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Diangle groups

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Diangle groups should make you think of *triangle groups*. Let $\Gamma \subseteq PSL(2,\mathbf{R})$ be a discrete co-compact group with presentation

$$\Gamma \cong \langle \alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g,\gamma_1,\dots,\gamma_k \mid \prod[\alpha_i,\beta_i]=1, \prod \gamma_j = 1, \gamma_j^{m_j}=1\rangle.$$

Here, γ_i are elliptic elements, the other ones being hyperbolic. Such a group acts by fractional transformations on the upper half plane $\mathbf{H}$, and one can compute the hyperbolic volume of a fundamental domain

$$\mu(\Gamma) := \mathrm{vol}(\Gamma\backslash\mathbf{H}) = 2\pi(2g-2+\sum_{j=1}^{k}(1-\frac{1}{m_j}).$$

If S is a Riemann surface of genus $g \geq 2$, its universal covering space is $\mathbf{H}$, and S is isomorphic to a space of the form $\Gamma_S\backslash\mathbf{H}$ for such a group Γ_S without elliptic elements (its fundamental group). If N_S denotes the normalizer of Γ_S in $PSL(2,\mathbf{R})$, then the automorphism group of S is isomorphic to $N_S\backslash\Gamma_S$, so we have

$$|\mathrm{Aut}(S)| = |\frac{N_S}{\Gamma_S}| = \frac{\mu(\Gamma_S)}{\mu(N_S)} = \frac{4\pi(g-1)}{\mu(N_S)} \leq 84(g-1),$$

the latter since one can estimate that $\mu(\Gamma) \geq \frac{\pi}{21}$ for any such group Γ (in particular for $\Gamma = N_S$). This bound is known as Hurwitz's bound, and in order to maximize $|\mathrm{Aut}(S)|$, it now seems reasonable to minimize $\mu(N_S)$, so $g(N_S\backslash\mathbf{H})$ and k. As it turns out that $k \geq 3$ always holds, one arrives at *triangle groups*, which are defined to be groups Γ as above with $(g=0,\, k=3)$ as signature. By fixing a free normal subgroup Γ_S of Γ, they correspond to coverings of $\mathbf{P}^1$ ramified above three points with ramification indices (m_1, m_2, m_3) given by $\Gamma_S \backslash \mathbf{H} \to \Gamma \backslash \mathbf{H} = \mathbf{P}^1$. A favourite example of such a triangle group is the fundamental group of the modular curve $X(7)$ of genus 3, also known as Klein's quartic, which has $84(3-1) = 168$ automorphisms. The appetizer for this talk is that, although $k \geq 3$ in characteristic zero, **diangle groups exist in positive characteristic** (i.e., signature $(g = 0, k = 2)$ groups).

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We will review a few known facts in the following table. Let k be an algebraically closed field. In the table, bounds on $|\mathrm{Aut}(X)|$ for a projective curve X over k of genus $g \geq 2$ will be given in terms of the data in the side bars.

	X= any curve	X = ordinary curve	X=Mumford curve
$\mathrm{char}(k) = 0$	$84(g-1)$ (Hurwitz)	×× ××	$12(g-1)$ (Herrlich '80)
$\mathrm{char}(k) = p > 0$	$16g^4 + 1$ exc. (Stichtenoth '73)	$84g(g-1)$ (S. Nakajima '87)	?? ??

Let us explain the ingredients of this table. For a curve X over a field k of positive characteristic $p > 0$, one defines the p-rank of X to be

$$\gamma = \dim_{\mathbf{F}_p} \mathrm{Jac}(X)[p].$$

It is always true that $\gamma \leq g$, and if equality holds, one says that X is ordinary (which is a Zariski dense property in the moduli space of all curves over k). For a general curve X, there is the bound of Stichtenoth – where "+1 exc." means that there is one exception to the bound (a Fermat curve) – but it turns out that curves with many automorphisms tend to have a low p-rank. This is already reflected in Nakajima's bound, which is quadratic in the genus. However, the main observation is that *there is no known family* $\{X_i\}_{i=1}^{\infty}$ *of curves of strictly increasing genus* $g(X_{i+1}) > g(X_i)$ *whose number of automorphisms* $|\mathrm{Aut}(X_i)|$ *attains a quadratic polynomial in* $g(X_i)$. Rather, only such families are known that attain a third degree bound in $\sqrt{g(X_i)}$.

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The proofs of Stichtenoth and Nakajima use the generalization of Hurwitz's formula connecting the genera of X and $\mathrm{Aut}(X)\backslash X$ via the ramification divisor, to which, in positive characteristic, also the higher groups in the ramification filtration contribute. Thus, their proofs have a different flavour than the one for Riemann surfaces mentioned above using hyperbolic volumes, which is more analytic. To do this kind of analytical proof over other fields, one need rigid analysis. Although over a *non-archimedean* valued field k (with residue class field $\bar{k}$), compact rigid-analytic and projective algebraic curves can still be identified, the analogue of the dualism to discrete group theory is of quite a different nature, as not all curves admit the same universal topological covering space (*e.g.*, curves having good reduction are analytically simply connected). But Mumford has shown that curves whose stable reduction is split multiplicative (*i.e.*, a union of rational curves intersecting in $\bar{k}$-rational points) are isomorphic to an analytic space of the form $\Gamma\backslash(\mathbf{P}^1_k - \mathcal{L}_\Gamma)$, where Γ is a discontinuous group in $PGL(2,k)$ with $\mathcal{L}_\Gamma$ as set of limit points. Thus, the theory of non-archimedean discrete groups is both more restrictive than the complex analytic one (as it cannot be applied to any curve), and more powerful, as it can lead to stronger results for such so-called Mumford curves. This is already apparent in the result of Herrlich for p-adic curves (one has to assume $\mathrm{char}(\bar{k}) \geq 7$ in the above table).

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It turns out that, in positive characteristic, Mumford curves are ordinary (basically, because the Jacobian is uniformzable by $(k^*)^g/\Gamma^{\mathrm{ab}}$). The question remains what should be in place of the question marks in the above table, and our main result answers this question:

Theorem ([CoKaCo]). *Let X be a Mumford curve of genus $g \geq 2$ over a non-archimedean valued field of characteristic $p > 0$. Then*

$$|\mathrm{Aut}(X)| \leq \max\{12(g-1), F(g) := 2\sqrt{g}(\sqrt{g}+1)^2\}.$$

Let us first make a few remarks about this theorem. We use the notation $\mathbf{Z}_n, D_n$ for the cyclic and dihedral group of order n and $2n$, respectively.

• If Γ is the Schottky group of X, then $\mathrm{Aut}(X) = N/\Gamma$, where N is the normalizer of Γ in $PGL(2,k)$.

• We need the "max" in the theorem. Although $12(g-1) > F(g)$ only for $g \in \{5,6,7,8\}$, there does exists, e.g., a 1-dimensional family of icosahedral

Mumford curves of genus 6 (so $|\mathrm{Aut}(X)| = |A_5| = 12(6-1)$). These occur, e.g., for the normalizer $N = A_5 *_{\mathbf{Z}_5} D_5$.

• The bound is sharp. It is attained by the so-called *Artin-Schreier-Mumford* curves whose affine equation is given by

$$X_{t,c} \ : \ (x^{p^t} - x)(y^{p^t} - y) = c$$

for an integer t and $c \in k^*$ with $|c| < 1$.[1] The reduction of this curve is a "chess board" with $(p^t-1)^2$ holes, so the curve is Mumford and its genus is $g_t := (p^t-1)^2$. Its automorphism group is isomorphic to $A_t := \mathbf{Z}_p^{2t} \rtimes D_{p^t-1}$. The normalizer of its Schottky group is isomorphic to

$$N_t = (\mathbf{Z}_p^t \rtimes \mathbf{Z}_{p^t-1}) *_{\mathbf{Z}_{p^t-1}} D_{p^t-1}.$$

The Schottky group of $X_{t,c}$ is generated by the commutators $[\varepsilon, \gamma\varepsilon\gamma]$, for all $\varepsilon \in \mathbf{Z}_p^t$, where γ is a fixed involution in $D_{p^t-1} - \mathbf{Z}_{p^t-1}$. Note that $X_{t,c} \to A_{t,c}\backslash X_{t,c} = \mathbf{P}^1$ is ramified above three points in $p \geq 3$ and above two points if $p = 2$, so here is our first *diangle group*!

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Let us now say something about the proof, which actually yields more: it provides a kind of classification of those curves X for which $|\mathrm{Aut}(X)| > 12(g-1)$. There are three steps. The first one is to observe that, by Hurwitz's techniques, the bound is O.K. unless if $X \to \mathrm{Aut}(X)\backslash X$ is a cover of $\mathbf{P}^1$ ramified above ≤ 3 points (just one branch point cannot occur, since we assume the curve is ordinary). In the second step, all such covers are classified into families according to the abstract structure of the corresponding N. In the third step, the bounds are established in each of these cases using a mixture of combinatorial group theory and algebraic geometry.

In the second step, the structure of N can be studied by its action on the Bruhat-Tits tree $\mathcal{T}$ of $PGL(2,k)$. Although this principle of proof was also used by Herrlich in the p-adic case, the quintessence of our techniques is rather different. In the p-adic case the expected bound is linear in the genus, and this allows Herrlich to restrict to normalizers that are the amalgam of

[1] T. Sekiguchi asked during the talk whether the Artin-Schreier-Mumford curves are the unique curves that attain the (numerical) bound of the theorem. It turns out that this is correct – at least when $g \notin \{5,6,7,8\}$ – as will be shown in forthcomming work of Cornelissen and Kato.

two finite groups. In positive characteristic, the expected bound is not linear in the genus, so we have to consider more complicated normalizers. We therefore investigate directly the link between the ramification in $\pi : X \to S := \mathrm{Aut}(X)\backslash X$ and the combinatorial geometry of the analytic reduction of S.

Let $\mathcal{T}_N$ (respectively $\mathcal{T}_N^*$) be the subtree of $\mathcal{T}$ which is generated by the limit points of Γ, seen as ends of $\mathcal{T}$ (respectively the limit points together with the fixed points of torsion elements in N). The quotients $T_N = N\backslash\mathcal{T}_N$ and $T_N^* = N\backslash\mathcal{T}_N^*$ are homotopic to the intersection graph of the analytic reduction of S as a rigid analytic space, but T_N^* has finitely many ends attached to it, which are in bijection with the points above which π is branched. The advantage of using T_N^* instead of the usual T_N lies in the following key proposition, which replaces the "restriction to an amalgam of two groups" in the p-adic case: *If T is a subtree of T_N^* having the same ends as T_N^*, then T is a contraction of T_N^**, i.e., *every geodesic connecting a point from $T_N^* - T$ to T is a path on which the stabilizers of vertices are ordered increasingly w.r.t. inclusion in the direction of T. In particular, the amalgams associated to T and T_N^* are the same* $(= N)$. The proof of this result is very combinatorial and depends on the structure theorem for finite subgroups of $PGL(2,k)$. It is then enough to consider a simpler subtree T of $\mathcal{T}_N^*$ which is a "line" if $m = 2$ and a "star" if $m = 3$. We then show that only finitely many types of such trees T (and hence, of such groups N) exist, and this gives the classification of possible N. The analogue of hyperbolic volume in the case of a tree such as T_N is given by

$$\mu(T_N) = \sum_{[vw]\in E(T_N)} \frac{1}{|N_{vw}|} - \sum_{v\in V(T_N)} \frac{1}{|N_v|},$$

where E, V are the sets of edges and vertices of T_N, respectively, and N_x is the stabilizer of x (be it an edge or vertex) for the action of N. The main theorem is equivalent to

$$\mu(T_N) \geq \min\{\frac{1}{12}, \frac{\sqrt{g}-1}{2\sqrt{g}(\sqrt{g}+1)}\}.$$

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At the end of this talk, I want to finish by showing a typical diangle group! It corresponds to the normalizer

$$N = PGL(2,p^t) *_{\mathbf{Z}_p^t \rtimes \mathbf{Z}_{p^t-1}} \mathbf{Z}_p^{td} \rtimes \mathbf{Z}_{p^t-1},$$

which is one of the seven families of "diangle" N's. One can wonder what orbifold this is the normalizer of. It turns out to be an arithmetical object. Let $q = p^t$, $F = \mathbf{F}_q(T)$, and $A = \mathbf{F}_q[T]$; let $F_\infty = \mathbf{F}_q((T^{-1}))$ be the completion of F and C a completion of the algebraic closure of F_∞. On Drinfeld's "upper half plane" $\Omega := \mathbf{P}^1_C - \mathbf{P}^1_{F_\infty}$ (which is a rigid analytic space over C), the group $GL(2, A)$ acts by fractional transformations. Let $\mathcal{Z} \cong \mathbf{F}_q^*$ be its center. For $\mathfrak{n} \in A$, the quotients of Ω by congruence subgroups $\Gamma(\mathfrak{n}) = \{\gamma \in GL(2, A) : \gamma = \mathbf{1} \bmod \mathfrak{n}\}$ are open analytic curves which can be compactified to projective curves $X(\mathfrak{n})$ by adding finitely many cusps. These are moduli schemes for rank two Drinfeld modules with principal level structure. They turn out to be Mumford curves for the free group Γ which splits the inclusion $\Gamma(\mathfrak{n})_{\mathrm{tor}} \lhd \Gamma(\mathfrak{n})$, where $\Gamma(\mathfrak{n})_{\mathrm{tor}}$ is the subgroup generated by torsion elements. It turns out that *the normalizer of* Γ *is* N, *and for* $p \neq 2$ *or* $q \neq 3$, *the automorphism group of the Drinfeld modular curves* $X(\mathfrak{n})$ *is the "modular" automorphism group* $G(\mathfrak{n}) := \Gamma(1)/\Gamma(\mathfrak{n})\mathcal{Z}$. That these are genuine diangle groups can be seen from the fact that the T_N has only two ends, which are stabilized by $\mathbf{Z}_{q+1}$ and $\mathbf{Z}_p^{td} \rtimes \mathbf{Z}_{p^t-1}$, respectively. These correspond exactly to the ramification groups in $X(\mathfrak{n}) \to X(1) = \mathbf{P}^1$.

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Based on the above material, one can conjecture that for any *ordinary* curve of genus $g \geq 2$ over a field k of positive characteristic, the following bound should hold

$$|\mathrm{Aut}(X)| \leq \max\{84(g-1), F(g) := 2\sqrt{g}(\sqrt{g}+1)^2\}.$$

(notice the factor 84!). Can one prove such a result by a kind of deformation of an ordinary curve with many automorphisms to a totally degenerate curve?[2]

[2] At least, it is possible to compute the dimension of the first order equivariant deformation space of Mumford curves and of ordinary curves, as will be shown in forthcomming work of Cornelissen and Kato.