Poisson Geometry: Lecture 10-11-12

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Abstract

The topics of this lectures are symplectic groupoids, Poisson paths and Poisson homotopy, Poisson cohomology, linearization problems in Poisson geometry and variation of symplectic areas.

1 Symplectic Groupoids

Let M be a manifold and Σ be a groupoid over M, with target and source maps t and s. Recall that Σ is a symplectic groupoid if it is provided with a symplectic form $w \in \Omega^2(\Sigma)$, such that it is multiplicative. In general the notion of being multiplicative makes sense for forms of all degrees, for instance if $f \in \Omega^\circ(\Sigma) = C^\infty(\Sigma)$, it just means that

$$f(g_1 \cdot g_2) = f(g_1) + f(g_2) \ \ \forall g_1, g_2 \in \Sigma_2,$$

where

$$\Sigma_2 = \{(g_1, g_2) / \mathsf{s}(g_1) = \mathsf{t}(g_2)\};$$

and more generally if $\sigma \in \Omega^k(\sum)$, we call it multiplicative if

$$m^*\sigma = pr_1^*\sigma + pr_2^*\sigma,$$

as forms on Σ_2 where $pr_1, pr_2 : \Sigma_2 \to \Sigma$ are the natural projections and m denotes the multiplication in the groupoid. We cite some of the properties of the symplectic groupoid (Σ, w) that were demostrated the last lecture.

1. M is embbeded into Σ (viewing elements of M as units in Σ) as a Lagrangian submanifold. As a corollary of this fact,the condition $\dim \Sigma = 2 \dim M$ must be satisfied and it imposes a restriction.

2. The s-fibers are symplectic orthogonal to the t-fibers.

3. There is a canonical induced Poisson structure on M, which is uniquely determined by the condition of $t : \Sigma \to M$ being a Poisson map.

Remark 1.1 From the third property, it follows then that $\mathsf{t} : \Sigma \to M$ is a complete symplectic realization of M.

On the other hand, we have that every Poisson manifold (M, π) provides T^*M with a cannonical Lie algebroid structure, $\left(T^*M, [\cdot,\cdot]_\pi, \pi^{\#}\right)$. So if we let π be the Posisson bivector ensured by the third point above, we obtain

$$Lie\,(\Sigma) \cong \left(T^*M, [\cdot,\cdot]_\pi, \pi^{\#}\right),$$

as Lie algenbroids.

Two questions are natural here:

Question 1: Does any Poisson manifold (M, π) arise from a symplectic groupoid?

Question 2: Assuming that the Lie algebroid of (M, π) comes from a Lie groupoid; does (M, π) comes from a symplectic Lie groupoid?

The answer to the first question is negative, in general, because not all Poisson manifold admits a complete sympletic realization. Moreover, there are examples of Poisson manifolds such that $\left(T^*M, [\cdot,\cdot]_\pi\, \pi^{\#}\right)$ is not isomorphic to the Lie algebroid of a Lie groupoid.

Concerning to Question 2, we have a positive answer. Furthermore, one can show that the following three conditions are equivalent

- The Lie algebroid of (M, π) comes from a Lie groupoid.
- (M, π) comes from a Lie groupoid.
- (M, π) admits a complete symplectic realization.

Definition 1.2 *If one of the above three conditions holds then (M, π) is called integrable.*

One important question is how to construct the Lie groupoid out of the Poisson manifold, this is in connection with the concept of Poisson paths that we shall discuss in the following section. Another important point is the determination of multiplicative 2-forms on Lie groupoid, the next theorem characterizes them.

Theorem 1.3 *Let Σ be a Lie groupoid over M and assume that the s-fibers of Σ are connected and simply connected. Let $(A, [\cdot,\cdot]_A, \rho)$ be the Lie algebroid of Σ. Then, there is a one-to-one corresponde between the multiplicative closed 2-forms $w \in \Omega^2(\Sigma)$ and the bundle maps $\sigma : A \to T^*M$ satisfying:*

$$\langle \sigma(\alpha), \rho(\beta) \rangle = -\langle \sigma(\beta), \rho(\alpha) \rangle$$

and

$$\sigma\left([\alpha, \beta]_A\right) = \mathcal{L}_{\rho(\alpha)}\left(\sigma(\beta)\right) - \mathcal{L}_{\rho(\beta)}\left(\sigma(\alpha)\right) - d\left(\langle \sigma(\alpha), \rho(\beta) \rangle\right), \ \forall \alpha, \beta \in \Gamma(A).$$

Moreover the relation between w and σ is given by

$$\langle \sigma(\alpha), X \rangle = w\left(\tilde{\alpha}, \tilde{X}\right),$$

for $\alpha \in A_x$, $X \in T_xM$, where $\tilde{\alpha} \in T_{1_x}\Sigma$ and $\tilde{X} \in T_{1_x}\Sigma$.

We may apply the last theorem to the case when the Lie algebroid is the the associated Lie algebroid structure on the contangent bundle of a Poisson manifold. In such case we take σ above as the identity, consequently a multiplicative 2-form is obtained, this 2-form is symplectic, answering Question 2.

2 Poisson Paths and Poisson Homotopy

Let (M, π) be a Poisson manifold recall that a pair (γ, a), where $\gamma : [0,1] \to M$ and $a : [0,1] \to T^*M$ are paths such that a is above γ, is called a Poisson path if

$$\pi^{\#}(a(t)) = \frac{d\gamma}{dt}$$

The reader can think of a as a cotangent derivative of γ.

There is a concept of "Poisson homotopy," which we do not define here, but we spell out some of its properties and importance. First, we remark that using the Poisson paths and Poisson homotopy we can construct a groupoid

$$\Sigma(M) := \{Poisson\ paths\} / \{Poisson\ homotopy\}.$$

One has that $\Sigma(M)$ is indeed a groupoid with multiplication induced by concatenation of Poisson paths and source and target maps given by $\mathsf{s}([\gamma, a]) = \gamma(0)$ at $\mathsf{t}([\gamma, a]) = \gamma(1)$. The only thing that may fail in this construction is the smootheness.

Let E be a vector bundle over M and assume we have a contravariant Poisson connection ∇ on E. Then ∇ leads to parallel transport along Poisson paths in (M, π), given a Poisson path (γ, a) we have

$$T_a : E_{\gamma(0)} \to E_{\gamma(1)},$$

then, as in the classical case, if the connection is flat, T_a only depends on the Poisson homotopy of (γ, a).

Assume now that $\mu : S \to M$ is a complete symplectic realization, then as we have done before, we have an induced paprallel transport.

$$T_a : \mu^{-1}(x) \to \mu^{-1}(y),\ x = \gamma(0),\ y = \gamma(1),$$

therefore it only depends on the homotopy class of a.

Remark 2.1 If $\mu : S \to M$ is a complete symplectic realization of (M, π), then we have an action of T^*M on S

$$\Omega'(M) \times C^\infty(S) \to C^\infty(S)$$

$$(\alpha, f) \longmapsto X_{\mu^*(\alpha)}(f).$$

As a conclusion, we obtain an important canonical action of the groupoid $\Sigma(M)$ on the symplectic realization, which is induced by the parallel transport $g(\cdot) : \mu^{-1}(x) \to \mu^{-1}(y)$, for $g \in \Sigma(M)$.

Remark 2.2 What is important here is not that μ is a submersion, the only important property is the completeness. Then completeness is certainly guaranteed if we just have a proper Poisson map $\mu : S \to M$ (here again (S, w) is symplectic), so if this situation, the symplectic manifold S comes with a cononical action of $\Sigma(M)$ on S.

Remark 2.3 One clearly has for the case $M = \mathfrak{g}^*$, with the Poisson-Lie structure, that this will become a hamiltonian action of the connected simply connected group integrating $\mathfrak{g}^*$.

3 Poisson Cohomology

Recall that $H^*(M) = H^*_{dR}(M) = \ker d/\mathrm{im} d$ denotes the cohomology of the de Rham complex $(\Omega^*(M), d)$, where

$$\Omega^k(M) = \Gamma\left(\Lambda^k T^* M\right)$$

and

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

is given by

$$dw\left(X_1, \ldots, X_{k+1}\right) = \sum_{j=1}^{k+1} (-1)^{j-1} \mathcal{L}_{X_j} w\left(X_1, \ldots, X_{j-1}, \hat{X}_j, X_{j+1}, \ldots, X_{k+1}\right)$$

$$+ \sum_{i<j} (-1)^{i+j} w\left(\left[X_i, X_j\right], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}\right).$$

For the Poisson case, we replace TM by T^*M, $\mathfrak{X}(M)$ by $\Omega^1(M)$ and $[\cdot,\cdot]$ by $[\cdot,\cdot]_\pi$. So, we obtain the clain complex $(\mathfrak{X}^*(M), \delta)$, where

$$\mathfrak{X}^k(M) = \Gamma\left(\Lambda^k TM\right),$$

$$\delta : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M),$$

and

$$\delta(X)\left(w_1, \ldots, w_{k+1}\right) = \sum_{j=1}^{k+1} (-1)^{j-1} \mathcal{L}_{\pi^{\#}(w_i)}\left(X\left(w_1, \ldots, \hat{w}_j, \ldots, w_{k+1}\right)\right) +$$

$$\sum_{i<j} (-1)^{i+j} X\left(\left[w_i, w_j\right]_\pi, w_1, \ldots, \hat{w}_i, \ldots, \hat{w}_j, \ldots, w_{k+1}\right).$$

The resulting cohomology is called the Poisson cohomology of (M, π) and is denoted by $H^*_\pi(M)$. For instance for low degrees,

$$H^\circ_\pi(M) = \left\{f \in C^\infty(M)/f \text{ is constant on the symplectic leaves}\right\},$$

and

$$H^1_\pi(M) = \left\{X \in \mathfrak{X}(M) : X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}\right\} / \left\{Hamiltonias\ X_h/h \in C^\infty(M)\right\}.$$

Remark 3.1 If a is a Poisson path, and $[X] \in H'_\pi(M)$, then

$$\int_a X := \int_0^1 \left\langle a(t), X_{\gamma_a}(t)\right\rangle dt$$

only depends on $[X]$ and the Poisson homotopy class of a.

In general $H^2_\pi(M)$ contains $[\pi]$ as a non zero element. This cohomology group is especially important to the study of smooth deformations of Poisson structures on a fixed manifold M. Assume that π_t (or equivalently $\{\cdot,\cdot\}_t$) is a smooth family of Poisson structures on (M,π) such that $\pi_0=\pi$. Define

$$u(f,g)=\left(\frac{d}{dt}\{f,g\}_t\right)_{t=0}.$$

Since the $\{\cdot,\cdot\}_t$ satisfy the Jacobi identity, then u does as well. Now a very easy computation shows that Jacobi identity is equivalent to have $\delta(u)=0$, thus $[u]\in H^2_\pi(M)$.

If $\pi_t=\varphi_t^*(\pi)$, for some smooth family of diffeomorphisms, the deformation is called trivial deformation; in this case the associated $[u]\in H^2_\pi(M)$ is seen to be trivial. An open question is the following:

If $[u]=0$, does it follow that the deformation is trivial?

4 Linearization in Poisson Geometry

In this section, we discuss some problems on Linearization of a Poisson manifold (M,π) around singular points, i.e, those points $x\in M$ such that $\pi_x=0$. Take $\mathfrak{g}_x^*$, the isotropy Lie algebra of π at x, this is

$$\mathfrak{g}_x^*=\ker\left(\pi_x^\#\right)=T_x^*M,$$

with bracket given by

$$[(df)_x,(dg)_x]_{\mathfrak{g}_x^*}:=[df,dg]_\pi(x).$$

Theorem 4.1 *(J. Conn) If $\mathfrak{g}_{x_0}^*$ is semisimple of compact type, then (M,π) around x is Poisson diffeomorphic to $\mathfrak{g}_{x_0}^*$ with the linear Poisson structure.*

For this theorem and related topics see [2, 3]. The proof given by J.Conn is completely analytic. One geometric proof is in progress right now. Some of the arguments are as follows: One first proves a result about cohomology of Lie algebras

$$H^1(\mathfrak{g}_{x_0})=H^1\left(\mathfrak{g}_{x_0},\mathfrak{g}_{x_0}^*\right)=0$$

Second, one proves that for all neighborhood U of x_0, there exists a neighborhood V of x_0 such that $H^2_\pi(V)=0$.

We may assume that $M=\mathbb{R}^n$ and $x_0=0$. Consider the deformation $\{\pi_t\}$ given by

$$\pi_t(x)=\frac{1}{t}\pi(tx)=\sum\frac{1}{t}\pi^{ij}(tx)\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_j};$$

so at $t=1$ we have $\pi_1=\pi$ and at $t=0$ we have $\pi_0=\pi^{linear}$ (since $\pi(0)=0$), the linear Poisson stucture coming from $\mathbb{R}^n=\mathfrak{g}_{x_0}^*$.

We now look for $\varphi_t:M\to M$ such that

$$(\varphi_t)_*(\pi_t)=\pi_0,\ \forall t\in[0,1].$$

For it, look for φ_t as the flow of a t-dependent vector field X_t. Taking $\frac{d}{dt}$ we obtain the defining equation of X_t. In addition to this equation, we look for a vector field having the form

$$X_t(x)=\frac{1}{t^2}X(tx),$$

where $X \in \mathfrak{X}(M)$.

On the other hand $\left(\frac{d}{dt}\pi_t\right)_{t=1}$ defines an element in $H^2_\pi(M)$. So, one finds a vector field Y s.t

$$\left(\frac{d}{dt}\pi_t\right)_{t=1} = \delta(Y),$$

where here δ comes from the Poisson cohomology. Slighly changing Y, one gets X, which is basically Y modified in such a way that

$$X(0) = 0$$

and

$$(\delta X)_0 = 0.$$

5 Variation of Symplectic Areas

For this section see [4]. Let (M, π) be a regular Poisson manifold. Let L be a symplectic leaf, $x \in L$ and $\pi_2(L, x)$ be its second fundamental group. Let w_L be its symplectic form. Let

$$A_{w_L}(\gamma) = \int_{S^2} \gamma^* w,$$

where $\gamma : S^2 \to M$ with $\gamma(p_N) = x$ (p_N is the north pole of S^2).

Consider $\gamma_t : S^2 \to M$ a deformation for $t \in (-\epsilon, \epsilon)$ with $\gamma = \gamma_0$. Then

$$\left(\frac{d}{dt}A_w(\gamma_t)\right)_{t=0}$$

only depends on two things: the homotopy class of γ, $[\gamma] \in \pi_2(L, x)$ and on

$$\left(\frac{d}{dt}\gamma_t(p_N)\right)_{t=0} \in T_xM/T_xL.$$

Set $T_xM/T_xL := \gamma_x$. Hence one obtains a map

$$\partial : \pi_2(L, x) \to \gamma_x^*$$

In particular, a group $N_x := \mathrm{im}(\partial) \subseteq \gamma_x^*$, which is some sort of monodromy group. Take $\Sigma = \Sigma(M)$. Look at the connected component of the isotropy group at x, denoted by $\Sigma(x, x)$; then one has

$$(\Sigma(x, x))^\circ = \gamma_x^*/N_x.$$

In particular if (M, π) is integrable, then N_x must be a dicreate subgroup of γ_x^*.

Open problem Find a compact Poisson manifold M, not being symplectic, such that $\Sigma(M)$ is a compact manifold. The last is equivalent to find a non-trivial compact symplectic groupoid which has connected and simply connected $\mathsf{s}-$fibers.

6 Poisson Lie groups: definitions and examples

The study of Poisson Lie groups has two main motivations. The first arises from quantum groups theory, in fact taking the classical limit of a quantum group one gets a Poisson Lie group; the second come from integrable system and soliton equations.

Definition 6.1 *A* Poisson Lie group (G, π) *is a Lie group G together with a Poisson structure π on G such that the multiplication $m : (G \times G, \pi \times \pi) \to (G, \pi)$ is a Poisson map.*
In this case π is called multiplicative.

$$G \text{ is multiplicative} \iff graph(m) \subseteq G \times G \times \bar{G} \text{ is coisotropic}$$

Let (M_1, π_1), (M_2, π_2) be Poisson manifolds. $f : (M_1, \pi_1) \to (M_2, \pi_2)$ is Poisson if and only if $graph(f) \subseteq M_1 \times \bar{M_2}$ is coisotropic. It follows that π is multiplicative if and only if $graph(m) \subseteq G \times G \times \bar{G}$ is coisotropic.

Also, if L_g and R_g denote the usual left and right multiplication by $g \in G$, then

$$\pi \text{ is multiplicative} \iff \pi_{gh} = (L_g)_* \pi_h + (R_h)_* \pi_g \quad \forall g, h \in G. \tag{6.1}$$

Remark 6.2
- $\pi_e = 0$. This follows by $g = h = e$ in the multiplicative relation (6.1).
- Linearize π : There exists a linear Poisson structure $\pi^{(1)}$ on $T_e\mathfrak{g}$ if and only if $\mathfrak{g}^*$ has a Lie algebra structure.

Example 6.3
- $\pi = 0$, i.e. any Lie group with its trivial Poisson structure is a Poisson Lie group.
- Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ be its dual. Consider $\mathfrak{g}^*$ as an abelian Lie group, then the Lie-Poisson structure is multiplicative, and $\mathfrak{g}^*$ is a Poisson Lie groups.

7 Poisson Lie groups from "$r-$matrices"

Let Λ be a bivector on the Lie algebra $\mathfrak{g}$, i.e. $\Lambda \in \Lambda^2(\mathfrak{g})$. Define

$$\pi_g = (L_g)_* \Lambda - (R_g)_* \Lambda$$

Question: When is π_g is Poisson?
It is possible to answer in terms of the Schouten–Nijenhuis bracket of Λ.

Definition 7.1 *The* Schouten–Nijenhuis bracket *on $\Lambda^k \mathfrak{g}$ is the unique operator*

$$\Lambda^* \mathfrak{g} \times \Lambda^* \mathfrak{g} \to \Lambda^* \mathfrak{g}$$

that extends the Lie bracket on $\mathfrak{g}$ and satisfy:

i) $[A, B] = -(-1)^{(a-1)(b-1)}[B, A]$ $\forall A, B \in \Lambda^* \mathfrak{g}$, *deg* $A = a$, *deg* $B = b$;

ii) $[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C]$ $\forall A, B, C \in \Lambda^* \mathfrak{g}$, *deg* $A = a$, *deg* $B = b$, *deg* $C = c$.

Proposition 7.2 *π is a Poisson structure if and only if $[\Lambda, \Lambda] \in \Lambda^3\mathfrak{g}$ is Ad-invariant.*

PROOF: Set $\Lambda^L := (L_g)_*\Lambda$ and $\Lambda^R := (R_g)_*\Lambda$, so that $\pi = \Lambda^L - \Lambda^R$.

$$\begin{aligned} \pi \text{ is Poisson} &\iff [\pi, \pi] = 0 \\ &\iff [\Lambda^L - \Lambda^R, \Lambda^L - \Lambda^R] = [\Lambda^L, \Lambda^L] - [\Lambda^R, \Lambda^R] = 0 \\ &\iff [\Lambda, \Lambda]^L = [\Lambda, \Lambda]^R = 0 \\ &\iff Ad_g[\Lambda, \Lambda] = [\Lambda, \Lambda] \end{aligned}$$

as we wanted to prove. □

If $\pi = \Lambda^L - \Lambda^R$ we call (G, π) "exact" Poisson–Lie group (or "coboundary" Poisson–Lie group). If Λ is such that $[\Lambda, \Lambda]$ is Ad-invariant we call it an "r-matrix". In particular, if $[\Lambda, \Lambda] = 0$ (classical Yang–Baxter equation) Λ is called a "triangular r-matrix".

Example 7.3 Consider $G = SU(2)$ and let $\mathfrak{g} = \mathfrak{su}(2)$ be its Lie algebra. The elements

$$e_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

satisfying the relations

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = e_2, \quad [e_2, e_3] = e_1,$$

form a basis of $\mathfrak{g}$. Note that $\Lambda^3\mathfrak{g}$ is 1 dimensional and generated by $e_1 \wedge e_2 \wedge e_3$, which is Ad-invariant. So, for any Λ, $[\Lambda, \Lambda]$ is a multiple of $e_1 \wedge e_2 \wedge e_3$ and thus is Ad-invariant. So any Λ defines a Poisson–Lie structure. In particular, if $\Lambda = 2(e_1 \wedge e_2)$, the corresponding Poisson structure is $\pi_g = 2(R_g(e_1 \wedge e_2) - L_g(e_1 \wedge e_2))$.

8 Lie Bialgebras

Lie bialgebras can be thought as the infinitesimal version of Poisson Lie groups. In this section we define them and, using the idea of double Lie group, show how to integrate them to a Poisson Lie group.

Definition 8.1 *Let ρ be a representation of a Lie group G and $d\rho : \mathfrak{g} \to End(V)$ be its differential. $\varphi : G \to V$ is a* 1-cocycle on G *if it satisfies $\varphi(gh) = \varphi(g) + \mathrm{Ad}_g\varphi(h)$.*
$\phi : \mathfrak{g} \to V$ is a 1-cocycle on $\mathfrak{g}$ *if $\phi([u, v]) = u\phi(v) - v\phi(u)$.*

Claim 8.2 *φ 1-cocycle on $G \Rightarrow \Phi := d_e\varphi$ is a 1-cocycle on $\mathfrak{g}$. Moreover, if G is simply connected, then 1-cocycles on $\mathfrak{g}$ can be integrated to G.*

Let π be a bivector field (not necessarily Poisson) on a Lie group G and consider the linearization of π at e. Namely, if $F := \pi^{(1)} := \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is the linear part of π at e. then consider its dual $F^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$.

It is a antisymmetric bilinear map given by

$$F(\xi, \eta) = [\xi, \eta]_\pi = d_e(\pi(\bar{\xi}, \bar{\eta})),$$

where, $\xi, \eta \in \mathfrak{g}^*$ and $\bar{\xi}$ and $\bar{\eta}$ are any 1-form on G with $\bar{\xi}(e) = \xi$ and $\bar{\eta}(e) = \eta$.

When (G, π) is a Poisson Lie group then $[\,,]_\pi$ satisfy the Jacobi identity. So in this case F^* is a Lie algebra structure on $\mathfrak{g}^*$ and $(\mathfrak{g}, [\,,]_\pi)$ is the linearization of the Poisson structure at e.

Define $\tilde{\pi} : G \to \Lambda^2\mathfrak{g}$ to be the map such that $\tilde{\pi}(g) := (R_{g^{-1}})_*\pi_g$ and $\pi^{(1)} = d_e\tilde{\pi}$. Then

$$\pi \text{ is multiplicative} \iff \tilde{\pi} \text{ satisfies } \tilde{\pi}(gh) = \tilde{\pi}(g) + \mathrm{Ad}_g\tilde{\pi}(h). \tag{8.1}$$

Condition (8.1) says that $\tilde{\pi}$ is a 1-cocycle on G.

Because $\tilde{\pi}$ is a 1-cocycle on G, then F is a 1-cocycle on $\mathfrak{g}$ relative to $\Lambda^2\mathfrak{g}$, i.e.

$$F([u,v]) = ad_uF(v) - ad_vF(u).$$

To summarize: Let (G,π) be a Poisson Lie group, then the map $F = d_e\tilde{\pi} : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is such that:

(1) $F^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket (equivalent to π is Poisson).

(2) F is a 1-cocycle (equivalent to multiplicativity).

Definition 8.3 *Let $\mathfrak{g}$ be a Lie algebra with dual space $\mathfrak{g}^*$. We say that $(\mathfrak{g}, \mathfrak{g}^*)$ is a* Lie bialgebra *over $\mathfrak{g}$ if there is a given Lie algebra structure on $\mathfrak{g}^*$ such that the map $F : \mathfrak{g} \to \mathfrak{g}\wedge\mathfrak{g}$, dual to the Lie bracket map $\mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$, is a 1-cocycle on $\mathfrak{g}$ (relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$).*

The following theorem, due to Drinfeld, establishes an equivalence of categories between (connected and simply connected) Poisson Lie groups (G,π) and Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$.

Theorem 8.4 *(Drinfeld)*
If (G,π) is a Poisson Lie group, then the linearization of π at e defines a Lie algebra structure on $\mathfrak{g}^*$, such that $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, called the tangent Lie bialgebra to (G,π). Conversely, if G is connected and simply connected, then every Lie bialgebra defines a unique multiplicative Poisson structure π on G such that $(\mathfrak{g}, \mathfrak{g}^*)$ is the tangent Lie bialgebra to (G,π)

Another way to view Lie bialgebras is the following: given a Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$, define on their direct sum $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ the following scalar product $\langle\ ,\ \rangle$:

$$\langle (u,\mu), (v,\nu)\rangle = \mu(u) + \nu(v).$$

Suppose $[\,,]_*$ is a Lie bracket on $\mathfrak{g}^*$, define a bracket on $\mathfrak{d}$ as follows:

- $[(u,0),(v,0)] = ([u,v],0)$
- $[(u,0),(0,\mu)] = (-ad^*_\mu u, ad^*_u\mu)$
- $[(0,\mu),(0,\nu)] = (0,[\mu,\nu]_*)$.

Theorem 8.5 *(Drinfeld, Manin)*

(1) $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra $\iff$ $[\,,]$ is a Lie bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$;

(2) $[\,,]$ on $\mathfrak{g} \oplus \mathfrak{g}^*$ is uniquely determined by the conditions:

- *$[\,,]$ restricts to the brackets on $\mathfrak{g}$, $\mathfrak{g}^*$;*
- *$\langle\ ,\ \rangle$ is ad-invariant.*

$\big(\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*, [\,,]\big)$ is called *Drinfeld double* of $(\mathfrak{g}, \mathfrak{g}^*)$.

Corollary 8.6 $(\mathfrak{g}, \mathfrak{g}^*)$ *is a Lie bialgebra* $\iff$ $(\mathfrak{g}^*, \mathfrak{g})$ *is a Lie bialgebra.*

Definition 8.7 *Let* (G, π) *be a Poisson Lie group with associated Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ *(see 8.4). Let* G^* *be the connected and simply connected Lie group integrating* $\mathfrak{g}^*$*. By corollary 8.6, on* G^* *there is a Poisson Lie group structure and* G^* *is called the dual Poisson Lie group of* (G, π) *.*

Example 8.8
- (G, π), where $\pi = 0$. Its dual Poisson Lie group is $\mathfrak{g}^*$ with the linear Poisson structure.
- Let $G = SU(2)$, its dual Poisson Lie group is

$$G^* = \left\{ \begin{pmatrix} a & b+ic \\ 0 & a^{-1} \end{pmatrix} \mid a > 0, b, c \in \mathbb{R} \right\} := SB(2, \mathbb{C}).$$

Definition 8.9 *A* Manin triple *is a triple of Lie algebras* $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ *and a non-degenerate invariant symmetric scalar product* $\langle\, ,\, \rangle$ *on* $\mathfrak{d}$ *such that*

(1) $\mathfrak{g}, \mathfrak{h}$ *are subalgebras of* $\mathfrak{d}$;

(2) $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ *as vector spaces;*

(3) $\langle\, ,\, \rangle_{|_\mathfrak{g}} = 0$ *and* $\langle\, ,\, \rangle_{|_\mathfrak{h}} = 0$.

Theorem 8.10 *There exists a 1-1 correspondence between Lie bialgebras and Manin triples, precisely:*

$$\begin{array}{ccc} (\mathfrak{g}, \mathfrak{g}^*) & \rightarrow & (\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*) \\ (\mathfrak{g}, \mathfrak{g}^*) & \leftarrow & (\mathfrak{d}, \mathfrak{g}, \mathfrak{h}) \end{array}$$

Example 8.11 (Gram–Schmidt) Given a matrix $A \in GL(n, \mathbb{C})$, applying Gram–Schmidt to the columns we get a unitary matrix $U \in U(n)$, which can be realized as the product $U = A\Theta$, where Θ is a upper triangular matrix with real positive diagonal entries. If A is in $SL(n, \mathbb{C})$, then by applying Gram–Schimdt we get a special unitary matrix $U \in SU(n)$. In this case $A = UT$ with $T \in SB(n, \mathbb{C})$, i.e. T is a upper triangular matrix in $SL(n, \mathbb{C})$ with real positive diagonal entries. So

$$SL(n, \mathbb{C}) = SU(n) \cdot SB(n, \mathbb{C}).$$

On the level of Lie algebras, set $\mathfrak{d} = \mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{g} = \mathfrak{su}(n)$ and $\mathfrak{h} = \mathfrak{sb}(n, \mathbb{C})$. Then $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$; defining on $\mathfrak{d}$ the scalar product $\langle X, Y \rangle = Im\, tr(XY)$ we realize $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h}, \langle\, ,\rangle)$ as a Manin triple. It follows that $SU(n)$ and $SB(n, \mathbb{C})$ are Poisson Lie groups.

More generally, if K is a compact Lie group and $G = K^{\mathbb{C}}$ is its complexification, then it's possible to decompose $G = K \cdot AN$ where A is abelian and N is nilpotent (Iwasawa's decomposition).

A Poisson action is the action of a Poisson group (G, π) on a Poisson manifold (M, π_M) such that the action

$$G \times M \rightarrow M$$

is a Poisson map.

8.1 Dressing Action

We now give definition of dressing action of the dual group G^* on the Poisson Lie group (G, π) and few of its main properties. The name "dressing action" come from the theory of integrable systems where -in some cases- the " dressing transformation group" plays the role of "hidden symmetries group".

Recall that, given a Poisson manifold (P, π), the space $\Omega^1(P)$ of 1-forms on P has a Lie algebra structure and the map $-\pi^\sharp$ defines a Lie algebra homomorphism from $\Omega^1(P)$ to the space $\mathfrak{X}(P)$ of vector fields on P.

For each $\xi \in \mathfrak{g}^*$ let ξ^l and ξ^r be respectively the left and right invariant 1-form on G with value ξ at the identity e, and define the map $l : \mathfrak{g} \to \mathfrak{X}(G^*)$ such that $\lambda(\xi) = \pi^\sharp_{G^*}(\xi^l)$. Similarly, define the map $\rho : \mathfrak{g}^* \to \mathfrak{X}(G)$ such that $\rho(\xi) = \pi^\sharp_{G^*}(\xi^r)$.

Definition 8.12 $\lambda(\xi)$ *is called a* left dressing vector field on G. *Similarly,* $\rho(\xi)$ *is called a right dressing vector field on G. Integrating* λ *gives rise to a local (or global, if the dressing vector fields are complete) left action of* G^* *on* G, *called the* left dressing action of G^* on G. *Similarly, integrating* ρ, *it is possible to define the right dressing action of* G^* *on* G.

From the definition, it follows that the dressing orbits are precisely the symplectic leaves of the Poisson Lie group G.

If $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple, the Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ integrates to the double Lie group D; there exists a Poisson Lie structure on D. If the dressing action is complete then there exists a symplectic structure ω_D generalizing $T^*G = G \times \mathfrak{g}^*$ (this is called the Heisenbergh double).

9 Moduli Space of Flat Connections

Let Σ be a compact oriented surface of genus g having d boundary components. Let G be a connected Lie group with a fixed bilinear, symmetric, non-degenerate form

$$(\cdot \ , \ \cdot)$$

which is invariant, i.e.

$$([u, v], w) + (v, [u, w]) = 0$$

for all u,v and w in $\mathfrak{g}$. For example, $G = SU(n)$ and $(A, B) := \mathrm{Tr}(A\bar{B})$, or in any case where $\mathfrak{g}$ is semi-simple, one could use the Killing form.

Definition 9.1 *Let* P *be a principal* G*-bundle over* Σ. *Recall that a connection on* P *is the choice of* G*-invariant* $\mathfrak{g}$*-valued 1-form* θ *such that* $\iota_{v_P}(\theta) = v$ *for all* $v \in \mathfrak{g}$, *where* $v_P \in \mathfrak{X}(P)$ *is the infinitesmal action of* v,

$$v_P(p) := \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(tv) .$$

An equivalent description is as follows. Given the map $pr : P \to \Sigma$ one has its tangent map $T_pP \to T_{pr(p)}\Sigma$. A connection on P can be described as a choice of a subspace H_p of T_pP, varying smoothly with p, which projects isomorphically onto $T_{pr(p)}\Sigma$ via the tangent map, and is G-invariant in the sense that $H_p \cdot g = H_{pg}$. Given a connection 1-form θ, the corresponding distribution H is given by $H = \ker(\theta)$.

Now, let P be the trivial principal G-bundle $\Sigma \times G$ and let A denote the space of all connection on P. Note that since any connection 1-form θ is G-invariant, θ is uniquely determined by its restriction to Σ, since $\theta((x,g)) = \theta((x,e)\cdot g) = \theta((x,e))$, where e is the identity element. Hence, we may make the identification

$$A = \Omega^1(\Sigma, \mathfrak{g}).$$

Now, the Lie bracket of $\mathfrak{g}$ may be extended to A in the sense that:

$$\begin{aligned}
&[\ \cdot\ ,\ \cdot\ ] : \Omega^1(\Sigma,\mathfrak{g}) \times \Omega^1(\Sigma,\mathfrak{g}) \to \Omega^2(\Sigma,\mathfrak{g}) \\
&[\alpha,\beta](X,Y) := [\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)].
\end{aligned}$$

Definition 9.2 *The curvature of a connection 1-form θ is $F(\theta) = d\theta + \frac{1}{2}[\theta,\theta] \in \Omega^2(\Sigma, g)$. A connection form θ is called flat, if its curvature form is zero.*

Let A_{flat} denote the space of all flat connections on P.

Definition 9.3 *The gauge group of P, $\mathcal{G} = \mathrm{Aut}_\Sigma(P)$, is the group of all principal G-bundle automorphisms of P, i.e., G-equivariant fibre bundle automorphisms.*

Remark 9.4 Any homomorphism of principal G-bundles is in fact an isomorphism.

Exercise: Prove that there is a one-to-one correspondence between principal G-bundle automorphisms of any principal G-bundle P over Σ and smooth maps $\phi : \Sigma \to G$

Definition 9.5 *The Moduli space of flat connections of P is defined to be* $\mathcal{M} := {}^{A_{flat}}/_{\mathcal{G}}$.

10 Holonomy

Definition 10.1 *Given a principal G-bundle $pr : P \to \Sigma$ with connection form θ and a path $\gamma : I \to \Sigma$, a horizontal lift of γ (with respect to the connection form θ) is a path $\mu : I \to P$ over γ that is horizontal, i.e., $pr(\mu(t)) = \gamma(t)$ and $\iota_{\dot{\mu}(t)}\theta = 0$.*

Given such a path γ and a point p_0 over $\gamma(0)$ in P, there exists a unique horizontal lift of γ, μ, starting at p_0.

This allows one to define parallel transport along γ,

$$\begin{aligned}
T_\gamma : P_{\gamma(0)} &\to P_{\gamma(1)} \\
p_0 &\mapsto \mu_{p_0}(1)
\end{aligned}$$

which is moreover G-equivariant.

For flat connections, the parallel transport T_γ depends only on the homotopy class of the curve γ. So, any loop based at $x \in \Sigma$, $[\gamma] \in \pi_1(\Sigma, x)$, defines a diffeomorphism of the fibre above x, P_x, via its parallel transport T_γ. Since P is principal, there is a unique $g \in G$ such that $T_\gamma(p) = p \cdot g$ for all $p \in P_x$. Hence, given a flat connection θ, we get a homomorphism

$$h_\theta : \pi_1(\sigma) \to G.$$

Conversely, given any homorphism $\pi_1(\Sigma) \to G$, one can recover θ up to gauge-equivalence, i.e., one can determine its image in $\mathcal{M} = {}^{A_{flat}}/_{\mathcal{G}}$. Two such homorphisms specify the same element of $\mathcal{M}$ if and only if they are conjugate by an element of G, in the sense that

$$\varphi^1([\gamma]) = g\varphi^2([\gamma])\, g^{-1}.$$

So, in conclusion, we have established

$$\mathcal{M} = {}^{A_{flat}}/_{\mathcal{G}} \cong {}^{\mathrm{Hom}(\pi_1(\Sigma), G)}/_{\mathrm{Ad}(G)}.$$

Now, since Σ is of genus g and has d boundary components, its fundamental group can be described as

$$\pi_1(\Sigma) = \left\langle \alpha_1, \ldots \alpha_g, \beta_1, \ldots \beta_g, \mu_1, \ldots \mu_d : \prod_{i=1}^{g} [\alpha_i, \beta_i] = \prod_{j=1}^{d} \mu_j \right\rangle.$$

It follows then that

$$\mathcal{M} = \left\langle (A_1, \ldots A_g, B_1, \ldots B_g, M_1, \ldots M_d) \in G^{2d+g} : \prod_{i=1}^{g} [A_i, B_i] = \prod_{j=1}^{d} M_j \right\rangle \Big/ Ad(G)$$

Where the $\mathrm{Ad}(G)$ action on G^{2d+g} is component-wise.

In general the moduli space $\mathcal{M}$ is not a manifold. However, when G is compact, it is at least a Hausdorff space. Since

$$\mathrm{Hom}(\pi_1(\Sigma), G) \cong \left\langle (A_1, \ldots A_g, B_1, \ldots B_g, M_1, \ldots M_d) \in G^{2d+g} : \prod_{i=1}^{g} [A_i, B_i] = \prod_{j=1}^{d} M_j \right\rangle,$$

when $d = 0$, $\mathrm{Hom}(\pi_1(\Sigma), G)$ is not smooth but when d is non-zero, $\mathrm{Hom}(\pi_1(\Sigma), G) \cong G^{2d+g-1}$.

However, the action of G is not in general free, so the quotient $\mathcal{M}$ can still fail to be smooth whether d is non-zero or not.

Remark 10.2 When $G = SU(2)$ and $d = 0$, $\mathcal{M}$ is only smooth when Σ has genus 2.

Furthermore, different surfaces can have the same moduli spaces.

Definition 10.3 *Denote by $\mathcal{M}_{g,d}(G)$ the moduli space of flat connections of the trivial principal G-bundle over the surface Σ of genus g that has d boundary components.*

For example,

$$\mathcal{M}_{0,3}(G) \cong \mathcal{M}_{1,1}(G). \tag{10.1}$$

However, as we will see, each moduli space $\mathcal{M}$ comes equipped with a natural Poisson structure and the isomorphism (10.1) is not a Poisson map.

Example 10.4 Let $\mathcal{M} = \mathcal{M}_{0,3}(SU(2))$. Then

$$\mathcal{M} = \left\{(M_1, M_2, M_3) \in SU(2)^3 : M_1 M_2 M_3 = Id\right\} / \mathrm{Ad}(SU(2)),$$

since the set of generators is already a group. In $SU(2)$, every element is conjugate to something of the form

$$M_\alpha = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

with $\alpha \in [0, \pi]$. Hence, we can restrict our attention to triples of the form (M_{α_1}, M_2, M_3). Moreover, one can always conjugate this triple with a diagonal matrix without changing M_{α_1}. Now, M_2 is in the general form

$$M_2 = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

with $|a|^2 + |b|^2 = 1$. Let U be an arbitrary diagonal matrix in $SU(2)$,

$$U = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix},$$

with of course $|\lambda|^2 = 1$. Since

$$\mathrm{Ad}_U(M_2) = \begin{pmatrix} a & \lambda^2 b \\ -\bar{\lambda}^2 \bar{b} & \bar{a} \end{pmatrix},$$

we may reduce to the case where

$$M_2 = \begin{pmatrix} a & b \\ -b & \bar{a} \end{pmatrix},$$

with $b > 0$ and $|a|^2 + b^2 = 1$. So, we can write $b = \sin(\alpha_3)$ and $a = e^{i\alpha_2}\cos(\alpha_3)$, with α_2 and $\alpha_3 \in [0, \pi]$.

Exercise: Show that the triple $(M_1, M_2, M_3) \in SU(2)^3$, with

$$\begin{aligned}
M_1 &= \begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{-i\alpha_1} \end{pmatrix} \\
M_2 &= \begin{pmatrix} e^{i\alpha_2}\cos(\alpha_3) & \sin(\alpha_3) \\ -\sin(\alpha_3) & e^{-i\alpha_2}\cos(\alpha_3) \end{pmatrix} \\
M_3 &= \begin{pmatrix} e^{i\alpha_2}\cos(\alpha_3) & \sin(\alpha_3) \\ -\sin(\alpha_3) & e^{-i\alpha_2}\cos(\alpha_3) \end{pmatrix}^{-1} \begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{-i\alpha_1} \end{pmatrix}^{-1}
\end{aligned}$$

is not conjugate to anything else of this form.

We conclude that

$$\mathcal{M}_{0,3}\left(SU(2)\right) \cong \left\{ (t_1, t_2, t_3) \in R^3 : 0 \leqslant t_i \leqslant 1, |t_1 - t_2| \leqslant t_3 \leqslant t_1 + t_2, t_1 + t_2 + t_3 \leqslant 2 \right\},$$

by letting $t_i = \frac{\alpha_i}{\pi}$. This is a tetrahedron, so, a manifold with corners.

We know that $\mathcal{M}_{0,3}\left(SU(2)\right) \cong \mathcal{M}_{1,1}\left(SU(2)\right)$, as spaces. However, the Poisson structure associated to $\mathcal{M}_{0,3}\left(SU(2)\right)$ is trivial, while for $\mathcal{M}_{1,1}\left(SU(2)\right)$, it is not.

11 Poisson Structures on Moduli Spaces

Let us return to the surface Σ of genus g and assume that it has no boundary components. Recall that A was the space of all connections forms on the trivial principal G-bundle over Σ.

Recall, that we have made the identification $A = \Omega^1\left(\Sigma, \mathfrak{g}\right)$, so A is an affine space, hence, its tangent space is itself.

A comes equipped with a constant symplectic form ω, which is canonical with respect to the fixed form $\left(\cdot \quad , \quad \cdot \right)$ on $\mathfrak{g}$,

$$\omega\left(\theta_1, \theta_2\right) := \int\limits_{\Sigma} (\theta_1, \theta_2),$$

where θ_1 and $\theta_2 \in \Omega^1\left(\Sigma, \mathfrak{g}\right) \cong TA$. Note, that the inner product of two elements of $\Omega^1\left(\Sigma, \mathfrak{g}\right)$ in $\mathfrak{g}$ results in a 2-form in $\Omega^2\left(\Sigma\right)$, so this integral yields real numbers.

This symplectic form is invariant under the action of the gauge group $\mathcal{G}$. In fact, it is a Hamiltonian action, as will be shown. Recall that we can describe $\mathcal{G}$ as $\mathcal{G} \cong C^\infty\left(\Sigma, G\right)$. Hence, $\mathcal{L}ie(\mathcal{G}) \cong C^\infty\left(\Sigma, \mathfrak{g}\right)$.. Additionally, we have the nice embedding

$$\begin{array}{rcl} \Omega^2\left(\Sigma, \mathfrak{g}\right) & \hookrightarrow & \mathcal{L}ie(\mathcal{G})^* \\ \alpha & \mapsto & \left\{ \varphi \mapsto \int\limits_{\Sigma} (\varphi, \alpha) \right\}. \end{array}$$

We define the map $\mu : A \to \Omega^2\left(\Sigma, \mathfrak{g}\right) \subseteq \mathcal{L}ie(\mathcal{G})^*$ to be the map determined by the curvature form, i.e.,

$$\mu\left(\theta\right) = d\theta + \frac{1}{2}\left[\theta, \theta\right].$$

This map serves as a moment map of the gauge group action. Notice that $A_{flat} = \mu^{-1}\left(0\right)$. Hence the moduli space,

$$\mathcal{M} = \mu^{-1}\left(0\right)\!/\mathcal{G},$$

inherits a symplectic structure.

In the case where Σ has boundary components, the gauge action is no longer Hamiltonian. However, the moduli space still inherits a Poisson structure. The symplectic leaves of the moduli space can be parameterized by d-tuples, $(C_1, \ldots, C_d)$ of conjugacy classes of G. The corresponding leaf is made up of the image of those connections whose associated holonomy map h_θ satisfies $h_\theta(M_i) \in C_i$ for $i = 1, \cdots, d$, where M_i is the generator of the fundamental group corresponding to the circle around the i^{th}-boundary component.

References

[1] M. Audin, Torus actions on symplectic manifolds. Germany. Springer (2000).

[2] J. Conn, Normal forms for analytic Poisson structures. *Annals of Math.,* **119** (1984), 577-601.

[3] J. Conn, Normal forms for smooth Poisson structures. *Annals of Math.,* **121** (1985), 565-593.

[4] M. Crainic and R.L. Fernandes, Integrability of Poisson brackets, *J. Differential Geom.*, **66** (2004), 71-137.

[5] *Multiplicative and Affine Poisson Structures on Lie Groups* by Jiang-Hua Lu