_1,u_2,u_3) = (\bar{\zeta}u_1, \bar{\zeta}u_2, \zeta u_3)$, where $\zeta = e^{2\pi i/3}$. Composing the isomorphism of Lemma 1 with the conjugation $(u_1,u_2,u_3) \mapsto (v_1,v_2,v_3) = (\bar{u}_1,\bar{u}_2,\bar{u}_3)$ (which is an isomorphism of Lie groups $H_{\mathbb{C}} \to H_{\mathbb{C}}$ leaving Γ_H invariant), we have that the action of ρ becomes $\rho(v_1,v_2,v_3) = (\zeta v_1, \zeta v_2, \zeta^2 v_3)$. This is the action used in [6].*

We take the basis $\{\alpha_i, \beta_i, \gamma_i, \eta_i; 1 \leq i \leq 2\}$ of the 1–forms on M considered above. The 1–forms dy_i, dz_i, dv_i, $1 \leq i \leq 2$, on G satisfy the following conditions similar to (4): $\rho^*(dy_1) = -dy_1 - dy_2$, $\rho^*(dy_2) = dy_1$, $\rho^*(dz_1) = -dz_1 - dz_2$, $\rho^*(dz_2) = dz_1$, $\rho^*(dv_1) = -dv_1 - dv_2$, $\rho^*(dv_2) = dv_1$. So

$$\begin{array}{ll} \rho^*(\alpha_1) = -\alpha_1 - \alpha_2, & \rho^*(\alpha_2) = \alpha_1, \\ \rho^*(\beta_1) = -\beta_1 - \beta_2, & \rho^*(\beta_2) = \beta_1, \\ \rho^*(\gamma_1) = -\gamma_1 - \gamma_2, & \rho^*(\gamma_2) = \gamma_1, \\ \rho^*(\eta_1) = -\eta_1 - \eta_2, & \rho^*(\eta_2) = \eta_1. \end{array} \tag{5}$$

Remark 4 *If we define the* 1*–forms* $\alpha_3 = -\alpha_1 - \alpha_2$, $\beta_3 = -\beta_1 - \beta_2$, $\gamma_3 = -\gamma_1 - \gamma_2$ *and* $\eta_3 = -\eta_1 - \eta_2$*, then we have* $\rho^*(\alpha_1) = \alpha_3$, $\rho^*(\alpha_2) = \alpha_1$, $\rho^*(\alpha_3) = \alpha_2$*, and analogously for the others.*

Define the quotient space

$$\widehat{M} = M/\mathbb{Z}_3,$$

and denote by $\varphi : M \to \widehat{M}$ the projection. It is an orbifold, and it admits the structure of a symplectic orbifold (see [4] for a general discussion on symplectic orbifolds).

Proposition 5 *The* 2*–form* ω *on* M *defined by*

$$\omega = \alpha_1 \wedge \alpha_2 + \eta_2 \wedge \beta_1 - \eta_1 \wedge \beta_2 + \gamma_1 \wedge \gamma_2$$

is a $\mathbb{Z}_3$*-invariant symplectic form on* M*. Therefore it induces* $\widehat{\omega} \in \Omega^2_{\text{orb}}(\widehat{M})$*, such that* $(\widehat{M}, \widehat{\omega})$ *is a symplectic orbifold.*

Proof Clearly $\omega^4 \neq 0$. Using (5) we have that $\rho^*(\omega) = (-\alpha_1 - \alpha_2) \wedge \alpha_1 + \eta_1 \wedge (-\beta_1 - \beta_2) + (\eta_1 + \eta_2) \wedge \beta_1 + (-\gamma_1 - \gamma_2) \wedge \gamma_1 = \omega$, so ω is $\mathbb{Z}_3$-invariant. Finally,

$$d\omega = d\eta_2 \wedge \beta_1 - d\eta_1 \wedge \beta_2 = (\beta_2 \wedge \gamma_1 - \beta_2 \wedge \gamma_2) \wedge \beta_1 - (-\beta_1 \wedge \gamma_1 + \beta_1 \wedge \gamma_2) \wedge \beta_2 = 0.$$

□

It can be seen (cf. proof of Proposition 2.3 in [6]) that $\widehat{M}$ is simply connected. Moreover, its cohomology can be computed using that

$$H^*(\widehat{M}) = H^*(M)^{\mathbb{Z}_3}.$$

We get

$$\begin{array}{rcl} H^1(\widehat{M}) & = & 0, \\ H^2(\widehat{M}) & = & \langle [\alpha_1 \wedge \alpha_2], [\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1], [\alpha_1 \wedge \beta_1 + \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_2], \\ & & [\alpha_1 \wedge \gamma_2 - \alpha_2 \wedge \gamma_1], [\alpha_1 \wedge \gamma_1 + \alpha_1 \wedge \gamma_2 + \alpha_2 \wedge \gamma_2], [\beta_1 \wedge \beta_2], [\beta_1 \wedge \gamma_2 - \beta_2 \wedge \gamma_1], \\ & & [\beta_1 \wedge \gamma_1 + \beta_1 \wedge \gamma_2 + \beta_2 \wedge \gamma_2], [\beta_1 \wedge \eta_2 - \beta_2 \wedge \eta_1], [\beta_1 \wedge \eta_1 + \beta_1 \wedge \eta_2 + \beta_2 \wedge \eta_2], \\ & & [\gamma_1 \wedge \gamma_2], [\gamma_1 \wedge \eta_2 - \gamma_2 \wedge \eta_1], [\gamma_1 \wedge \eta_1 + \gamma_1 \wedge \eta_2 + \gamma_2 \wedge \eta_2] \rangle, \\ H^3(\widehat{M}) & = & 0. \end{array}$$

Remark 6 *The Euler characteristic of $\widehat{M}$ can be computed via the formula for finite group action quotients: let Π be the cyclic group of order n, acting on a space X almost freely. Then*

$$\chi(X/\Pi) = \frac{1}{n}\chi(X) + \sum_p \left(1 - \frac{1}{\#\Pi_p}\right),$$

where $\Pi_p \subset \Pi$ is the isotropy group of $p \in X$. In our case $\chi(\widehat{M}) = \frac{1}{3}\chi(M) + 81(1-\frac{1}{3}) =$ 54.

Using this remark and the previous calculation, we get that $b_1(\widehat{M}) = b_7(\widehat{M}) = 0$, $b_2(\widehat{M}) = b_6(\widehat{M}) = 13$, $b_3(\widehat{M}) = b_5(\widehat{M}) = 0$ and $b_4(\widehat{M}) = 26$. Note that $\widehat{M}$ satisfies Poincaré duality since $H^*(\widehat{M}) = H^*(M)^{\mathbb{Z}_3}$ and $H^*(M)$ satisfies Poincaré duality.

NON-FORMALITY OF THE SYMPLECTIC ORBIFOLD

Formality is a property of the rational homotopy type of a space which is of great importance in symplectic geometry. This is due to the fact that compact Kähler manifolds are formal [5] whilst there are compact symplectic manifolds which are non-formal [11, 3, 6]. A general discussion of the property of formality can be found in [11].

The non-formality of a space can be detected by means of Massey products. Let us recall its definition. The simplest type of Massey product is the triple (also known as ordinary) Massey product. Let X be a smooth manifold and let $a_i \in H^{p_i}(X)$, $1 \le i \le 3$, be three cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. The (triple) Massey product of the classes a_i is defined as the set

$$\langle a_1, a_2, a_3 \rangle = \{[\alpha_1 \wedge \eta + (-1)^{p_1+1}\xi \wedge \alpha_3] \mid a_i = [\alpha_i],\ \alpha_1 \wedge \alpha_2 = d\xi,\ \alpha_2 \wedge \alpha_3 = d\eta\}$$

inside $H^{p_1+p_2+p_3-1}(X)$. We say that $\langle a_1, a_2, a_3 \rangle$ is trivial if $0 \in \langle a_1, a_2, a_3 \rangle$.

The definition of higher Massey products is as follows (see [8, 11]). The Massey product $\langle a_1, a_2, \dots, a_t \rangle$, $a_i \in H^{p_i}(X)$, $1 \le i \le t$, $t \ge 3$, is defined if there are differential forms $\alpha_{i,j}$ on X, with $1 \le i \le j \le t$, except for the case $(i,j) = (1,t)$, such that

$$a_i = [\alpha_{i,i}], \qquad d\,\alpha_{i,j} = \sum_{k=i}^{j-1} \bar{\alpha}_{i,k} \wedge \alpha_{k+1,j}, \tag{6}$$

where $\bar{\alpha} = (-1)^{\deg(\alpha)}\alpha$. Then the Massey product is

$$\langle a_1, a_2, \dots, a_t \rangle = \left\{ \left[\sum_{k=1}^{t-1} \bar{\alpha}_{1,k} \wedge \alpha_{k+1,t} \right] \mid \alpha_{i,j} \text{ as in (6)} \right\} \subset H^{p_1+\cdots+p_t-(t-2)}(X).$$

We say that the Massey product is trivial if $0 \in \langle a_1, a_2, \dots, a_t \rangle$. Note that for $\langle a_1, a_2, \dots, a_t \rangle$ to be defined it is necessary that $\langle a_1, \dots, a_{t-1} \rangle$ and $\langle a_2, \dots, a_t \rangle$ are defined and trivial.

The existence of a non-trivial Massey product is an obstruction to formality, namely, if X has a non-trivial Massey product then X is non-formal.

In the case of an orbifold, Massey products are defined analogously but taking the forms to be *orbifold forms* (see [4, Section 2]).

Now we want to prove the non-formality of the orbifold $\widehat{M}$ constructed in the previous section. By the results of [11], M is non-formal since it is a nilmanifold which is not a torus. We shall see that this property is inherited by the quotient space $\widehat{M} = M/\mathbb{Z}_3$. For this, we study the Massey products on $\widehat{M}$.

Lemma 7 *$\widehat{M}$ has a non-trivial Massey product if and only if M has a non-trivial Massey product with all cohomology classes $a_i \in H^*(M)$ being $\mathbb{Z}_3$-invariant cohomology classes.*

Proof We shall do the case of triple Massey products, since the general case is similar. Suppose that $\langle a_1, a_2, a_3 \rangle$, $a_i \in H^{p_i}(\widehat{M})$, $1 \leq i \leq 3$ is a non-trivial Massey product on $\widehat{M}$. Let $a_i = [\alpha_i]$, where $\alpha_i \in \Omega^*_{\mathrm{orb}}(\widehat{M})$. We pull-back the cohomology classes α_i via $\varphi^* : \Omega^*_{\mathrm{orb}}(\widehat{M}) \to \Omega^*(M)$ to get a Massey product $\langle [\varphi^*\alpha_1], [\varphi^*\alpha_2], [\varphi^*\alpha_3] \rangle$. Suppose that this is trivial on M, then $\varphi^*\alpha_1 \wedge \varphi^*\alpha_2 = d\xi$, $\varphi^*\alpha_2 \wedge \varphi^*\alpha_3 = d\eta$, with $\xi, \eta \in \Omega^*(M)$, and $\varphi^*\alpha_1 \wedge \eta + (-1)^{p_1+1}\xi \wedge \varphi^*\alpha_3 = df$. Then $\tilde{\eta} = (\eta + \rho^*\eta + (\rho^*)^2\eta)/3$, $\tilde{\xi} = (\xi + \rho^*\xi + (\rho^*)^2\xi)/3$ and $\tilde{f} = (f + \rho^*\eta + (\rho^*)^2\eta)/3$ are $\mathbb{Z}_3$-invariant and $\varphi^*\alpha_1 \wedge \tilde{\eta} + (-1)^{p_1+1}\tilde{\xi} \wedge \varphi^*\alpha_3 = d\tilde{f}$. Writing $\tilde{\eta} = \phi^*\hat{\eta}$, $\tilde{\xi} = \phi^*\hat{\xi}$, $\tilde{f} = \phi^*\hat{f}$, for $\hat{\eta}, \hat{\xi}, \hat{f} \in \Omega^*_{\mathrm{orb}}(\widehat{M})$, we get $\alpha_1 \wedge \hat{\eta} + (-1)^{p_1+1}\hat{\xi} \wedge \alpha_3 = d\hat{f}$, contradicting that $\langle a_1, a_2, a_3 \rangle$ is non-trivial.

Conversely, suppose that $\langle a_1, a_2, a_3 \rangle$, $a_i \in H^{p_i}(M)^{\mathbb{Z}_3}$, $1 \leq i \leq 3$, is a non-trivial Massey product on M. Then we can represent $a_i = [\alpha_i]$ by $\mathbb{Z}_3$-invariant differential forms $\alpha_i \in \Omega^{p_i}(M)$. Let $\hat{\alpha}_i$ be the induced form on $\widehat{M}$. Then $\langle [\hat{\alpha}_1], [\hat{\alpha}_2], [\hat{\alpha}_3] \rangle$ is a non-trivial Massey product on $\widehat{M}$. For if it were trivial then pulling-back by φ, we would get $0 \in \langle \varphi^*[\hat{\alpha}_1], \varphi^*[\hat{\alpha}_2], \varphi^*[\hat{\alpha}_3] \rangle = \langle a_1, a_2, a_3 \rangle$. □

In our case, all the triple and quintuple Massey products on $\widehat{M}$ are trivial. For instance, for a Massey product of the form $\langle a_1, a_2, a_3 \rangle$, all a_i should have even degree, since $H^1(\widehat{M}) = H^3(\widehat{M}) = H^5(\widehat{M}) = H^7(\widehat{M}) = 0$. Therefore the degree of the cohomology classes in $\langle a_1, a_2, a_3 \rangle$ is odd, hence they are zero.

Since the dimension of $\widehat{M}$ is 8, there is no room for sextuple Massey products or higher, since the degree of $\langle a_1, a_2, \ldots, a_s \rangle$ is at least $s+2$, as $\deg a_i \geq 2$. For $s = 6$, a sextuple Massey product of cohomology classes of degree 2 would live in the top degree cohomology. For computing an element of $\langle a_1, \ldots, a_6 \rangle$, we have to choose $\alpha_{i,j}$ in (6). But then adding a closed form ϕ with $a_1 \cup [\phi] = \lambda[\widehat{M}] \in H^8(\widehat{M})$ to $\alpha_{2,6}$ we can get another element of $\langle a_1, \ldots, a_6 \rangle$ which is the previous one plus $\lambda[\widehat{M}]$. For suitable λ the we get $0 \in \langle a_1, \ldots, a_6 \rangle$.

The only possibility for checking the non-formality of $\widehat{M}$ via Massey products is to get a non-trivial quadruple Massey product.

From now on, we will denote by the same symbol a $\mathbb{Z}_3$-invariant form on M and that induced on $\widehat{M}$. Notice that the 2 forms $\gamma_1 \wedge \gamma_2$, $\beta_1 \wedge \beta_2$ and $\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2$ are $\mathbb{Z}_3$-invariant forms on M, hence they descend to the quotient $\widehat{M} = M/\mathbb{Z}_3$. We have the following:

Proposition 8 *The quadruple Massey product*

$$\langle[\gamma_1\wedge\gamma_2],[\beta_1\wedge\beta_2],[\beta_1\wedge\beta_2],[\alpha_1\wedge\gamma_1+\alpha_2\wedge\gamma_1+\alpha_2\wedge\gamma_2]\rangle$$

is non-trivial on $\widehat{M}$. Therefore, the space $\widehat{M}$ is non-formal.

Proof First we see that

$$\begin{aligned}(\gamma_1\wedge\gamma_2)\wedge(\beta_1\wedge\beta_2) &= d\xi,\\ (\beta_1\wedge\beta_2)\wedge(\alpha_1\wedge\gamma_1+\alpha_2\wedge\gamma_1+\alpha_2\wedge\gamma_2) &= d\varsigma,\end{aligned}$$

where ξ and ς are the differential 3–forms on $\widehat{M}$ given by

$$\begin{aligned}\xi &= -\frac{1}{6}\left(\gamma_1\wedge(\beta_1\wedge\eta_2+\beta_2\wedge\eta_2+\beta_2\wedge\eta_1)+\gamma_2\wedge(\beta_1\wedge\eta_2+\beta_1\wedge\eta_1+\beta_2\wedge\eta_1)\right),\\ \varsigma &= \frac{1}{3}\left(-\alpha_1\wedge(\eta_2\wedge\beta_1+\eta_1\wedge\beta_1+\eta_1\wedge\beta_2)+\alpha_2\wedge(\eta_2\wedge\beta_2-\eta_1\wedge\beta_1)\right).\end{aligned}$$

Therefore, the triple Massey products $\langle[\gamma_1\wedge\gamma_2],[\beta_1\wedge\beta_2],[\beta_1\wedge\beta_2]\rangle$ and $\langle[\beta_1\wedge\beta_2],[\beta_1\wedge\beta_2],[\alpha_1\wedge\gamma_1+\alpha_2\wedge\gamma_1+\alpha_2\wedge\gamma_2]\rangle$ are defined, and they are trivial because all the (triple) Massey products on $\widehat{M}$ are trivial. (Notice that the forms ξ and ς are $\mathbb{Z}_3$-invariant on M and so descend to $\widehat{M}$.) Therefore, the quadruple Massey product $\langle[\gamma_1\wedge\gamma_2],[\beta_1\wedge\beta_2],[\beta_1\wedge\beta_2],[\alpha_1\wedge\gamma_1+\alpha_2\wedge\gamma_1+\alpha_2\wedge\gamma_2]\rangle$ is defined on $\widehat{M}$. Moreover, it is trivial on $\widehat{M}$ if and only if there are differential forms $f_i\in\Omega^3(\widehat{M})$, $1\leq i\leq 3$, and $g_j\in\Omega^4(\widehat{M})$, $1\leq j\leq 2$, such that

$$\begin{aligned}&(\gamma_1\wedge\gamma_2)\wedge(\beta_1\wedge\beta_2)=d(\xi+f_1),\\ &(\beta_1\wedge\beta_2)\wedge(\beta_1\wedge\beta_2)=d\,f_2,\\ &(\beta_1\wedge\beta_2)\wedge(\alpha_1\wedge\gamma_1+\alpha_2\wedge\gamma_1+\alpha_2\wedge\gamma_2)=d(\varsigma+f_3),\\ &(\gamma_1\wedge\gamma_2)\wedge f_2-(\xi+f_1)\wedge(\beta_1\wedge\beta_2)=dg_1,\\ &(\beta_1\wedge\beta_2)\wedge(\varsigma+f_3)-f_2\wedge(\alpha_1\wedge\gamma_1+\alpha_2\wedge\gamma_1+\alpha_2\wedge\gamma_2)=dg_2,\end{aligned}$$

and the 6–form given by

$$\Psi=-(\gamma_1\wedge\gamma_2)\wedge g_2-g_1\wedge(\alpha_1\wedge\gamma_1+\alpha_2\wedge\gamma_1+\alpha_2\wedge\gamma_2)+(\xi+f_1)\wedge(\varsigma+f_3)$$

defines the zero class in $H^6(\widehat{M})$. Clearly f_1, f_2 and f_3 are closed 3–forms. Since $H^3(\widehat{M})=0$, we can write $f_1=d\,f_1'$, $f_2=d\,f_2'$ and $f_3=d\,f_3'$ for some differential 2–forms f_1', f_2' and $f_3'\in\Omega^2(\widehat{M})$. Now, multiplying $[\Psi]$ by the cohomology class $[\sigma]\in H^2(\widehat{M})$, where $\sigma=2\alpha_1\wedge\gamma_2-\alpha_2\wedge\gamma_1+\alpha_1\wedge\gamma_1+\alpha_2\wedge\gamma_2$ we get

$$\sigma\wedge\Psi=-\frac{1}{3}(\alpha_1\wedge\alpha_2\wedge\beta_1\wedge\beta_2\wedge\gamma_1\wedge\gamma_2\wedge\eta_1\wedge\eta_2)+d(\sigma\wedge\xi\wedge f_3'+\sigma\wedge\varsigma\wedge f_1'+\sigma\wedge f_1'\wedge d\,f_3').$$

Hence, $[2\alpha_1\wedge\gamma_2-\alpha_2\wedge\gamma_1+\alpha_1\wedge\gamma_1+\alpha_2\wedge\gamma_2]\cup[\Psi]\neq 0$, which implies that $[\Psi]$ is non-zero in $H^6(\widehat{M})$. This proves that the Massey product $\langle[\gamma_1\wedge\gamma_2],[\beta_1\wedge\beta_2],[\beta_1\wedge\beta_2],[\alpha_1\wedge\gamma_1+\alpha_2\wedge\gamma_1+\alpha_2\wedge\gamma_2]\rangle$ is non-trivial, and so $\widehat{M}$ is non-formal. □

Finally, there is a way to desingularize $(\widehat{M},\widehat{\omega})$ to get a smooth symplectic manifold.

Theorem 9 *There is a smooth compact symplectic* 8*-manifold* $(\widetilde{M},\widetilde{\omega})$ *which is simply-connected and non-formal.*

Proof By [4, Theorem 3.3], there is a symplectic resolution $\pi:(\widetilde{M},\widetilde{\omega})\to(\widehat{M},\widehat{\omega})$, which consists of a smooth symplectic manifold $(\widetilde{M},\widetilde{\omega})$ and a map π which is a diffeomorphism outside the singular points.

To prove the non-formality of $\widetilde{M}$, we work as follows. All the forms of the proof of Proposition 8 can be defined on the resolution $\widetilde{M}$. Take a $\mathbb{Z}_3$-equivariant map $\psi: M\to M$ which is the identity outside small balls around the fixed points, and contracts smaller balls onto the fixed points. Substitute the forms ϑ, τ_i, κ, $\xi,\dots$ by $\psi^*\vartheta$, $\psi^*\tau_i$, $\psi^*\kappa$, $\psi^*\xi$, $\dots$ Then the corresponding elements in the quadruple Massey product are non-zero, but these forms are zero in a neighbourhood of the fixed points. Therefore they define forms on $\widetilde{M}$, by extending them by zero along the exceptional divisors $E_p=\pi^{-1}(p)$ ($p\in\widehat{M}$ singular point). Now the proof of Proposition 8 works for $\widetilde{M}$ with these forms.

Finally, the manifold $\widetilde{M}$ is simply connected as it is proved in [6, Proposition 2.3] (basically, this follows from the simply-connectivity of $\widehat{M}$). □

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