Poisson geometry
Lectures 1, 2, 3

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Abstract

The definition of a Poisson manifold is given in terms of the Poisson bracket and some basic examples are shown. Symplectic manifolds as certain examples of Poisson manifolds are treated in more detail. After that the Poisson bivector is introduced and its correspondence to the Poisson structure is shown. The first lecture ends with considerations about the Jacobi identity and the Jacobiator. In the second lecture Poisson manifolds are defined in terms of the Poisson tensor and we show how their cotangent bundles can be viewed as Lie algebroids via the Courant bracket. After proving the local splitting theorem of a Poisson manifold, we introduce the notion of symplectic foliation and give several examples, most notably the coadjoint orbits for the Lie Poisson structure on the dual of a lie algebra. In the third lecture, we prove the existence and uniqueness of manifold structures of symplectic leaves, with some examples of symplectic leaves and poisson maps from Hamiltonian actions. Finally the Marsden- Weinstein theorem is also discussed.

1 Basic Properties of Poisson Manifolds

In this chapter M will denote a C^∞-manifold.

Definition 1.1 (Poissonstructure) *A **Poisson structure** on M is a $\mathbb{R}$-bilinear bracket*

$$\{,\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M),$$

such that

$$\{f,g\} = -\{g,f\} \quad \textit{(skew symmetry)}$$
$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0 \quad \textit{(Jacobi Identity)}$$
$$\{f,gh\} = \{f,g\}h + g\{f,h\} \quad \textit{(Leibniz rule)}$$

hold.

*(M,$\{,\}$) is then called **Poisson manifold**.*

Example 1.2 (First examples) 0. Trivial Poisson structure $\{,\} \equiv 0$. Any manifold is a Poisson manifold in this way.

1. Classical Poisson bracket:
   $M = \mathbb{R}^{2n} = \{(q_1, ..., q_n, p_1, ..., p_n)\}$
   $\{f,g\} := \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$

2. Let (S,ω) be a symplectic manifold (i.e. $\omega \in \Omega^2(S),\ d\omega = 0,\ \omega$ nondegenerate).

- $f \in C^\infty(M),\ X_f \in \mathfrak{X}(M)$ hamiltonian vector field, uniquely defined by: $i_{X_f}\omega = -df$
- Poisson bracket: $\{f,g\} := \omega(X_f, X_g) = -df(X_g) = dg(X_f) = L_{X_f}g = X_f(g)$
This bracket clearly is skew symmetric and satisfies the Leibniz identity.

Exercise: Consider the classical Poisson bracket in mechanics, $\{f,g\} := \sum\limits_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$, $M = \mathbb{R}^{2n} = \{(q_1,\dots,q_n,p_1,\dots,p_n)\}$. Check that the bracket comes from the canonical form $\omega = \sum\limits_{i=1}^{n} dq_i \wedge dp_i$ on $\mathbb{R}^{2n}$.
Solution:
The poisson bracket defined by the symplectic form is given by $\{f,g\}_\omega = \omega(X_f,X_g)$, where X_f and X_g are the Hamiltonian vector field associated with $f,g \in \mathcal{C}^\infty(M)$. These are defined by $\omega(X_f,Y) = Yf$ for all vector field $Y \in \mathfrak{X}(M)$.
We first check that

$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{1.1}$$

holds.
Consider $\omega(X_f, \frac{\partial}{\partial p_i}) = \frac{\partial f}{\partial p_i}$. Now realize that $\frac{\partial}{\partial q_i}$ and $\frac{\partial}{\partial p_i}$ form a basis of the tangent space at each point satisfying the relation

$$\omega(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_j}) = \delta_{ij}$$

and all other pairings give zero. This shows that $\frac{\partial f}{\partial p_i} = \omega(X_f, \frac{\partial}{\partial p_i}) = dq_i(X_f)$. So $\frac{\partial f}{\partial p_i}$ is the $\frac{\partial}{\partial q_i}$ component of X_f. Similarly, $-\frac{\partial f}{\partial q_i}$ is the $\frac{\partial}{\partial p_i}$ component of X_f. This gives 1.1. Using these results, we get

$$\begin{aligned}
\omega(X_f,X_g) &= \sum_{i=1}^{n} dq_i(X_f)dp_i(X_g) - dp_i(X_f)dq_i(X_g) \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}(-\frac{\partial g}{\partial q_i}) - (-\frac{\partial f}{\partial q_i})\frac{\partial g}{\partial p_i} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \\
&= \{f,g\}
\end{aligned}$$

This solves the exercise.

Proposition 1.3 *Let (S,ω) be a symplectic manifold. $\{,\}$ in Example1.2.2 satisfies the Jacobi identity.*

Proof: $d\omega = 0 \Rightarrow d\omega(X_f,X_g,X_h) = 0$. Then using $d\omega(X,Y,Z) = X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y) - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X)$ in the second and skew symmetry in the last equation gives:

$$\begin{aligned}
0 &= d\omega(X_f, X_g, X_h) \\
&= X_f\{g,h\} - X_g\{f,h\} + X_h\{f,g\} + [X_f, X_g]h - [X_f, X_h]g + [X_g, X_h]f \\
&= \{f,\{g,h\}\} - \{g,\{f,h\}\} + \{h,\{f,g\}\} + \{f,\{g,h\}\} - \{g,\{f,h\}\} \\
&\quad - \{f,\{h,g\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} - \{h,\{g,f\}\} \\
&= -3 \cdot (\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\})
\end{aligned}$$

□

Remark 1.4 If $\omega \in \Omega^2(S)$ is nondegenerate, then $\{f,g\} := \omega(X_f, X_g)$ still makes sense. The proof of Propositon1.3 shows:

$$d\omega = 0 \Longleftrightarrow \{,\} \text{ is a Poisson bracket}$$

Example 1.5 Let (S,ω) be a symplectic manifold. Let G be a Lie Group acting on S, such that $\sigma^*\omega = \omega$ holds $\forall \sigma \in G$. Assume the action is free and proper. Then S/G has a unique structure of a smooth manifold such that the natural projection

$$p : S \to S/G = M$$

is a submersion.

Note:

$$\begin{aligned}
C^\infty(M) &\cong C^\infty(S)^G \\
f &\mapsto f \circ p
\end{aligned}$$

where $C^\infty(S)^G \subseteq C^\infty(S)$ is the space of G-invariant functions on S (i.e. $\sigma^* f = f$, for $f \in S$).

Claim 1.6 *$C^\infty(S)^G$ is closed under $\{,\}_S$.*

Proof: Let $\sigma \in G$ and $f, g \in C^\infty(S)^G$. We have to show, that $\sigma^*\{f,g\} = \{f,g\}$. Now

$$\begin{aligned}
\sigma^*(i_{X_f}\omega) &= \sigma^*(-df) \\
&= -d(\sigma^* f) \\
&= -df \\
&= i_{X_f}\omega.
\end{aligned}$$

and we have

$$\begin{aligned}
\sigma^*(i_{X_f}\omega) &= i_{(d\sigma)^{-1}(X_f)}\sigma^*\omega \\
&= i_{(d\sigma)^{-1}(X_f)}\omega
\end{aligned}$$

So it follows $i_{X_f}\omega = i_{(d\sigma)^{-1}(X_f)}\omega$. Now view ω as a map $\omega : TS \to T^*S$. This map is injective, because ω is symplectic. So it follows, that

$$\begin{aligned}
(d\sigma)^{-1}(X_f \circ \sigma) &= X_f \\
\Rightarrow d\sigma(X_f) &= X_f \circ \sigma \qquad (1.2)
\end{aligned}$$

At a point $p \in S$ we have

$$(\sigma^*(\{f,g\}))_p = (\sigma^*(\omega(X_f, X_g)))_p = \omega_{\sigma(p)}(X_f \circ \sigma, X_g \circ \sigma)$$

using (1.2) we get

$$= \omega_{\sigma(p)}((d\sigma)_p(X_f), (d\sigma)_p(X_g)) = (\sigma^*\omega)_p(X_f, X_g)$$

and because of $\sigma^*\omega = \omega$:

$$= \omega_p(X_f, X_g) = \{f,g\}_p$$

and therefore

$$\sigma^*\{f,g\} = \{f,g\}.$$

□

Claim (1.6) shows, that $C^\infty(M)$ is Poisson.

Remark 1.7 Let G be a Lie Group, acting on itself by multiplication. This lifts to an action of G on $S = T^*G \cong G \times \mathfrak{g}^*$, which then makes $S/G \cong \mathfrak{g}^*$ into a Poisson manifold.

Example 1.8 Suppose S is a smooth manifold and $\omega_t \in \Omega^2(S)$ a smooth family of symplectic structures, $t \in \mathbb{R}$. Then for $f, g \in C^\infty(S) \times \mathbb{R}, x \in S$,

$$\{f,g\}(x,t) := \{f_t, g_t\}_{\omega_t}(x)$$

where $f_t, g_t \in C^\infty(S)$ for fixed t, defines a Poisson bracket on $S \times \mathbb{R}$.

1.1 Hamiltonian Vector Fields

Let $(M, \{,\})$ be a Poisson manifold. Because $\{f, \cdot\}$ satisfies the Leibniz rule, we can think of $\{f, \cdot\}$ as a derivation of $C^\infty(M)$. There is a 1-1 correspondense between vector fields and derivations:

$$\begin{aligned} \mathfrak{X}(M) &\longleftrightarrow \text{derivations of } \; C^\infty(M) \\ X &\longmapsto L_X(fg) = L_X(f)g + fL_X(g) \end{aligned}$$

Hence $\forall f \in C^\infty(M)$ there exists a well defined vector field X_f such that

$$\{f,g\} = L_{X_f}(g) = X_f(g).$$

Definition 1.9 (Hamiltonian vector field) *Let $(M, \{,\})$ be a Poisson manifold, $f, g \in C^\infty(M)$. The vector field X_f that satisfies*

$$\{f,g\} = L_{X_f}(g)$$

*is called the **Hamiltonian vector field of f** .*

If (S, ω) is a symplectic manifold and $f, g \in C^\infty(S)$. Then S has a Poisson structure given by $\{f,g\} = \omega(X_f, X_g)$ where the Hamiltonian vector fields are defind by $i(X_f)\omega = -df$. This definition coincides with definition (1.9), since

$$\{f,g\} = \omega(X_f, X_g) = -df(X_g) = dg(X_f) = L_{X_f}(g) = X_f(g).$$

Properties:

- If $f \in C^\infty(M)$, then $L_{X_f}(f) = \{f, f\} = 0$ by skew symmetry.
- If $f, g \in C^\infty(M)$, then $X_{\{f,g\}} = [X_f, X_g]$.
  That means in other words, that the mapping

$$\begin{aligned} C^\infty(M) &\longleftrightarrow \mathfrak{X}(M) \\ f &\longmapsto X_f \end{aligned}$$

  is a Lie algebra homomorphism.

Proof: Let be $f, g, h \in C^\infty(M)$. Then

$$\begin{aligned} [X_f, X_g]h &= X_f(X_g h) - X_g(X_f h) \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\} \\ &= \{f, \{g, h\}\} + \{g, \{h, f\}\} \\ &= -\{h, \{f, g\}\} \\ &= \{\{f, g\}, h\} \\ &= X_{\{f,g\}} h. \end{aligned}$$

Where skew symmetry was used in the second and fourth equation and the Jacobi idetity in the third equation. □

1.2 Poisson Tensor

Recall: Multivector Fields

Let M be a n-dimensional smooth manifold and $x \in M$. We call a section π of the natural projection $p : \Lambda^k TM \to M$ a k-vector field on M. For $\xi_i \in T_x^* M$ we have $\pi_x(\xi_1, ..., \xi_k) \in \mathbb{R}$. We denote by $\mathfrak{X}^k(M) := \Gamma(\Lambda^k TM)$ the space of all k-vector fields on M.

This is the dual to differential forms $\eta \in \Omega^k(M) = \Gamma(\Lambda^k T^* M)$, where $\eta_x(x_1, ..., x_k) \in \mathbb{R}$, for $x_i \in T_x M$.

Let $(x_1, ..., x_n)$ be local coordinates on M. Then $\Lambda^k T_x M$ has a linear basis $\{\frac{\partial}{\partial x_{i_1}} \wedge ... \wedge \frac{\partial}{\partial x_{i_k}}, i_1 < ... < i_k\}$. Where $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$, again are the duals to $dx_1, ..., dx_n$. $\pi \in \mathfrak{X}^k(M)$ is then in these local coordinates given by:

$$\pi = \sum_{i_1 < ... < i_k} \pi_{i_1, ..., i_k} \frac{\partial}{\partial x_{i_1}} \wedge ... \wedge \frac{\partial}{\partial x_{i_k}}\}$$

.

Given a k-vector field $\pi \in \mathfrak{X}^k(M)$, it defines a skew symmetric, k-linear bracket:

$$C^\infty(M) \times ... \times C^\infty(M) \to C^\infty(M)$$

$$\{f_1, ..., f_k\} := \pi(df_1, ..., df_k)$$

Proposition 1.10 *Any k-linear bracket*

$$\{\cdot, ..., \cdot\} : C^\infty(M) \times ... \times C^\infty(M) \to C^\infty(M),$$

which is skew symmetric and satisfies the following Leibniz rule:

$$\{fg, f_2, ..., f_k\} = f\{g, f_2, ..., f_k\} + \{f, f_2, ..., f_k\}g$$

arises from a k-vector field $\pi \in \mathfrak{X}^k(M)$.

PROOF:
Fix $x \in M$. We will show, that $\{f_1, ..., f_k\}(x)$ depends only on $d_x f_1$. In other words: If $d_x f_1 = 0$, then $\{f_1, ..., f_k\}(x) = 0$.

Assume that $d_x f_1 = 0$. Then follows from Taylor's theorem, that $f_1 = c + \sum_{i=1}^{n} x_i g_i$, s.t. $x_i(x) = 0, g_i(x) = 0$, where c is constant. Then

$$\{f_1, ..., f_k\} = \{c, f_2, ..., f_k\} + \sum_{i=1}^{n} \{x_i g_i, f_2, ..., f_k\}$$

But since the bracket is a derivation and derivations are 0 on constants it is $\{c, f_2, ..., f_k\} = 0$. Then

$$\{f_1, ..., f_k\} = \sum_{i=1}^{n} x_i\{g_i, f_2, ..., f_k\} + \{x_i, f_2, ..., f_k\}g_i$$

And this is 0 at x since $x_i(x) = 0, g_i(x) = 0$.

Now for $\forall \alpha_i \in T_x^* M$ we can find f_i such that $\alpha_i = d_x f_i$. So define the desired k-vector field π as follows:

$$\pi_x(\alpha_1, ..., \alpha_k) := \pi_x(d_x f_1, ..., d_x f_k) = \{f_1, ..., f_k\}(x)$$

□

Corollary 1.11 *Let $(M, \{,\})$ be a Poisson manifold. Then there is a unique bivector $\pi \in \mathfrak{X}^2(M)$, such that*

$$\{f, g\} = \pi(df, dg).$$

This bivector π is called ***Poisson bivector*** *or* ***Poisson tensor****.*

Corollary (1.10) means in other words:
Poisson structure on $M \iff \exists! \pi \in \mathfrak{X}^2(M)$, s.t. $\pi(df, dg)$ satisfies the Jacobi identity
This bivector is given in local coordinates as

$$\pi = \sum_{i<j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

The bracket applied to $f, g \in C^\infty(M)$ is then given by

$$\{f, g\} = \pi(df, dg) = \sum_{i<j} \pi_{ij} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}\right)$$

Example 1.12 Let be $(q_1, ..., q_n, p_1, ..., p_n) \in \mathbb{R}^{2n}$. Then

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$$

is the Poisson bracket associated to

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$$

Exercise: Let (S, ω) be a symplectic manifold and ω locally given as

$$\omega = \sum_{ij} \omega_{ij} dx_i \wedge dx_j.$$

Check that for the coefficients of the corresponding bivector π the following holds $\pi_{ij} = -(\omega_{ij})^{-1}$.
Solution:
Consider the map $\omega^\sharp : TM \to T^*M, X \mapsto i_X\omega$ and the map $\pi^\sharp : T^*M \to TM, \alpha \mapsto \pi^\sharp(\alpha)$, where $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$ and π is the corresponding bivector to ω. Take $\alpha, \beta \in T^*M$, such that $\alpha = i_X\omega, \beta = i_Y\omega$. Then we have $\pi(\alpha, \beta) = \omega((\omega^\sharp)^{-1}(\alpha), (\omega^\sharp)^{-1}(\beta)) = \omega(X, Y)$. For the left side of the equation we get

$$\begin{aligned}
\pi(\alpha, \beta) &= -\pi(\beta, \alpha) \\
&= - < \alpha, \pi^\sharp(\beta) > \\
&= - < i_X\omega, \pi^\sharp(\beta) > \\
&= -\omega(X, \pi^\sharp(\beta)) \\
&= -\omega(X, \pi^\sharp(i_Y\omega)) \\
&= -\omega(X, \pi^\sharp(\omega^\sharp(Y))
\end{aligned}$$

So we have $-\omega(X, \pi^\sharp(\omega^\sharp(Y)) = \omega(X, Y)$. By the nondegeneracy of ω, it follows that $\pi^\sharp = -(\omega^\sharp)^{-1}$ and therefore $\pi_{ij} = -(\omega_{ij})^{-1}$ holds for the coefficients.

Let M be a smooth manifold, $\pi \in \mathfrak{X}^2(M)$. Then (M, π) is called an "almost" Poisson manifold, if π does not satisfy the Jacobi identity.

Definition 1.13 (the Jacobiator) *The map*

$$J : C^\infty(M) \times C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

$$(f, g, h) \mapsto \{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\}$$

is called ***Jacobiator****.*

Proposition 1.14 *J is skew symmetric and is a derivation in each argument.*

Proof: Let $a, f, g, h \in C^\infty(M)$. We will show the skew symmetry of J regarding the first two arguments. The proof for the other arguments is a similar calculation.

$$\begin{aligned}
J(g, f, h) &= \{\{g, f\}, h\} + \{\{h, g\}, f\} + \{\{f, h\}, g\} \\
&= -\{\{f, g\}, h\} - \{\{g, h\}, f\} - \{\{h, f\}, g\} \\
&= -(\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\}) \\
&= -J(f, g, h)
\end{aligned}$$

So J is skew symmetric in the first two arguments. We will show now that J is a derivation in the first argument. The proof for the other arguments is a similar calculation.

$$
\begin{aligned}
J(af,g,h) &= \{\{af,g\},h\}+\{\{h,af\},g\}+\{\{g,h\},af\} \\
&= \{a\{f,g\},h\}+\{\{a,g\}f,h\}+\{a\{h,f\},g\}+\{\{h,a\}f,g\}+a\{\{g,h\},f\}+\{\{g,h\},a\}f \\
&= a\{\{f,g\},h\}+\{a,h\}\{f,g\}+\{a,g\}\{f,h\}+\{\{a,g\},h\}f+a\{\{h,f\},g\}+\{a,g\}\{h,f\} \\
&\quad +\{h,a\}\{f,g\}+\{\{h,a\},g\}f+a\{\{g,h\},f\}+\{\{g,h\},a\}f \\
&= a(\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,h\},f\}) \\
&\quad +(\{\{a,g\},h\}+\{\{h,a\},g\}+\{\{g,h\},a\})f \\
&\quad +(\{a,h\}+\{h,a\})\{f,g\}+\{a,g\}(\{f,h\}+\{h,f\}) \\
&= a(J(f,g,h))+(J(a,g,h))f++(\{a,h\}-\{a,h\})\{f,g\}+\{a,g\}(\{f,h\}-\{f,h\}) \\
&= a(J(f,g,h))+(J(a,g,h))f
\end{aligned}
$$

So J is a derivation in the first argument. □

The last proposition shows, that $J \in \mathfrak{X}^3(M)$. In local coordinates J is given by

$$J(f,g,h)=\sum_{i,j,k} J_{ijk}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial x_k}$$

where $J_{ijk}=J(x_i,x_j,x_k)$.

So to check the Jacobi identity on $C^\infty(M)$, it suffices to check, that it holds on coordinate functions, i.e. $J_{ijk}=0$.

In the following examples we show some applications of our discussion about the Jacobiator.

Example 1.15

1. $dim(M)=2$. Then any bivector field $\pi \in \mathfrak{X}^2(M)$ is Poisson. $J=0$, because $J \in \mathfrak{X}^3(M)$, but $dim(M)=2$.

2. Let $\mathfrak{g}^*$ be a finite dimensional Lie algebra with basis $v_1,...,v_n$, such that $[v_i,v_j]=\sum_{k=1}^{n} c_{ijk}v_k$. Here the c_{ijk} are the structure constants of the Lie algebra. Let $f,g \in C^\infty(\mathfrak{g}^*), \mu \in \mathfrak{g}^*$. Then define a bracket

$$\{f,g\}:=<[df(\mu),dg(\mu)],\mu>$$

This definition uses the identification $(\mathfrak{g}^*)^* \cong \mathfrak{g}$.

This bracket is clearly skew symmetric and the Leibniz rule holds. Now let us look at coordinates $\mu_1,...,\mu_n \in (\mathfrak{g}^*)^*$, s.t. the μ_i correspond to the v_i and $\mu_i(\mu)=\mu_i$, to check the Jacobi identity. Then

$$\{f,g\}=\sum_{ijk} c_{ijk}\mu_k\frac{\partial f}{\partial \mu_i}\frac{\partial g}{\partial \mu_j}$$

And on the coordinate functions we have:

$$\{\mu_i,\mu_j\}=\sum_k c_{ijk}\mu_k$$

So $\{,\}$ is a Lie bracket and $J(\mu_i,\mu_j,\mu_k)=0$. So the Jacobi identity holds on the whole $\mathfrak{g}^*$.

2 Poisson manifold

A Poisson manifold is a pair (M, π) , $\pi \in \mathfrak{X}^2(M) = \Gamma(\Lambda^2 TM)$ such that $\{f, g\} := \pi(df, dg)$ is a Lie bracket. (Only thing needed to check is Jacobi identity)

A nontrivial example is given by $M = \mathfrak{g}^*$ where $\mathfrak{g}$ is a finite dimensional Lie algebra. Given $\mu \in \mathfrak{g}^*$, $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$, define the bracket by $\{f, g\}(\mu) := \langle \mu, [df(\mu), dg(\mu)] \rangle$.

If $\mathfrak{g}$ has the following structural constants, $[v_i, v_j] = \sum c_{ij}^k v_k$, for $v_1, \ldots, v_n$ a basis of $\mathfrak{g}$, then in terms of the dual basis $\mu_1, \ldots, \mu_n \in \mathfrak{g}^*$, $\{f, g\} = \sum c_{ij}^k \mu_k \frac{\partial f}{\partial \mu_i} \wedge \frac{\partial g}{\partial \mu_j}$. This shows that $\{\mu_i, \mu_j\} = \sum c_{ij}^k \mu_k$ which implies $J_{ijk} = 0$ where recall $J_{ijk} = \{\mu_i, \{\mu_j, \mu_k\}\} + \{\mu_j, \{\mu_k, \mu_i\}\} + \{\mu_k, \{\mu_i, \mu_j\}\}$ is the Jacobiator. Thus $\{.,.\}$ is indeed a Poisson bracket, and $\mathfrak{g} \to \mathcal{C}^\infty(\mathfrak{g}^*)$ is a Lie algebra homomorphism.

Let V be a vector space, $x_1, \ldots, x_n \in V^*$ coordinates in V. A Poisson structure is said to be linear if $\{x_i, x_j\} = \sum_k c_{ij}^k x_k$ i.e., linear functions are closed under bracket. In that case, the restriction of $\{.,.\}$ from $\mathcal{C}^\infty(V)$ to V^* makes V^* into a Lie algebra.So we conclude that linear Poisson structures on V are in 1-1 correspondence with Lie algebra structures on V^*.

The simplest examples of Poisson manifolds are given by Poisson vector space V, $\pi \in \Lambda^2 V$, which can be viewed as a constant (with respect to the obvious connection) Poisson structure on the manifold V. Then it is easy to see that π automatically satisfies the Jacobi identity by checking it on coordinate functions x_i and notice that $d\pi(dx_i, dx_j) = 0$ by constancy. By definition, this is equivalent to $\pi^\# : V^* \to V$ being linear (see definition below), and $(\pi^\#)^* = -(\pi)^\#$.

Now for a general Poisson manifold (M, π), we have a bundle map (i.e., fiberwise linear): $\pi^\# : T^*M \to TM$ defined by $\beta(\pi^\#(\alpha)) = \pi(\alpha, \beta)$. $\pi^\# : \Omega^1(M) \to \mathfrak{X}(M)$. Note that if $f \in \mathcal{C}^\infty(M)$, $df \in \Omega^1(M)$; $\pi^\#(df) = X_f$, the hamiltonian vector field generated by f, which generalizes the same notion in symplectic geometry.

Definition 2.1 *$D = \pi^\#(T^*M) \subset TM$ is called the symplectic distribution (not necessarily of constant rank).*

Question: can we "integrate" D to a "symplectic foliation"?

Definition 2.2 *(M, π) is a regular Poisson manifold if rank of $\pi^\#$ is constant. (i.e., D is a regular distribution)*

Note: $M = \mathfrak{g}^*$, then D is regular if and only if $\mathfrak{g}$ is abelian. The reason is that it has rank zero at $0 \in \mathfrak{g}^*$, so to be regular it must have rank 0 everywhere.

Bringing Lie algebroids into the picture: there exists a natural Lie bracket on $\Omega^1(M)$, (M, π) Poisson manifold: For α, $\beta \in \Omega^1(M)$, $[\alpha, \beta] = \mathcal{L}_{\pi^\#(\alpha)}\beta - \mathcal{L}_{\pi^\#(\beta)}\alpha - d(\pi(\alpha, \beta)) = i_{\pi^\#(\alpha)} d\beta - i_{\pi^\#(\beta)} d\alpha + d(\pi(\alpha, \beta))$ (*) , where the last equality uses Cartan's formula: $\mathcal{L}_X = i_X d + d i_X$.

Proposition 2.3 *The bracket (*) is a Lie bracket (in particular satisfies the Jacobi identity), and satisfies the following:*

(1) d-naturality: $[df, dg] = d\{f, g\}$;

(2) Leibniz property: $[\alpha, f\beta] = f[\alpha, \beta] + (\mathcal{L}_{\pi^\#(\alpha)} f)\beta$, $f \in \mathcal{C}^\infty(M)$;

(3) $\pi^\# : \Omega^1(M) \to \mathfrak{X}(M)$ *preserves the bracket.*

These conditions imply that $(T^*M, [,]_\pi, \pi^\#)$ *is a Lie algebroid.*

PROOF: The first property, namely $[df, dg] = d\{f, g\}$ can be seen by using the last identity for the bracket, $[\alpha, \beta] = i_{\pi^\#(\alpha)} d\beta - i_{\pi^\#(\beta)} d\alpha + d(\pi(\alpha, \beta))$, in which we see only the last term survive upon substitution of two exact one forms.

For the Leibniz property, we simply compute: $[\alpha, f\beta] = i_{\pi(\alpha)} d(f\beta) - f i_{\pi(\beta)} d\alpha + d(f\pi(\alpha, \beta)) = f[\alpha, \beta] + i_{\pi(\alpha)}(df \wedge \beta) + \pi(\alpha, \beta)df = f[\alpha, \beta] + (\mathcal{L}_{\pi(\alpha)} f)\beta$.

Next we show that $\pi^\#$ preserves the bracket: It suffices to check for locally exact forms, because in general every 1-form can be written as a finite sum of functions times exact 1-forms; and using the Leibniz rule we could conclude the same thing for these non-exact 1-forms. Here are the details:

$\pi^\#([df, dg]) = \pi^\#(d\{f, g\}) = X_{\{f,g\}} = [X_f, X_g]$ and in general, $\pi^\#([hdf, dg]) = \pi^\#(h[df, dg] + (\mathcal{L}_{\pi^\#(dg)} h)df) = h[X_f, X_g] + \mathcal{L}_{X_g} hX_f = [hX_f, X_g]$.

Finally for the Jacobi identity, it also suffices to check for only the exact 1-forms, which is clear by the Jacobi identity for Poisson bracket. To see that it holds in general, we only need to check that $J(fdx, dy, dz) = 0$, all the other possibilities can be obtained by linear combinations, permutation symmetry of J, or induction on the number of nonexact entries. So let's simply do that:

$$
\begin{aligned}
[fdx, [dy, dz]] &= f[dx, [dy, dz]] + \mathcal{L}_{\pi^\#([dy,dz])}(f)dx; \\
[dy, [dz, fdx]] &= [dy, f[dz, dx] + \mathcal{L}_{\pi^\#(dz)}(f)dx] \\
&= f[dy, [dz, dx]] + \mathcal{L}_{\pi^\#(dy)}(f)[dz, dx] + \mathcal{L}_{\pi^\#(dz)}(f)[dy, dx] + \mathcal{L}_{\pi^\#(dy)}\mathcal{L}_{\pi^\#(dz)}(f)dx; \\
[dz, [fdx, dy]] &= [dz, f[dx, dy] - (\mathcal{L}_{\pi^\#(dy)}(f)dx] \\
&= f[dz, [dx, dy]] + \mathcal{L}_{\pi^\#(dz)}(f)[dx, dy] - \mathcal{L}_{\pi^\#(dz)}\mathcal{L}_{\pi^\#(dy)}(f)dx - \mathcal{L}_{\pi^\#(dy)}(f)[dz, dx].
\end{aligned}
$$

Now it's clear that everything cancels either by antisymmetry of the bracket (*), or jacobi identity on exact 1-forms, plus the fact that

$$\mathcal{L}_{\pi^\#(dy)}\mathcal{L}_{\pi^\#(dz)} - \mathcal{L}_{\pi^\#(dz)}\mathcal{L}_{\pi^\#(dy)})(f) = \mathcal{L}_{\pi^\#([dy,dz])}(f).$$

□

Corollary 2.4 *If $D := \mathrm{im}\pi^\#$ is regular, then* $[\Gamma(D), \Gamma(D)] \subset \Gamma(D)$

PROOF: Let $X, Y \in \Gamma(D)$, $\pi^\#(\alpha) = X$, $\pi^\#(\beta) = Y$, then

$$[\pi^\#(\alpha), \pi^\#(\beta)] = \pi^\#([\alpha, \beta]) \in \Gamma(D).$$

So by the theorem of Frobenius, D is integrable. □

The above computation also follows from the general fact that the image of the anchor map of a Lie algebroid integrates to a singular foliation in the sense of Stefan.

2.1 Splitting theorem

Let (M, π) be a Poisson manifold, with D the image of $\pi^\#$ as defined above. At each $x \in M$, D_x is a symplectic vector space (see exercise below). Notice also that $\ker(\pi_x^\#) \subset T_x^*M$ has a Lie algebra structure given by the ristriction of the bracket (*). We have also

$$[\alpha, f\beta] = f[\alpha, \beta] + \mathcal{L}_{\pi^\#(\alpha)} f\alpha = f[\alpha, \beta]$$

since $\pi^{\#}(\alpha) = 0$, being in the kernel. So the bracket operation can be localized to individual tangent vector spaces.

We can also define $Ann(D_x) := \ker \pi_x^{\#}$ to be the Annihilator of the distribution, which terminology is justified by the fact that $\pi^{\#}(\alpha) = 0$ if and only if $\beta(\pi^{\#}(\alpha)) = 0$ for all β.

In open subsets of M where $\pi^{\#}$ is regular, i.e., of locally constant rank, D is integrable with leaves $\mathcal{L} \hookrightarrow M$, $T\mathcal{L} = D|_{\mathcal{L}}$. $\omega_x(X_f, X_g) = \{f, g\}(x)$.

In general, let $x, y \in (M, \pi)$. Declare $x\ y$ if $\exists X_1, \ldots, X_r \in \Gamma(D)$, $t_1, \ldots, t_r \in \mathbb{R}$ such that $X = \phi_{X_1}^{t_1} \circ \ldots \circ \phi_{X_r}^{t_r}(y)$. Let $\mathcal{L}_x \subset M$ be the equivalence classes under this relation, which form the set-theoretic leaves. We will sketch a proof of the fact that $\mathcal{L} \hookrightarrow M$ are immersed submanifolds with unique differential structure, $T\mathcal{L} = D|_{\mathcal{L}}$ and $\mathcal{L}$ is symplectic. Thus the image of $\pi^{\#}$ integrates to a singular foliation with symplectic leaves. This will follow from

Theorem 2.5 *(Splitting Theorem, Weinstein, 83') Let (M, π) be a Poisson manifold, $x \in M$. $Rank \pi_x = 2k$. Then there exists a neighborhood centered at x with coordinates*

$$(q_1, \ldots, q_k, p_1, \ldots, p_k, y_1, \ldots, y_l)$$

such that

$$\pi = \sum \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + 1/2 \sum \phi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j},$$

with $\phi_{ij}(0) = 0$. In other words, locally we can write M as a product $M = N \times S$ with coordinate $\{y_i\}$ on N and $\{(q_i, p_i)\}$ on S. $\dim M = n = 2k + l$.

PROOF: In the case of $k = 0$, $rank(\pi_x) = 2k$ and nothing is to be proved.

So assume $k \neq 0$, which implies that $\exists f, g \in \mathcal{C}^\infty(M)$ such that $\{f, g\}(x) \neq 0$.

Label $p_1 = g$. $X_{p_1}(f) = \{g, f\}(x) \neq 0$ hence $X_{p_1}(x) \neq 0$. So we can find coordinates $(x_1, \ldots, x_n)$ such that $X_{p_1} = \frac{\partial}{\partial x_1}$ for example by using the local immersion theorem applied to the integral curve of X_{p_1}. This then leads to $\{p_1, x_1\} = X_{p_1} x_1 = \frac{\partial}{\partial x_1} x_1 = 1$. Label $q_1 = x_1$. Note that X_{p_1}, X_{q_1} are linearly independent at x (which is an open condition), therefore linearly independent in a neighborhood of x.

Use Frobenius to get a 2-dimensional foliation, i.e., find functions $(y_1, \ldots, y_{n-2}, y_{n-1}, y_n)$, $\frac{\partial}{\partial y_{n-1}} = X_{p_1}$, $X_{q_1} = \frac{\partial}{\partial y_n}$. Then

- $dy_1, \ldots, dy_{n-2}$ are linearly independent.
- $X_{q_1} y_1 = 0$, $X_{p_1} y_j = 0$, $j = 1, \ldots, n-2$.
- $dp_1, dq_1, dy_1, \ldots, dy_{n-2}$ are linearly independent.

These imply that $(q_1, p_1, y_1, \ldots, y_{n-2})$ are local coordinates around x, and $\{q_1, p_1\} = 1$, $\{q_1, y_j\} = 0$, $\{p_1, y_j\} = 0$.

Using Jacobi's identity, $\{q_1, \{y_i, y_j\}\} = 0$, $\{p_1, \{y_i, y_j\}\} = 0$. (**)

Thus in the coordinates $(q_1, p_1, y_1, \ldots, y_{n-2})$, Π has the form $\Pi = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \frac{1}{2} \sum_{ij} \phi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$, where ϕ_{ij} only depends on y by equations (**).

If $k = 1$ we are done. Otherwise repeat the construction. $\square$

Consequences of the splitting theorem: Let $x \in M$ and $U = \{(q_i, p_j, y_l)\}$ be a splitting chart around x. Then we can endow smooth structure on the leaves $L = \{y = 0\}$ by defining its tangent space to be $TL = D|_L = Span\{\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_j}\}$.

Furthermore, from the form of Π we see that the leaves are naturally equipped with symplectic structures induced from the Poisson structure on M. Thus we get in general singular symplectic folation on M.

Example 2.6 on $\mathbb{R}^2 = \{(x,y)\}$, take $\Pi = f(x,y)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, for any smooth f. As concrete examples, take

(a) $f = x^2 + y^2$, and

(b) $f = x$.

The leaves have to be 0 or 2 dimensional since they are symplectic. And in fact the 0 dimensional leaves correspond exactly to the zero locus of the function f. Thus in case (a) we get a 0-dimensional leaf at the origin, and the rest of $\mathbb{R}^2$ forms a single 2-dimensional leaf, whereas in case (b) we have 0-dimensional leaves along the y-axis and two 2-dimensional leaves taking the left and right open half-planes.

Example 2.7 let $M = \mathfrak{g}^*$ with the Lie Poisson structure. Then the symplectic distribution $D := \Pi^\#(T^*M)$ is generated at each point by the Hamiltonian vector fields of linear functions on $\mathfrak{g}^*$. Viewing $u, v \in \mathfrak{g} \hookrightarrow \mathcal{C}^\infty(\mathfrak{g}^*)$, we have

$X_u v(\mu) = \langle \mu, [u,v] \rangle = \langle ad_u^*(\mu), v \rangle$ according to the definition of Lie Poisson structure. But ad_u^* is the infinitesimal generator of the coadjoint action of G on $\mathfrak{g}^*$. Since v is a linear function on $\mathfrak{g}^*$, we have $X_u v(\mu) = \frac{d}{dt}|_{t=0} v(\mu + tX_u) = v(X_u) = \langle X_u, v \rangle$. So basically we have $X_u(\mu) = ad_u^*(\mu) = \frac{d}{dt}|_{t=0} Ad^*_{exp(tu)}(\mu)$, showing that X_u is tangent to the coadjoint orbit. Thus the symplectic leaves are the coadjoint orbits.

Exercise: let $W = \mathrm{im}(\pi^{\#}) = \pi^{\#}(V^*) \subset V$. Show there exists a nondegenerate skew symmetric bilinear form $\omega \in \Lambda^2 W^*$ defined by: for $u, v \in W$, $\omega(u,v) := \pi(\alpha,\beta)$ where $\pi^{\#}(\alpha) = u$, $\pi^{\#}(\beta) = v$ for arbitrary choice of $\alpha, \beta \in V^*$. Show this is well-defined. As a consequence, $Im\pi^{\#}$ is even dimensional.
Conversely, given a vector space V, the data (W,ω) where $W \subset V$, $\omega \in \Lambda^2 W^*$ nondegenerate defines a Poisson structure. Moreover, π is uniquely determined by (W,ω).
Conclusion: For a vector space V, there is a 1-1 correspondence between constant Poisson structure and collection of pairs (W,ω) where $W \subset V$ is a subspace and ω is a symplectic form on W.
Solution:
suppose $\pi^{\#}(\alpha') = u$ also. Then $\pi^{\#}(\alpha - \alpha') = 0$ and $\pi(\alpha - \alpha', \beta) = 0$ for all β; similarly for β. This proves ω is well-defined.
Next we show it's nondegenerate. So suppose $\omega(u,v) = 0$ for all $v \in W$, then $\pi(\alpha,\beta) = 0$ for all $\beta \in W^*$. But this means $\pi^{\#}(\alpha) = 0$ which equals u by definition.
Conversely, given the data (W,ω) we define a Poisson structure on V as follows. Let $\alpha, \beta \in V^*$, then they restricts to elements in W^*, which we call $\bar{\alpha}, \bar{\beta} \in W^*$. There we can take $\bar{\alpha}^b, \bar{\beta}^b \in W$, i.e., the vectors corresponding to the covectors α, β under the symplectic isomorphism $\omega : W \to W^*$. Now define $\pi(\alpha,\beta) := \omega(\bar{\alpha}^b, \bar{\beta}^b)$, and finally for functions $f, g \in \mathcal{C}^\infty(V)$, define $\{f,g\}(x) = \pi(df_x, dg_x)$. Clearly the structure is constant, because π is independent of x. It is well-defined since we did not make any choice in the definition. Antisymmetry and linearity come for free. The Leibniz rule is a consequence of the Leibniz rule for differential of functions, i.e., $\{f, gh\} = \pi(df_x, h(x)dg_x + g(x)dh_x) = h(x)\pi(df_x, dg_x) + g(x)\pi(df_x, dh_x) = h\{f,g\} + g\{f,h\}$. The Jacobi identity follows from the vanishing of Jacobiator on every triple of standard basis covectors in V^*. But since π is a constant bivector on V, $d(\pi(e_i^*, e_j^*)) = 0$, hence each term of the Jacobiator vanishes.
Now we show that the form defined on W from π coincides with ω if π is constructed from ω by the above procedure. Define $\omega'(u,v) := \pi(\alpha,\beta)$ with $\pi^{\#}(\alpha) = u$ and $\pi^{\#}(\beta) = v$ as before. Then $\pi(\alpha, beta) = \omega(\bar{\alpha}^b, \bar{\beta}^b)$. Furthermore, every $u \in W$ can be written in the form $u = \bar{\alpha}^b$ for some $\alpha \in W^*$. So it suffices to check $\omega'(\bar{\alpha}^b, \bar{\beta}^b) = \omega(\bar{\alpha}^b, \bar{\beta}^b)$, which further reduces to checking $\pi^{\#}(\alpha) = \bar{\alpha}^b$.
By definition, $\omega(\bar{\alpha}^b, v) = \alpha(v)$ for $v \in W$. If $v^{\#} = i_v(\omega)$ and $\tilde{v^{\#}}$ is any extension of $v^{\#} \in W^*$ to V^*, then we have $\omega(\pi^{\#}(\alpha), v) = v^{\#}(\pi^{\#}(\alpha)) = \pi(\alpha, \tilde{v^{\#}}) = \omega(\bar{\alpha}^b, \bar{\bar{\tilde{v^{\#}}}}^b) = \omega(\bar{\alpha}^b, v) = \bar{\alpha}(v)$, where for the second to the last equality, we used the fact that extension followed by restriction is the identity W^* and $\#$ followed by ^{b} is the identity on W. Therefore $\omega(\bar{\alpha}^b, v) = \omega(\pi^{\#}(\alpha), v)$ for all $v \in W$. Since ω is nondegenerate, $\bar{\alpha}^b = \pi^{\#}\alpha$.
Finally, $\omega'(\bar{\alpha}^b, \bar{\beta}^b) = \omega'(\pi^{\#}(\alpha), \pi^{\#}(\beta)) = \pi(\alpha,\beta) = \omega(\bar{\alpha}^b, \bar{\beta}^b)$, as desired.
Given (V,π) a constant Poisson structure, define ω on $W = \pi^{\#}(V)$ as before. We show $\pi' \in \Lambda^2(V^*)$ defined by $\pi'(\alpha,\beta) = \omega(\bar{\alpha}^b, \bar{\beta}^b)$ coincides with π. Again it suffices to show $\pi^{\#}(\alpha) = \bar{\alpha}^b$, but this follows from the computation above. So there is a 1-1 correspondence between constant Poisson structures on V and symplectic subspace $(W,\omega) \subset V$.

3 Symplectic leaves

Let (M, π) be a poisson manifold and $D = \pi^\sharp(T^*M) \subseteq TM$ be the associated vector subbundle, then we define a equivalence relation R on Mas follows:

$$xRy \Longleftrightarrow \exists X_1, X_2, ..., X_r \in \Gamma(D) \ s.t. \ y = \phi^{t_1}_{X_1} \circ ... \circ \phi^{t_r}_{X_r}(x) \tag{3.1}$$

We define leaves (set theoretically) as the equivalence classes of this relation.

Definition 3.1 *A plaque of a leaf $\mathcal{L}$ in a neighborhood U is a connected component of $\mathcal{L}$ in U.*

Lemma 3.2 *If $\mathcal{L}$ is a leaf of a poisson manifold (M, π) as above, then:*
i)$\mathcal{L}$ has a unique C^∞ structure of immersed connected submanifold $\mathcal{L} \xrightarrow{i} M$ s.t. $T\mathcal{L} = D|\mathcal{L}$
ii) $\mathcal{L}$ has a natural symplectic structure determined by:

$$\forall f, g \in C^\infty \qquad \{i^*f, i^*g\} = i^*\{f, g\} \tag{3.2}$$

Proof: *Given $x \in \mathcal{L}$ pick $\mathcal{U}_x = S_x \times N_x$ (as in the splitting theorem).*
We have

$$\forall y \in S_x, \pi^\sharp(T^*_y M) = span\{(\partial/\partial q_1)_y ..., (\partial/\partial q_k)_y, (\partial/\partial p_1)_y, ..., (\partial/\partial p_k)_y\} = T_y S_x$$

so $S_x \in \mathcal{L}$.
Take the set $\{open\ sets\ of\ S_x | x \in \mathcal{L}\}$ as basis of the topology of $\mathcal{L}$ (we will show that this topology is 2^{nd} countable) and $\{(S_x, (q_1, ..., q_n, p_1, .., p_n)) | x \in \mathcal{L}\}$ as an atlas on it. This atlas defines a C^∞-structure of a manifold on $\mathcal{L}$ and

$$\forall x \in \mathcal{L}, T_x\mathcal{L} = \mathcal{D}_x$$

Because this holds locally, so $T\mathcal{L} = \mathcal{D}|_{\mathcal{L}}$.

$\{\mathcal{U}_x | x \in m\}$ such that $\mathcal{U}_x$ as in the splitting theorem, is a covering for M.By paracompactness of M we may select a locally finite, hence countable refinement like $\mathcal{A} = \{\mathcal{U}_{x_i}\}_{i \in I}$ where I is countable.

Now choose $x \in \mathcal{L}$ and $\mathcal{U}_x$ as in the splitting theorem. We want to show that the intersection $\mathcal{L} \cap \mathcal{U}_x$ consists of at most countably many plaques of $\mathcal{L}$. By definition $\mathcal{L} \cap \mathcal{U}_x$ consists of the disjoint union of plaques of $\mathcal{L}$. Let P_x be the unique plaque containing x. We define an increasing sequence $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots$ of collections of plaques inductively as follows. First $\mathcal{P}_0 = \{P_x\}$. Next, let $k \geq 1$, and assume $\mathcal{P}_{k-1}$ has been defined. For each $\mathcal{U}_{x_i}$ of $\mathcal{A}$ we denote by $\mathcal{P}(\mathcal{U}_{x_i})$ the collection of plaques of $\mathcal{L}$ in $\mathcal{U}_{x_i}$ that have a point with $\cup\mathcal{P}_{k-1}$ in common. In addition, define $\mathcal{P}_k$ to be the union of the sets $\mathcal{P}(\mathcal{U}_{x_i})$ for $i \in I$. By induction, each set $\mathcal{P}_k$ is at most countable. Therefore, the union $\mathcal{P} = \cup_{k \geq 1}\mathcal{P}_k$ is at most countable. Clearly $\mathcal{L}$ is the union of the sets from $\mathcal{P}$, then the intersection $\mathcal{L} \cap \mathcal{U}_x$ consists of countably many plaques of $\mathcal{L}$. By using this you can find a countable basis for the topology of $\mathcal{L}$.

For showing the uniqueness of the manifold structure on $\mathcal{L}$, let $(\mathcal{U}_x, \varphi)$ be as before, where $\varphi = (q_1, ..., q_n, p_1, ..., p_n, y_1, .., y_k)$ and π_1, π_2 are the projections on frist $2n$ and later k components, respectivly. If $\acute{\mathcal{L}}$ is another structure such that $\acute{i} : \acute{\mathcal{L}} \to M$ is an immersion, then $\acute{i}^{-1}(\mathcal{U}_x)$ is an open neighborhood of x in the new structure. Let $\mathcal{O}$ be the connected component of $\acute{i}^{-1}(\mathcal{U}_x)$ containing x. Since $\mathcal{L} \cap \mathcal{U}_x$ consists of at most countably many plaques and $\pi \circ \varphi$ is constant on plaques, it follows that the function $\pi_2 \circ \varphi \circ \acute{i}|_{\mathcal{O}}$ has at most countably many values. Since $\mathcal{O}$

is connected, it follows by the lemma below that the function $\pi_2 \circ \varphi \circ \acute{i}|_{\mathcal{O}}$ has a constant value $y \in R$. Hence, $\acute{i}$ maps $\mathcal{O}$ into a fixed plaque containing x, so into S_x.

The function $\pi_1 \circ \varphi \circ \acute{i}|_{\mathcal{O}}$ is an injective immersion from $\mathcal{O}$ into R^{2n}. For dimensional reasons, it follows that $\pi_1 \circ \varphi \circ \acute{i}|_{\mathcal{O}}$ is a diffeomorphism from $\mathcal{O}$ onto an open subset of R^{2n}. We conclude that $\varphi \circ \acute{i}|_{\mathcal{O}}$ is a diffeomorphism from $\mathcal{O}$ onto an open subset of $R^{2n} \times \{y\}$. Hence, $\acute{i}$ is a diffeomorphism from $\mathcal{O}$ onto an open neighborhood of x in S_x. So the map $\acute{i}^{-1} \circ i$ from $\acute{\mathcal{L}}$ to $\mathcal{L}$ must be a diffeomorphism. □

Lemma 3.3 *Let S be a non-empty connected subset of M. If S is at most countable, then S consists of a single point.*

Proof: *For $M = \mathbb{R}$ the result is obvious. Let $f \in C^\infty(M)$. Then $f(S)$ is connected and a at most countable subest of $\mathbb{R}$. Hence $f(S)$ consists of a single value. Since the functions from $C^\infty(M)$ separate the points of M, i.e. for every two points m_1, m_2 you can find f such that $f(m_1) \neq f(m_2)$, the results follows.* □

Example 3.4 As in example 2.6 part (a), for $(x, y) = (0, 0)$ the symplectic leaf is $\mathcal{L} = \{(0, 0)\}$ with the trivial symplectic form and for $(x, y) \neq (0, 0)$ the symplectic leaf is $\mathbb{R}/\{0\}$ with the canonical symplectic form.

Recall: Let G be a Lie group, we have the following actions of G on $\mathfrak{g}$ and $\mathfrak{g}^*$ where $\mathfrak{g}$ is the Lie algebra of G, and $\xi \in \mathfrak{g}, \mu \in \mathfrak{g}^*$

- Adjoint action $(g, \xi) \to Ad_g \xi$ and infinitesimally for $\xi \in \mathfrak{g}$

$$\xi \to \xi_{\mathfrak{g}} \quad where \quad \xi_{\mathfrak{g}}(u) = [\xi, u] \in \mathfrak{g} = T_u^* \mathfrak{g}$$

- Coadjoint action $(g, \mu) \to (Ad_g)^*(\mu)$ and infinitesimally for $\xi \in \mathfrak{g}$

$$\xi \to \xi_{\mathfrak{g}^*} \quad where \quad \xi_{\mathfrak{g}^*}(\mu) \in \mathfrak{g}^* = T_\mu \mathfrak{g}^* \quad s.t. \quad \xi_{\mathfrak{g}^*}(\mu)(u) =< \mu, [u, \xi] >$$

Example 3.5 Let $M = \mathfrak{g}^*$ with the Lie-Poisson bracket on it, i.e. for $f, g \in C^\infty(\mathfrak{g}^*)$

$$\{f, g\}(\mu) =< \mu, [d_\mu f, d_\mu g] >$$

in particular for $u, v \in \mathfrak{g} \subseteq C^\infty(\mathfrak{g}^*)$

$$\{f, g\}(\mu) =< \mu, [u, v] >$$

now for $\mu \in \mathfrak{g}^*$

$$\mathcal{D}_\mu = \{\pi_\mu^\sharp(\xi) | \xi \in \mathfrak{g}\} = \{< \mu, [., \xi] >, \xi \in \mathfrak{g}\} = \{\xi_{\mathfrak{g}^*}(\mu) | \xi \in \mathfrak{g}\}$$

this vector space is the tangent space at the point μ of the coadjoint orbit through μ so the symplectic leaf is a connected component of the coadjoint orbit through μ.

If $\mathcal{O}$ is a coadjoint orbit and $X, Y \in T_\mu \mathcal{O}$ for $\mu \in \mathcal{O}$ then

$$\exists \xi, \xi' \in \mathfrak{g} \;\; s.t \;\; X = \xi_{\mathfrak{g}^*}(\mu) \;\; and \;\; Y = \xi'_{\mathfrak{g}^*}(\mu)$$

and

$$\omega(X, Y) = \omega(\xi_{\mathfrak{g}^*}(\mu), \xi'_{\mathfrak{g}^*}(\mu)) =< \mu, [\xi, \xi'] >$$

This form is called the "KKS"(Kirillov-Kostant-Souriau) form on coadjoint orbits.
As a concrete example take $G = SO(3)$ then we have $(\mathfrak{so}(3), [.,.]) \cong (\mathbb{R}^3, \times)$ by using the following map

$$\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \to (v_1, v_2, v_3)$$

Identify $\mathfrak{so}^*(3)$ with R^3 via the natural pairing given by the Euclidean product. Let

$$A \in SO(3), \hat{\xi}, \hat{\eta} \in \mathfrak{so}(3), \xi, \eta \in \mathbb{R}^3, \hat{\mu} \in \mathfrak{so}^*(3), \mu \in \mathbb{R}^3$$

Then we have $Ad_A(\hat{\xi}) = \widehat{A\xi}$ so

$$< Ad_A^* \hat{\mu}, \hat{\xi} >=< \hat{\mu}, Ad_A \hat{\xi} >=< \hat{\mu}, \hat{A\xi} >= \mu.A\xi = A^T\mu.\xi \Longrightarrow Ad_A^*\hat{\mu} = A^T\mu$$

So the coadjoint orbit at point μ is $\mathcal{O}_\mu = \{A^T\mu | A \in SO(3)\}$ which is the sphere in $\mathbb{R}^3$ with radius $||\mu||$. We have also $[\hat{\xi}, \hat{\eta}] = \xi \times \eta$, so

$$< \hat{\xi}_{\mathfrak{so}^*(3)}(\hat{\mu}), \hat{\eta} >=< \hat{\mu}, [\hat{\xi}, \hat{\eta}] >= \mu.(\xi \times \eta) = (\mu \times \xi).\eta$$

so

$$T_\mu \mathcal{O}_\mu = \{\hat{\xi}_{\mathfrak{so}^*(3)}(\hat{\mu}) | \xi \in R^3\} = \{\xi \times \mu | \xi \in \mathbb{R}^3\}$$

Then for $X, Y \in T_\mu\mathcal{O}_\mu$ we have

$$\omega_\mu(X, Y) = \omega_\mu(\xi \times \mu, \eta \times \mu) = \mu.(\xi \times \eta)$$

Definition 3.6 *Let $(M, \{.,.\})$ be a poisson manifold, $f \in C^\infty(M)$ such that $\{f, .\} \equiv 0$ is called Casimir function.*

Exercise: *Check that if f is a casimir function, then the restriction of f on every symplectic leaf is a constant function.*
Solution: Let $x \in \mathcal{L}$ and $\mathcal{U}_x = S_x \times N_x$. Since $\Gamma(D) = span\{X_{q^1}, ..., X_{q^k}, X_{p^1}, ..., X_{p^k}\}$ on S_x, then $\{f, q^i\} = \{f, p^i\} = 0$ means that f is constant on S_x and since $\mathcal{L}$ is connected, f is constant on $\mathcal{L}$.

Exercise: *If $\{\mu_1, \mu_2, \mu_3\}$ is the standard basis for $\mathfrak{so}(3) \cong \mathbb{R}^3$, we can view μ_i as a function on $\mathfrak{g}$ also, which sends every element to the ith component. Then show that $\mu_1^2 + \mu_2^2 + \mu_3^2$ is casimir. We have seen that the symplectic leaves are the level sets of this function.*
Solution: Since $(df)_\alpha \in \mathfrak{g}$ for every $\alpha \in \mathfrak{g}$, it is enough to show that $< [\Sigma_{i=1}^3 \mu_i(\alpha).\mu_i, \mu_j], \alpha >= 0$ for $j = 1, 2, 3$ and $\alpha \in \mathfrak{g}$. Let $j = 1$

$$\begin{aligned} < [\Sigma_{i=1}^3 \mu_i(\alpha).\mu_i, \mu_1], \alpha > &=< \mu_1(\alpha)[\mu_1, \mu_1] + \mu_2(\alpha)[\mu_2, \mu_1] + \mu_3(\alpha)[\mu_3, \mu_1], \alpha > \\ &=< \mu_1(\alpha).o - \mu_2(\alpha)\mu_3 + \mu_3(\alpha)\mu_2, \alpha > \\ &= -\mu_2(\alpha).\mu_3(\alpha) + \mu_3(\alpha).\mu_2(\alpha) = 0 \end{aligned}$$

and the similar calculation for $j = 2, 3$.

Definition 3.7 *Let (M, π_M) and (N, π_N) be poisson manifolds, then $\Psi : M \to N$ is a poisson map if it preserves the poisson bracket i.e.*

$$\{\Psi^* f, \Psi^* g\}_M = \Psi^*\{f, g\}_N \ \ where \ \ f, g \in C^\infty(N) \tag{3.3}$$

Recall: Let (S, ω) be a symplectic manifold along with a symplectic, free and proper action of a lie group G. As we saw before $M = S/G$ is a poisson manifold and $P : S \to M/G$ is a poisson map. We denote the associated infinitesimal action by $\rho : \mathfrak{g} \to \chi(S)$.

The action is Hamiltonian if there exists a momentum map $J : S \to \mathfrak{g}^*$ such that:
1)$\rho(v) = X_{J_v}$ where $J_v(x) =< J(x), v >$ and X_{J_v} is the Hamiltonian vector field associated to J_v
2)J is $G-$equivariant that is $J(gx) = Ad^*_g J(x)$

Lemma 3.8 *J is a poisson map.*

PROOF: *Let $\{x_1, ..., x_n, y_1, ..., y_n\}$ be a canonical chart for S at point $x \in S$ and $\{v_1, ..., v_k\}$ be a basis for $\mathfrak{g}$. View v_i as a coordinate function on $\mathfrak{g}^*$ such that $v_i(\mu) =< \mu, v_i >$. For every $V, W \in \mathfrak{g}$ we have*

$$\begin{aligned}
\{J^*V, J^*W\}(x) &= \{J_V, J_W\}(x) \\
&= X_{J_V}(J_W)(x) \\
&= \rho(V)(J_W)(x) \\
&= \frac{d}{dt}|_{t=0} < J(\exp(tV).x), W > \\
&= \frac{d}{dt}|_{t=0} < Ad^*_{\exp tx} J(x), W > \\
&=< J(x), \frac{d}{dt}|_{t=0} Ad_{\exp tx} W > \\
&=< J(x), [V, W] > \\
&= J^*(\{V, W\})
\end{aligned}$$

Now for $f, g : \mathfrak{g}^* \to R$ we have (all the calculations are pointwise)

$$
\begin{aligned}
\{J^*f, J^*g\} &= \{f \circ J, g \circ J\} \\
&= \sum_i (\partial f \circ J/\partial x_i \cdot \partial g \circ J/\partial y_i - \partial g \circ J/\partial x_i \cdot \partial f \circ J/\partial y_i) \\
&= \sum_i (\sum_j \partial f/\partial v^j \cdot \partial J_{v_j}/\partial x_i \cdot \sum_k \partial g/\partial v^k \cdot \partial J_{v_k}/\partial y_i \\
&- \sum_j \partial g/\partial v^j \cdot \partial J_{v_j}/\partial x_i \cdot \sum_k \partial f/\partial v^k \cdot \partial J_{v_k}/\partial y_i) \\
&= \sum_{k,j} \partial f/\partial v^j \cdot \partial g/\partial v^k (\sum_i \partial J_{v_j}/\partial x_i \cdot \partial J_{v_k}/\partial y_i - \partial J_{v_j}/\partial y_i \cdot \partial J_{v_k}/\partial x_i) \\
&= \sum_{k,j} \partial f/\partial v^j \cdot \partial g/\partial v^k \cdot \{J_{v_j}, J_{v_k}\} \\
&= \sum_{k,j} \partial f/\partial v^j \cdot \partial g/\partial v^k < J, [v_j, v_k] > \\
&=< J, [\sum_j \partial f/\partial v^j \cdot v_j, \sum \partial f/\partial v^k \cdot v_k] > \\
&=< J, [df, dg] >= J^*\{f, g\}
\end{aligned}
$$

□

Remark: If we have a Hamiltonian action of G on (s, ω) and μ is a regular value for the momentum map J, i.e. $J^{-1}(\mu)$ is a submanifold of S, then $J^{-1}(\mu)$ is G_μ-invariant, where $G_\mu = \{g \in G, Ad_g(\mu) = \mu\}$.

Theorem 3.9 *(Marsden-Weinstein) If the G_μ-action on $J^{-1}(\mu)$ is free and proper then $M_\mu = J^{-1}(\mu)/G_\mu$ has a natural symplectic structure ω_μ, defined by*

$$
P^*\omega_\mu = i^*\omega \tag{3.4}
$$

$$
\begin{array}{ccc}
J^{-1}(\mu) & \xrightarrow{i} & S \\
P \downarrow & & \\
M_\mu & &
\end{array}
$$

where $i : J^{-1}(\mu) \to S$ is the inclusion map and $P : J^{-1}(\mu) \to M_\mu$ is the projection map.

Proof: *Let $x \in J^{-1}(\mu)$ such that $J(x) = \mu$. We want to show that*

$$
T_x J^{-1}(\mu) = (T_x(G.x))^\omega
$$

By definition we have $J_X(x) =< J(x), X. >$ for $x \in S$ and $X \in \mathfrak{g}$. So by using $i_{X_S}\omega = -dJ_X$ we have

$$
\begin{aligned}
T_x J^{-1}(\mu) &= ker(T_x J : T_x S \to \mathfrak{g}^*) \\
&= \{v \in T_x S | < (dJ_X)_x, X >= 0, \forall X \in \mathfrak{g}\} \\
&= \{v \in T_x S | \omega_x(X_S(x), v) = 0, \forall X \in \mathfrak{g}\} \\
&= \{X_S(x) | X \in \mathfrak{g}\}^\omega = (T_x(G.x))^\omega
\end{aligned}
$$

Now we have

$$ker(i^*\omega)_x = T_xJ^{-1}(\mu) \cap (T_xJ^{-1}(\mu))^\omega$$
$$= T_xJ^{-1}(\mu) \cap T_x(G.x)$$

We want to show that

$$T_x(G_\mu.x) = ker(i^*\omega)$$

Let $v \in ker(i^*\omega)$*, then by the above, there is an* $X \in \mathfrak{g}$ *such that*

$$v = X_S(x)$$

and

$$T_xJ(X_S(x)) = 0$$

Because J *is equivariant, we have*

$$T_xJ(X_S(x) = \frac{d}{dt}|_{t=0}J(\exp(tX).x)$$

$$= \frac{d}{dt}|_{t=0}Ad^*(\exp(tX).\mu) = X_{\mathfrak{g}^*}(\mu) = 0$$

Hence

$$ker(i^*\omega)_x = \{X_S(x) \in T_xS | X_{\mathfrak{g}^*}(\mu) = 0\}$$

and by using Lie group theory, we get our claim. So the 2-form ω_μ *on* M_μ *induced by* $i^*\omega$ *is nondegenrate.* □

Lemma 3.10 *Suppose that the action of* G *on* (S, ω) *is free and proper, then* S/G *is a poisson manifold, and* $\pi : S \to M = S/G$ *is a poisson map. If the action is Hamiltonian then*
1) J *is a submersion.*
2) $\pi(J^{-1}(\mu)) = J^{-1}(\mu)/G_\mu$
3)$J^{-1}(\mu)/G_\mu$ *is a leaf of* S/G *with the same symplectic structure.* Proof: *(1)Let* $x \in S$ *be given. We claim that* T_xJ *is surjective if and only if the linear map*

$$\rho : \mathfrak{g} \to T_xS$$

$$X \to X_S(x)$$

is injective. Since the action is free, cleary this map is injective. So if we prove our claim, the proof will complete. Let $X \in \mathfrak{g}$*. We will show that the implication*

$$X_S(x) = 0 \to X = 0 \tag{3.5}$$

holds if and only if the tangent map T_xJ *is surjective. Indeed, because the form* ω *is nondegenerate, we have*

$$X_S(x) = 0$$

if and only if

$$\omega(X_S(x), v) = 0$$

for all $v \in T_xS$*. So*

$$(dJ_X)_x = 0$$

consider the linear map

$$i_X : \mathfrak{g}^* \to \mathbb{R}$$

$$i_X(\eta) :=< \eta, X >$$

then

$$(dJ_X)_x = (d(i_X \circ J))_x = i_X \circ T_x J$$

Hence $(dJ_X)_x = 0$ *iff* i_X *is zero on the image of* T_xJ*. Therefore, the implication (3.5) holds for all* $X \in \mathfrak{g}$ *iff* T_xJ *is surjective. (2) Let* $x, y \in J^{-1}(\mu)$ *and* $[x]_G = [y]_G$ *then* $\exists g \in G$ *such that* $x = g \cdot y$*. Then*

$$\mu = J(x) = J(g \cdot y) = Ad_g J(y) = Ad_g(\mu) \Rightarrow g \in G_\mu$$

so $[x]_{G_\mu} = [y]_{G_\mu}$*. Therefore, the map* $[x]_{G_\mu} \mapsto [x]_G$ *is injective and since it is clearly surjective we have the claim. (3) For every* $x \in S$ *we have the following diagram, where* Λ *is the induced poisson bivector.*

$$\begin{array}{ccc} T_x^*S & \xrightarrow{\omega^{-1}} & T_xS \\ \pi_x^* \uparrow & & \downarrow T_x\pi \\ T_{\pi(x)}^*S/G & \xrightarrow{\Lambda^\sharp} & T_{\pi(x)}S/G \end{array} \tag{3.6}$$

Let $\alpha \in T^*_{\pi(x)}S/G$*, for every* $v \in T_x\{G.x\}$ *we have* $\pi^*(\alpha(v)) = 0$ *so* $\omega^{-1}(\pi^*(\alpha)) \in (T_x\{G.x\})^\omega$*. Since* $T_xJ^{-1}(\mu) = (T_x\{G.x\})^\omega$ *so* $T_x\pi(\omega^{-1}(\pi^*(\alpha))) \in T_{\pi(x)}J^{-1}(\mu)/G_\mu$ *and for every* $w \in T_{\pi(x)}J^{-1}(\mu)/G_\mu$ *you can come back and find a* v *so* $\Lambda^\sharp(T^*_{\pi(x)}S/G) = T_{\pi(x)}J^{-1}(\mu)/G_\mu$*. Since* π *and* P *act similarly on* $J^{-1}(\mu)$*, the symplectic structures are the same also.* $\square$