Non-archimedean Geometry

Symmetries of equations. A popular viewpoint on mathematics is that its mechanism of formalization allows one to reduce many a problem to "solving equations". An example which at the same time displays a sufficiently rich structure is given by a (homogeneous) polynomial equation of three variables, e.g.,

$$x^3y + y^3z + z^3x = 0. \tag{1}$$

Typically, one applies in such a case the process of "geometric visualization": one looks at the real solutions (x, y, z) to the equation by plotting the curve of solutions $(x/z, y/z)$ in the plane (Fig. 1). One can do even better by visualizing the complex solutions in the complex (projective) plane – the result is a real connected surface S, called a Riemann surface (not to be confused with the above picture; up to deformation, it looks like Fig. 2 for equation (1)).
The first thing which catches the eye is that certain of these pictures reveal a symmetry, e.g., there are movements of the ambient space which leave the picture invariant. With this consideration, genuine mathematics enters the stage: the questions shift to finding a theory about the structure of equations and their Riemann surfaces.
The set of symmetries of a Riemann surface form a mathematical object called the *automorphism group* of the equation (it is a "group" since symmetries can be composed in a nice way). One then typically asks what these groups can be, or: how many elements they can have. Formulated as such, the question is ill-posed, since it turns out that for any number n, there is a Riemann surface with n automorphisms, and the sphere even has infinitely many. However, A. Hurwitz discovered in 1893 that a Riemann surface S has at most $84(g-1)$ automorphisms if $g > 1$. Here, the *genus* g is a number which can be read off from the picture of S immediately: it is the "number of holes on S". The curve (1) has genus $g = 3$ and exactly $84(3-1) = 168$ automorphisms. But this is more of an exception: in some well-defined sense, "most" curves have no automorphisms at all for $g > 2$ (if all curves of fixed genus are put into their "moduli space", those having automorphisms are almost always singular points of that space).

Uniformization. The above mentioned pictures are also the playground of two powerful analytical theories: that of local and global uniformization. Local uniformization is like making an atlas of the surface, choosing coordinates in any sufficiently small part of the surface to represent it in some "faithful" way. These charts can (and should) be chosen to be biholomorphic maps. As a typical example, projections from the north- and south-pole respectively produce plain charts of the sphere.
The real mystery, however, is the existence of global uniformization. Let us look at a completely different construction first: all real 2×2 matrices γ of unit determinant $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form a group $SL(2, \mathbf{R})$ that acts on the set H of complex numbers $z = a + bi$ with $b > 0$ by fractional transformations $z \mapsto \gamma \cdot z = \frac{az+b}{cz+d}$.

Now if Γ is a "nice" infinite subgroup of $SL(2, \mathbf{R})$, then we can form the quotient $\Gamma \backslash H$, identifying z with $\gamma \cdot z$ for any $\gamma \in \Gamma$. The result turns out to be a Riemann surface. What is more, for any equation with Riemann surface S, there is a corresponding group Γ such that $S = \Gamma \backslash H$. In the above example (1), a possible group is called $\Gamma(7)$; it consists of all matrices with a, b, c, d integers, such that $a-1, b, c$ and $d-1$ are divisible by 7 (though technically speaking, some points will be missing). Fig. 3 displays the 168 triangles in H (itself depicted as a disk), of which each point represents a distinct point of $\Gamma(7) \backslash H$, and each triangle corresponds to $SL(2, \mathbf{Z}) \backslash H$. From now on, denote this surface by its canonical name, $Y(7)$.
The existence of global uniformization lead Hermann Weyl to proclaim that "in dem Symbol des zweidimensionalen Nicht-Euclidischen Kristalls wird das Urbild der Riemannschen Flächen selbst, rein und befreit von allen Verdunklungen, erschaubar". The morale is that all the information about the equation lies hidden in a huge infinite subgroup Γ of $SL(2, \mathbf{R})$. In particular, it is possible to compute the automorphisms of S only knowing Γ. For the cognoscenti: the automorphism group is N/Γ, where N is the normalizer of Γ. In our example, it is the group $SL(2, \mathbf{Z})/\Gamma(7) = PSL(2, 7)$. Actually, since H carries a non-Euclidean metric (like space-time), it makes sense to look at the hyperbolic volume of $\Gamma \backslash H$. Hurwitz's theorem says exactly that $\pi/21$ is a lower bound for it.

Non-archimedean structures. Rational numbers, when expanded in powers of 10, have a "repeating decimal part"(like the 3 which repeats itself endlessly in $1/3 = 0.3333...$). If one allows any decimal part (not only repeating), one arrives at all real numbers (like $\pi = 3.1415926...$). K. Hensel realized that one might equally well expand rational numbers in powers of any fixed prime p, for example the current year

$$2000 = 2^4 + 2^6 + 2^7 + 2^8 + 2^9 + 2^10 = 2 + 2 \cdot 3^3 + 2 \cdot 3^5 + 2 \cdot 3^6 = 5^3 + 3 \cdot 5^4$$

in powers of 2, 3 or 5. Just like one gets real number by allowing any decimal part, one gets the so-called p-adic numbers by allowing infinitely many powers of p in such expansions. As a matter of fact, it is necessary for number theory to treat all these fields on an equal footing.
The nice thing about allowing p-adic numbers is that one has a kind of "analytic" or "metric" control of prime divisibility. Just like there is the usual "distance" $|x - y|$ between real numbers x and y, there is a metric on such p-adic numbers, namely, $|x|_p$ is the inverse of the least power of p occurring in the expansion of x (so $|2000|_2 = 2^{-4}, |2000|_3 = 1, |2000|_5 = 5^{-3}$). However, this metric exhibits a strange geometric behaviour. Whereas for real numbers x and y, $|x + y| \leq |x| + |y|$, here it holds true that

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}. \tag{2}$$

This looks perhaps less annoying if one realizes that anyone actually knows metrics with such properties: we measure how "large" a one-variable polynomial $F(X)$ is by its degree. The "measure" $|F| = 2^{\deg(F)}$ behaves as in (2): the degree of the sum of two polynomials F and G is the maximum of the degrees of F and G, unless if both are of the same degree and have inverse leading coefficients. Any "measure" $x \mapsto |x|$ satisfying property (2) (and some more common things) is called a *non-archimedean* metric.

Around 1970, people realized that it would be of considerable interest to have a geometric theory similar to the one sketched above for Riemann surfaces, but valid in the non-archimedean situation – in this way, classical phenomena in number theory could be accessed by geometrical or analytical means. That such a thing is possible is not so obvious, because the metric properties are quite different (e.g., all triangles are isosceles). But a successful theory was worked out by several people, using fresh ideas from algebraic geometry.

Both MPI-directors Yuri Manin and Gerd Faltings have contributed to this still active field of research and several visitors of the MPI specialize in it. During his 1999 visit to the MPI, Takeshi Saito constructed a theory of higher ramification groups using such non-archimedean analysis: these form a chain of easy-to-understand groups that describe geometrically possibly very complicated branching structures. During her 1998 stay at the MPI, Annette Werner used non-archimedean concepts in her study of local heights, i.e., the p-adic "nearness" of, say, sets of points on curves. At the same time, Jiandong Guo studied higher dimensional p-adic periodic functions (famous in the real case ever since Fourier's work on oscillations).

Non-archimedean uniformization. Around 1975, after J. Tate discovered local uniformization for "non-archimedean curves" (viz., the set of solutions to equation such as (1) in non-archimedean structures), D. Mumford realized that *not* all such spaces allow a nice global uniformization. The curves that do (called "Mumford curves") can be characterized as follows. The non-archimedean world has one feature which is absent in the "real" world, namely, there is a process of *reduction*. Typically, this means that a p-adic number is mapped to the p-term in its expansion (so 2000 reduces to 0 and 2 in the 2-adic and 3-adic world, respectively). When one reduces such a Mumford curve, then it becomes an arrangement of intersecting lines, say, in the plane, and it is this line arrangement that encodes all the information about the curve. To any Mumford curve, one can associate a "nice" matrix group, which acts on a "universal arrangement of lines", called the Bruhat-Tits tree (it plays the role of H).

Here is an example: consider the polynomial ring $R = \mathbf{F}_{41}[T]$ in the variable T with coefficients in the field $\mathbf{F}_{41}$, i.e., we add and multiply coefficients modulo 41 (so $23 + 40 = 22$, $23 \times 40 = 18$, and $a^{41} = a$ for any a). A typical curve of which one wants to study the solutions in R is the following:

$$(x^{41} - x)(y^{41} - y) = T. \tag{3}$$

The non-archimedean metric in this case is given by $|F| = 41^{\deg(F)}$ for $F \in R$ (as in the previous paragraph). After applying reduction (which maps T to zero) we arrive at a "chess-board" of lines as in Fig. 4. Observe that the number of holes in this picture is $g = (41 - 1)^2 = 1600$.

One can wonder again how many symmetries such a non-archimedean curve can have (in terms of its genus g). We see for example (3) that the whole picture is invariant under shifting the lines vertically and horizontally in a cyclic way, and turning the whole square around. These symmetries are given by $x \mapsto x + a, y \mapsto y + b$ for $a, b \in \mathbf{F}_{41}$ and interchanging x and y. There is one more symmetry given by $x \mapsto cx, y \mapsto c^{-1}y$ for non-zero c. In total, we find $41 \times 41 \times 2 \times 40 = 134480$ symmetries, which equals $2\sqrt{1600}(\sqrt{1600} + 1)^2$. The strange thing to notice is that this number exceeds Hurwitz's bound for Riemann surfaces, which would be only $84(g - 1) = 3360$.
Recent studies of these phenomena at the MPI by Gunther Cornelissen, Fumiharu Kato and Aristeides Kontogeorgis revealed that the picture of this example persists: *any Mumford curve of genus $g \geq 9$ over a non-archimedean field has at most $2\sqrt{g}(\sqrt{g}+1)^2$ automorphisms, and this bound is reached for any square g.*
Many curves and higher dimensional geometrical objects that are especially relevant to questions in number theory turn out to possess a global non-archimedean uniformization. One such family which is somewhat similar to $Y(7)$, called "Shimura varieties", was studied in detail at the MPI by Gerd Faltings and Thomas Haines. They gave general recipes to simplify the study of uniformization and reduction of such spaces. At the same time, Fumiharu Kato showed that one such variety is, strangly enough, a "fake plane". This means that it has all the reasonable topological characteristics of the ordinary plane, without being equal to it.
Another typical class of such curves are "Drinfeld modular curves", studied at the MPI by Gunther Cornelissen and Douglas Ulmer. Their construction is a mixture of $Y(7)$ and curve (3), taking "the best of both worlds". Their most amazing property is their symmetry: not only do they have many more than $84(g - 1)$ automorphisms (where g is their genus), but also does their reduction produce the best known (up to now) natural bounded concentrators. This means the following: if one draws a dot in the plane for every line in the arrangement of the reduction of such a curve, and connects any two dots for which the corresponding lines intersect, then the resulting "network" has optimal properties for the transmission of information (as is required, e.g., in switching networks – Fig. 5).
The above results and the techniques used to prove it are part of a much larger project to make non-archimedean methods as omnipresent in algebraic geometry as complex analysis has always been. The philosophy behind this seems to be that the primitive human act of "measurement" should be varied as much as possible, as each measure sheds a different light on the object of study. That the particular "discrete" aspects of non-archimedean theory pay off so well in applications, comes as no surprise in our networking age. (*G. Cornelissen*)