# Poisson geometry: lectures 4, 5, 6

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May 17, 2014

**Abstract**

In these lectures we discuss transversal Poisson structures. Our aim is to relate the transversals of different points in the same symplectic leaf. For that we introduce the notions of Dirac, almost Dirac structures and the Gauge transformations. Finally, we state the question of the symplectic realization of a Poisson manifold and give some examples.

## Lecture 4

## 1 Submanifolds in Poisson geometry

Let $(M, \pi)$ be a Poisson manifold.

**Definition 1.1** *A transversal of $(M, \pi)$ through $x$ is a submanifold $N \subset M$ such that $T_x(M) = T_x(N) \oplus T_x(S)$, where $S$ is the symplectic leaf through $x$.*

Consider Poisson manifolds $(P, \pi)$ together with a point $x \in P$ where $\pi$ vanishes.

**Definition 1.2** *Two such triples $(P_1, \pi_1, x_1)$ and $(P_2, \pi_2, x_2)$ are called isomorphic if there exist open subsets $U_1 \in P_1$ and $U_2 \in P_2$ containing $x_1$ and $x_2$, respectively, and a Poisson diffeomorphism $\phi : U_1 \to U_2$, such that $\phi(x_1) = x_2$.*

A germ of a Poisson manifold is an equivalence class with respect to isomorphisms.

**Definition 1.3** *Let $N \subset M$ be a submanifold. Define*

$$(T_xN)^{\perp} = \pi^{\sharp}((T_xM/T_xN)^*) \subset T_xM.$$

*We say that $N$ is of constant rank in $(M, \pi)$ if the dimension of $(T_xN)^{\perp}$ does not depend on $x$.*

**Remarks**. 1. If $(M, \pi)$ is a symplectic manifold with $\pi = -\omega^{-1}$ and $N$ is a submanifold of $M$, then

$$(T_xN)^{\perp} = \{x \in T_xM : \omega(x, y) = 0 \ \forall y \in T_xN\}.$$

2. $(T_xN)^{\perp}$ is the usual symplectic orthogonal of $T_xN \cap T_xS$ inside the symplectic space $T_xS$.

**Definition 1.4** *A submanifold $N \subset M$ is called cosymplectic in $(M, \pi)$ if $T_xM = T_xN + (T_xN)^{\perp}$ for all $x \in N$.*

**Lemma 1.5** *For each point $x \in M$, there is at least one cosymplectic transversal that passes through $x$. More precisely, for each transversal $\tilde{N}$ through $x$, there exists an open subspace $N \subset \tilde{N}$ containing $x$ such that $N$ is a cosymplectic transversal.*

**Proposition 1.6** *If $N$ is cosymplectic in $(M, \pi)$, then $T_xM = T_xN \oplus (T_xN)^\perp$ for all $x \in N$. In particular, $N$ is of constant rank.*

PROOF: Consider the following sequence

$$K \to (T_xM/T_xN)^* \xrightarrow{\pi^\sharp} (T_xN)^\perp,$$

where $K$ is the kernel of $\pi^\sharp$ and the left arrow is the natural embedding. We have

$$\dim(M) - \dim(N) = \dim(T_xN)^\perp + \dim(K),$$

and so

$$\dim(M) = \dim(T_xN)^\perp + \dim(K) + \dim(T_xN) \geq \dim(T_xN) + \dim(T_xN)^\perp \geq \dim(T_xN + (T_xN)^\perp).$$

However, $M = T_xN + (T_xN)^\perp$, which implies that

$$\dim(T_xN) + \dim(T_xN)^\perp = \dim(T_xN + (T_xN)^\perp),$$

and thus the sum is direct. □

Now we prove Lemma 1.5.

PROOF: We have

$$T_x\tilde{N} + T_xS_x = T_xM, \tag{1.1}$$

where $S_x$ is the symplectic leaf through $x$. We must prove that

$$T_y\tilde{N} + (T_y\tilde{N})^\perp = T_yM \tag{1.2}$$

for all $y$ in a sufficiently small neighborhood of $x$ in $\tilde{N}$. At the point $y = x$, equation (1.2) follows from equation (1.1). On the other hand, condition (1.2) is an open condition, so (1.2) holds for every $y$ in a neighborhood of $x$. □

**Definition 1.7** *A submanifold $N$ of a Poisson manifold $(M, \pi)$ satisfies the Poisson–Dirac condition at $x$ ($PD_x$) if $T_xN \cap (T_xN)^\perp = \{0\}$.*

**Remarks.** 1. A cosymplectic subspace satisfies $PD_x$ at every $x \in N$.

2. There exist submanifolds that satisfy $PD_x$ but are not of constant rank.

3. $PD_x$ insures that there exists an induced bivector $\pi_{N,x} \in \wedge^2 T_xN$ defined as follows. For each $\xi$ in the cotangent space $T^*_xN$, we consider its extensions: $\tilde{\xi} \in T^*_xM$ such that $\tilde{\xi}|_{T^*_xN} = \xi$. We call an extension good if $\tilde{\xi}|_{(T^*_xN)^\perp} = 0$. Since $T_xN \cap (T_xN)^\perp = \{0\}$, every $\xi$ has a good extension. Define $\pi_{N,x}$ as $\pi_{N,x}(\xi, \eta) = \pi_x(\tilde{\xi}, \tilde{\eta})$ for $\xi, \eta \in T^*_xN$. To prove that $\pi_{N,x}$ is well defined, we must show that if $\bar{\xi}$ and $\hat{\xi}$ are two good extensions of $\xi$, then $\pi(\bar{\xi}, \tilde{\eta}) = \pi(\hat{\xi}, \tilde{\eta})$. Denote $\tilde{\xi} = \bar{\xi} - \hat{\xi}$. We have

$$\tilde{\xi}|_{T^*_xN} = 0, \tag{1.3}$$

$$\tilde{\eta}|_{(T^*_xN)^\perp} = 0. \tag{1.4}$$

We must show that $\pi(\tilde{\xi}, \tilde{\eta}) = 0$. From equation (1.3) it follows that $\pi^\sharp(\tilde{\eta}) \in T^*_xN$. We have $\pi(\tilde{\xi}, \tilde{\eta}) = \langle \tilde{\xi}, \pi^\sharp(\tilde{\eta}) \rangle$. Together with (1.4) this gives $\pi(\tilde{\xi}, \tilde{\eta}) = 0$.

**Definition 1.8** *A submanifold $N \subset M$ is called a Poisson–Dirac submanifold of $M$ if*
*(i) $PD_x$ holds for all $x \in N$,*
*(ii) the induced bivectors $\{\pi_{N,x}\}_{x \in N}$ define a smooth bivector $\pi_N \in \Gamma(\wedge^2 TN)$.*

**Proposition 1.9** *If $N$ is a Poisson–Dirac submanifold of $(M, \pi)$, then $(N, \pi_N)$ is a Poisson manifold.*

The proof of this proposition will be given later.

**Proposition 1.10** *If $N$ satisfies $PD_x$ at all $x \in N$ and $N$ is of constant rank, then $N$ is a Poisson–Dirac submanifold.*

Proof: Denote $(TM)|_N$ by $T_N M$. We have

$$T_N M \supseteq TN \oplus (TN)^\perp.$$

We can find a compliment of the vector subbundle $(TN)^\perp$, i.e., a projection $p : T_N M \to TN$ such that
(i) $p|_{TN} = \mathrm{Id}$,
(ii) $p|_{(TN)^\perp} = 0$.

Then the map $\wedge^2 p : \wedge^2 T_N M \to \wedge^2 TN$ takes $\pi$ to $\pi_N$, i.e., $\pi_N(\xi, \eta) = \pi(\xi \circ p, \eta \circ p)$ for all $\xi$ and $\eta$. Both $\pi$ and $p$ are smooth, so $\pi_N$ is smooth. Thus $N$ is a Poisson–Dirac submanifold. □

**Corollary 1.11** *Cosymplectic submanifolds are Poisson–Dirac submanifolds.*

## 2 Dirac geometry

Let $M$ be a manifold of dimension $n$. We introduce the following objects.

- Generalized tangent bundle $\mathcal{T}M = TM \oplus T^*M$,
- Nondegenerate symmetric bilinear pairing $\langle ., . \rangle$ defined as

$$\langle (x, \xi), (x', \xi') \rangle = \xi'(x) + \xi(x'),$$

- Bracket $[., .]$ on $\Gamma(\mathcal{T}M)$ defined as

$$[(x, \xi), (x', \xi')] = ([x, x'], \mathcal{L}_x(\xi') - \mathcal{L}_{x'}(\xi) + d(\xi(x'))).$$

**Definition 2.1** *An almost Dirac structure on $M$ is a subbundle $L \subset \mathcal{T}M$ of rank $n$ such that $\langle \alpha, \beta \rangle = 0$ for all $\alpha, \beta \in \Gamma(L)$.*

*An almost Dirac structure on $M$ is called a Dirac structure if $[\alpha, \beta] \in \Gamma(L)$ for all $\alpha, \beta \in \Gamma(L)$.*

**Example 2.2 (Closed 2-form type)** Let $\omega$ be a 2-form on $M$. Then $L_\omega = \{(X, \iota_X \omega) : X \in TM\} \subseteq \mathcal{T}M$ is an almost Dirac structure.

**Claim 2.3** *This structure is Dirac if $d\omega = 0$.*

Proof: Let $\alpha = (x, \iota_x\omega)$ and $\beta = (y, \iota_y\omega)$ be two sections of $L_\omega$. Consider the bracket of these sections:

$$[\alpha, \beta] = ([x, y], \mathcal{L}_x(\iota_y\omega) - \mathcal{L}_y(\iota_x\omega) + d((\iota_x\omega)(y))).$$

The structure is Dirac if the $T^*M$-part of the right-hand side of the above equation is equal to $\iota_{[x,y]}\omega$. We use Cartan's magic formula to rewrite $d((\iota_x\omega)(y))$ as

$$\begin{aligned} d((\iota_x\omega)(y)) = d(\iota_y\iota_x\omega) = \\ =\mathcal{L}_y(\iota_x\omega) - \iota_y(d\iota_x\omega) = \\ =\mathcal{L}_y(\iota_x\omega) - \iota_y\mathcal{L}_x\omega + \iota_y\iota_x d\omega. \end{aligned}$$

The first term of this expression cancels with the second term in the $T^*M$-part, and we obtain the following condition:

$$\iota_{[x,y]}\omega + \iota_y\iota_x d\omega = \omega([x, y]),$$

which is equivalent to $d\omega = 0$. □

Note that a Dirac structure $L$ on $M$ is of this type if $\mathrm{pr}_1 : L \to TM$ is an isomorphism.

**Example 2.4 (Poisson bivector type)** Let $\pi$ be a bivector on $M$. Then $L_\pi = \{(\pi^\sharp(\xi), \xi) : \xi \in T^*M\}$ is an almost Dirac structure.

**Claim 2.5** *This structure is Dirac if* $[\pi, \pi] = 0$*.*

Proof: Let $\alpha = (\pi^\sharp(\xi), \xi)$ and $\beta = (\pi^\sharp(\eta), \eta)$ be two sections of $L_\pi$. Consider their bracket:

$$[\alpha, \beta] = ([\pi^\sharp(\xi), \pi^\sharp(\eta)], \mathcal{L}_{\pi^\sharp(\xi)}(\eta) - \mathcal{L}_{\pi^\sharp(\eta)}(\xi)) + d(\xi(\pi^\sharp(\eta))) = ([\pi^\sharp(\xi), \pi^\sharp(\eta)], [\xi, \eta]_\pi).$$

The structure is Dirac if $\pi^\sharp[\xi, \eta]_\pi = [\pi^\sharp(\xi), \pi^\sharp(\eta)]$, i.e., if $\pi^\sharp$ preserves brackets which is equivalent to the desired condition. □

Note that a Dirac structure $L$ on $M$ is of this type if $\mathrm{pr}_2 : L \to T^*M$ is an isomorphism.

Let $L$ be a Dirac structure on $M$, and let $N$ be a submanifold of $M$. Then

$$(L|_N)_x = \{(X_x, \xi_x) \in T_xN \oplus T_x^*N : \exists (X_x, \tilde{\xi}_x) \in L, \tilde{\xi}|_{T_xN} = \xi\}. \tag{2.1}$$

**Definition 2.6** *A submanifold $N$ of a Dirac manifold $(M, L)$ is called a Dirac submanifold of $(M, L)$ if $L|_N$ is smooth.*

**Proposition 2.7** *Let $(M, \pi)$ be a Poisson manifold and $L_\pi$ the associated Dirac structure. Let $N \subset M$. Then $N$ is a Poisson–Dirac submanifold of $(M, \pi)$ if and only if $N$ is a Dirac submanifold of $(M, L_\pi)$ and $(L_\pi)|_N$ is of bivector type.*

Proof: From (2.1) we see that

$$(L_\pi)|N = \{(\pi^\sharp(\tilde{\xi}), \tilde{\xi}|_{T_xN}) : \tilde{\xi} \in T_x^*M, \tilde{\xi}|_{(T_xN)^\perp} = 0\}$$

is a Dirac structure on $N$.

We know that $(L_\pi)|_N$ is of bivector type if and only if $pr_2 : (L_\pi)|_N \to T^*N$ is an isomorphism (or, equivalently, if $pr_2$ is injective, because $(L_\pi)|_N$ and $T^*N$ have the same dimension). However,

$$\ker(pr_2) = \{(\pi^\sharp(\tilde{\xi}), \tilde{\xi}|_{TN}) : \tilde{\xi}|_{TN} = 0, \tilde{\xi}|_{(TN)^\perp} = 0\}.$$

Thus, the map $pr_2$ is injective if and only if $\pi^\sharp(\tilde{\xi}) = 0$ for all $\tilde{\xi}$ such that $\tilde{\xi}|_{TN} = 0$ and $\tilde{\xi}|_{(TN)^\perp} = 0$, which is equivalent to the condition $TN \cap (TN)^\perp = \{0\}$. □

**Exercise:** (i) Let $L$ be a Dirac structure on $M$. We consider its first projection $\mathcal{S} = \mathrm{pr}_1(L) \subset TM$. We define leaves of $L$ as leaves of $\mathcal{S}$, i.e., maximal integrals of $\mathcal{S}$. Show the existence of leaves.

**Solution:** Direct calculation shows that $(L, \mathrm{pr}_1, [.,.])$ is a Lie algebroid. Using Theorem 2.3.4 [1] we conclude that $L$ is integrable and $\mathrm{pr}_1(L)$ generates a singular foliation of $M$. This foliation is called the presymplectic foliation of $L$.

**Exercise:** (ii) Let $S$ be a presymplectic leaf of $L$. We define $\omega_S$ in the following way. For every $x \in S$ and for every $X_x, Y_x \in T_xS = \mathcal{S}_x$ we choose $\xi_x, \eta_x \in T^*M$ such that $(X_x, \xi_x), (Y_x, \eta_x) \in L$. We define

$$\omega_S(X_x, Y_x) = -\xi_x(Y_x) = \eta_x(X_x).$$

Show that $\omega_S$ is a well-defined closed 2-form on $S$. Check that if $L$ is of Poisson bivector type, then the presymplectic leaves of $L$ coincide with the symplectic leaves of $\pi$ and for such a leaf $S$, the form $\omega_S$ is the symplectic form.

**Solution:** Let us choose $\tilde{\xi}_x, \tilde{\eta}_x \in T^*M$ such that $(X_x, \tilde{\xi}_x), (Y_x, \tilde{\eta}_x) \in L$. Since $L$ is a Dirac structure, the pairing $< .,. >$ vanishes. Applying this to the pairs $(X_x, \xi_x), (Y_x, \eta_x)$ and $(X_x, \tilde{\xi}_x), (Y_x, \tilde{\eta}_x)$ we obtain that

$$\xi_x(Y_x) = \tilde{\xi}_x(Y_x) = -\eta_x(X_x) = -\tilde{\eta}_x(X_x),$$

which shows that the form is well defined. Using the explicit formula for the exterior derivative, direct calculation shows that the form is closed.
Suppose $L = L_\pi = \{(\pi^\sharp(\xi), \xi) | \xi \in T^*M\}$. The fact that $\mathrm{pr}_2 : L \to T^*M$ is an isomorphism shows that $\mathrm{pr}_1 = \pi^\sharp(T^*M)$, which is exactly the distribution of the Poisson structure $\pi$.
We have

$$\omega_S(X_x, Y_x) = -\xi_x(Y_x) = -\xi_x(\pi^\sharp(\eta_x)) = -\pi(\eta_x, \xi_x) = \pi(\xi_x, \eta_x),$$

which is exactly the symplectic form induced by $\pi$.

**Exercise:** (iii) Show that two Dirac structures coincide if and only if they have the same presymplectic leaves and the same induced 2-forms.

**Solution:** Let $V$ be a vector space, $\dim V = n$. A linear Dirac structure $L \subset V \oplus V^*$ is determined by $\langle \alpha, \beta \rangle = 0$ for all $\alpha, \beta \in L$ and $\dim L = n$. Consider $W = \mathrm{pr}_1(L)$. Define $\theta(v_1, v_2) = -\xi_1(v_2) = \xi_2(v_1)$ for $(v_1, \xi_1), (v_2, \xi_2) \in L$. We see that the form $\theta \in \Omega^2(W)$ is well defined (use part (ii) of the exercise). Note that there is a one to one correspondence between linear Dirac structures and the pairs $(W, \theta)$.
In the general case, suppose we have a Dirac structure $L \subset \mathcal{T} = TM \oplus T^*M$ on a manifold $M$. Let $S$ be a presymplectic leaf. For every $x \in S$ we have the following equality

$$(\omega_S)_x(v_1, v_2) = \theta(v_1, v_2)$$

for each $v_1, v_2 \in W = \mathrm{pr}_1(L_x)$. However, the pair $(W, \theta)$ uniquely determines the linear Dirac structure on $L_x$ which build up an almost Dirac structure on $M$. The closedness of the forms $\omega_S \in \Omega^2(S)$ for every presymplectic leaf $S$ implies the closedness of the bracket defined on the sections of $L$.

Literature: 1. T.J. Courant. *Dirac Manifolds.* Transactions of the American Mathematical Society, Vol. 319, Number 2, June 1990

# Lecture 5

## 3 Main question

Recall that if $(M, \pi)$ is a Poisson manifold and $x \in M$, after restricting to an open neighborhood of $x$ in a transversal $N$, this neighborhood $(N, x)$ becomes a Poisson manifold. We saw this in two ways:

- Using the splitting theorem: in a neighborhood of x the manifold $M$ looks like $N \times \mathbb{R}^{2k}$.
- Intrinsic way: first we assume that N is co-symplectic: $TN + (TN)^\perp = TM$ at all points of $N$. For every $\xi_x, \eta_x \in T^*M$ we define an induced Poisson bivector $\pi_N \in \mathfrak{X}^2(M)$ as $\pi_N(\xi_x, \eta_x) = \pi(\tilde{\xi}_x, \tilde{\eta}_x)$, where we choose $\tilde{\xi}, \tilde{\eta} \in T^*_x M$ such that

$$\begin{cases} \tilde{\xi}_x|_{T_xN} = \xi_x \\ \tilde{\xi}_x|_{(T_xN)^\perp} = 0 \end{cases}, \begin{cases} \tilde{\eta}_x|_{T_xN} = \eta_x \\ \tilde{\eta}_x|_{(T_xN)^\perp} = 0 \end{cases}$$

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The main question is if for two transversals $N, N'$ through $x$ the manifolds $(N, x)$ and $(N', x)$ are Poisson diffeomorphic.

**Remark 3.1** This will imply that for every $x_0, x_1 \in M$ and for every two transversals $N_0$ and $N_1$ through $x_0$ and $x_1$ respectively, if $x_0$ and $x_1$ are in the same symplectic leaf, then $(N_0, x_0)$ and $(N_1, x_1)$ are Poisson diffeomorphic.

Now we can reformulate the main question. Let $(N,\pi)$ be a Poisson manifold and $M = N \times \mathbb{R}^{2k} = (N,\pi) \times (\mathbb{R}^{2k}, \pi_{can})$, where $\pi_{can} = \sum_{i=1}^{k} \partial/\partial x_i \wedge \partial/\partial y_i$ is the canonical 2-form and $(x_1,...,x_n,y_1,...,y_n) \in \mathbb{R}^{2k}$ are the coordinates. Let $N_1 \subset M$ be a cosymplectic transversal of the Poisson manifold $M$. We may assume that $N_1 = \mathrm{Graph}(\phi) = \{(x,\phi(x)|x \in N\}$ for some map $\phi : N \to \mathbb{R}^{2k}$.
Note that we have an inclusion $i_\phi : N \hookrightarrow M$, $x \mapsto (x,\phi(x))$ which identifies $N$ with $i_\phi(N) = N_1$. By the assumption on $N_1$ we have an induced Poisson structure $\pi_{N_1}$ on $N_1$ that we move to $N$ using the identification $i_\phi$. Thus we obtain a Poisson structure on $N$. Let us call it $\pi_\phi$. By construction $(N,\pi_\phi) \cong (N_1,\pi_{N_1})$.

Now we must prove that $(N_1,\pi_{N_1})$ and $(N,\pi_\phi)$ are Poisson diffeomorphic.

**Lemma 3.2** *Let $B = \phi^*(\omega_{can}) \in \Omega^2(N)$. Consider the map $B_\sharp : TN \to T^*N$, $X \mapsto i_X(B)$. Then $N_\phi$ is cosymplectic in $M$ if and only if $(\mathrm{Id}_{TN} - \pi^\sharp \circ B_\sharp) : TN \to TN$ is an isomorphism. Moreover, in this case*

$$\pi_\phi^\sharp = (\mathrm{Id} - \pi^\sharp \circ B_\sharp)^{-1} \circ \pi^\sharp.$$

Proof: We have

$$T_{(x,\phi(x))}(N_\phi) = \{(X_x, (d\phi)_x(X_x)) : X_x \in T_xN\}$$

and

$$(T_{(x,\phi(x))}M / T_{(x,\phi(x))}(N_\phi))^* = \{(\phi^*(\eta), \eta) : \eta \in T^*_{\phi(x)}\mathbb{R}^{2k}\}.$$

This gives

$$(T_{(x,\phi(x))}(N_\phi))^\perp = \{(\pi^\# \phi^*(\eta), \pi^*_{can}(\eta)) : \eta \in T^*_{\phi(x)}\mathbb{R}^{2k}\} =$$
$$= \{(\pi^\# \phi^*(\omega_{can})_\#(V), V) : V \in T_{\phi(x)}\mathbb{R}^{2k}\}.$$

Then

$$TN_\phi \cap (TN_\phi)^\perp = \{(X_x, (d\phi)_x(X_x)) : X_x \in T_xN \text{ such that } X_x = \pi^\# \phi^*(\omega_{can})_\#(d\phi)_x(X_x)\} =$$
$$= \{(X_x, (d\phi)_x(X_x)) : (\mathrm{Id}_{TN} - \pi^\# \circ B_\#)(X_x) = 0\} \cong$$
$$\cong \ker(\mathrm{Id}_{TN} - \pi^\# \circ B_\#)_x.$$

Dimensional count shows that $N_\phi$ is co-symplectic if and only if $(\mathrm{Id}_{TN} - \pi^\# \circ B_\#)$ is an isomorphism. □

# 4 Gauge transformations

**Definition 4.1** *Let $(N,\pi)$ be a Poisson manifold, and $B \in \Omega^2(N)$ be a closed 2-form on $N$. We say that $B$ is a Poisson gauging form for $(N,\pi)$ if $(\mathrm{Id}_{TN} - \pi^\# \circ B_\#) : TN \to TN$ is an isomorphism, where $B_\sharp$ is the map $TN \to T^*N$ such that $B_\sharp(v) = i_vB$.*

**Proposition 4.2** *If $B$ is a Poisson gauging for $(N,\pi)$ then $\pi_B^\# = (Id_{TN} - \pi^\# \circ B_\#)^{-1} \circ \pi^\# : T^*N \to TN$ defines a Poisson structure on $N$ which is called the gauge transformation of $\pi$ with respect to $B$.*

PROOF: We must check the Jacobi identity for $\pi_B^\sharp$. However $\pi_B(\xi,\eta) = \pi(\xi + i_X B, \eta)$, where $X = \pi_B^\sharp(\xi)$. The fact that $[\pi_B, \pi_B] = 0$ gives the desired identity. □

**Explanation 1.** Look at the symplectic leaves of $\pi$ and $\pi_B$.

**Proposition 4.3** *$B$ is a Poisson gauging for $(N,\pi)$ if and only if for each symplectic leaf $(S,\omega_S)$ of $(N,\pi)$, the form $\omega_S + B|_S \in \Omega^2(S)$ is symplectic. Moreover, $\pi_B$ is uniquely determined the following conditions:*

- *$\pi_B$ has the same symplectic leaves as $\pi$.*
- *If $S$ is such a leaf, then the symplectic form induced by $\pi_B$ on $S$ is $\omega_S + B|_S$.*

PROOF: We must show that $\ker(\mathrm{Id}_{TN} - \pi^\# \circ B_\#) = \ker(\omega_S + B|_S)$ at a point $x \in N$.
We have

$$\begin{aligned}
&\ker(\mathrm{Id}_{TN} - \pi^\# \circ B_\#) = \{X : X = \pi^\# \circ B_\# X,\ X \in T_xN\} = \\
=&\{\pi^\#(\xi) : \xi \in T_x^*N,\ \pi^\#\xi = \pi^\# B_\# \pi^\# \xi\} = \\
=&\{\pi^\#\xi : \xi \in T_x^*N,\ \langle \eta, \pi^\#\xi\rangle = \langle \eta, \pi^\# B_\# \pi^\#\xi\rangle\ \forall \eta \in T_x^*N\} = \\
=&\{\pi^\#\xi : \xi \in T_x^*N,\ \pi(\xi,\eta) = -\langle \pi^\#\eta, B_\#\pi^\#\xi\rangle,\ \forall \eta \in T_x^*N\} = \\
=&\{\pi^\#\xi : \xi \in T_x^*N,\ \omega_S(\pi^\#\xi, \pi^\#\eta) = -B(\pi^\#\xi, \pi^\#\eta),\ \forall \eta \in T_x^*N\} = \\
=&\{X : X \in T_xS,\ \omega_S(X,Y) = -B(X,Y),\ \forall Y \in TS\} = \\
=&\ker(\omega_S + B|_S).
\end{aligned}$$

□

**Explanation 2.** Via Dirac structures.
If we have a 2-form $B \in \Omega^2(N)$, then there is a map $\tau_B : \mathcal{T}N \to \mathcal{T}N$, $\tau_B(X,\xi) = (X, \xi - i_X(B))$ such that

- $\tau_B$ respects $\langle .,. \rangle_{\mathcal{T}}$: $\langle \tau_B(a), \tau_B(b)\rangle_{\mathcal{T}} = \langle a, b\rangle_{\mathcal{T}}$, $\forall a, b \in \mathcal{T}N$.
- $B$ is closed if and only if $\tau_B$ preserves $[.,.]_{\mathcal{T}}$, i.e., $[\tau_B(a), \tau_B(b)]_{\mathcal{T}} = \tau_B([a,b])$, for all $a, b \in \Gamma(\mathcal{T}N)$.

Hence, if $B$ is a closed 2-form, then $\tau_B$ preserves $\mathcal{T}N$ with all the relevant structures, which allows us to talk about Dirac structures. Hence for any Dirac structure $L$ on $N$, the structure $\tau_B(L) = \{(X, \xi - i_X(B) : (X,\xi) \in L\}$ is a Dirac structure.
We start with $(N,\pi)$. Consider $L_\pi$. Then we can talk about $\tau_B(L_\pi)$ as a Dirac structure defined for each closed $B \in \Omega^2(N)$. We are interested, when $\tau_B(L_\pi)$ is of Poisson type as well.

**Lemma 4.4** *$\tau_B(L_\pi)$ is of Poisson type if and only if $B$ is a Poisson gauging for $(N,\pi)$. Moreover $\pi_B$ is determined by $\tau_B(L_\pi) = L_B$.*

PROOF: ($\Leftarrow$) Let $B$ be a Poisson gauging for $(N,\pi)$. Then $\mathrm{Id} - \pi^\sharp \circ B_\sharp : TM \to TN$ is an isomorphism. Define $\pi_B^\sharp$ as $\pi_B^\sharp = (\mathrm{Id} - \pi^\sharp \circ B_\sharp)^{-1} \circ \pi^\sharp$. We must show that $\tau_B(L_\pi) = \{(\pi_B^\sharp(\eta), \eta) :\ \eta \in T^*N\}$. We have $\tau_B(L_\pi) = \{(\pi^\sharp(\xi), \xi - i_{\pi^\sharp(\xi)}(B))\}$, so we must show that if

$\eta = \xi - i_{\pi^\sharp(\xi)}(B)$, then $\pi^\sharp(\xi) = \pi^\sharp_B(\eta)$.
We have
$$\pi^\sharp_B(\xi - i_{\pi^\sharp(\xi)}(B)) = (\mathrm{Id} - \pi^\sharp \circ B_\sharp)^{-1}(\pi^\sharp(\xi) - \pi^\sharp(i_{\pi^\sharp(\xi)}B)).$$
Consider the map $\mathrm{Id} - \pi^\sharp \circ B_\sharp$. It maps $X$ to $X - \pi^\sharp(i_X B)$, so
$$\pi^\sharp(\xi) \mapsto \pi^\sharp(i_{\pi^\sharp(\xi)}B).$$
We obtain the desired equality
$$\pi^\sharp_B(\xi - i_{\pi^\sharp(\xi)}) = \pi^\sharp(\xi).$$

($\Rightarrow$) Let $\tau_B(L_\pi)$ be of Poisson type.
We use the exercise previous to this lecture. The presymplectic leaves coincide with the symplectic leaves of $\pi_B$. Let $S$ be such a leaf. Let $\tilde{\omega}_S$ be the form defined in the exercise. We must show that $\tilde{\omega}_S = \omega_S + B|_S$, where $\omega_S$ is the symplectic form induced by $\pi$. Then $B$ is the Poisson gauging of $(N, \pi)$.
Choose $(X, \xi), (Y, \eta) \in L_\pi$. Then $(X, \xi - i_{\pi^\sharp(\xi)}B), (Y, \eta - i_{\pi^\sharp(\eta)}B)$ are in $\tau_B(L_\pi)$. We have
$$\begin{aligned} \tilde{\omega}_S(X, Y) &= -(\xi - i_{\pi^\sharp(\xi)}B)(\pi^\sharp(\eta)) = -(\xi(\pi^\sharp(\eta))) + B(\pi^\sharp(\xi), \pi^\sharp(\eta)) = \\ &= \omega_S + B|_S(X, Y). \end{aligned}$$

□

Now let us recall the Moser path method for symplectic manifolds.

**Theorem 4.5** *Let $S$ be a compact manifold with $\omega_0, \omega_1 \in \Omega^2(S)$ as symplectic forms. Assume that there is a smooth path $\{\omega_t\}_{t \in [0,1]}$ such that*

- *each $\omega_t$ is symplectic,*
- *$d[\omega_t]/dt \in H^2(S)$ is zero, i.e., $d\omega_t/dt = d(\theta_t)$ for some $\theta_t \in \Omega^1(S)$.*

*Then $(S, \omega_0)$ and $(S, \omega_1)$ are symplectomorphisms.*

We use this theorem in the Poisson case.

**Theorem 4.6** *Let $M$ be a compact Poisson manifold and $\pi_0, \pi_1$ be a Poisson structure. Assume that there is a smooth family $\{B_t\}_{t \in [0,1]}$ of gauging forms for $\pi_0$ such that*

- *$B_0 = 0$,*
- *$\pi_1$ is a gauge transformation of $\pi_0$ with respect to $B_1$,*
- *$d[B_t]/dt = 0$ in $H^2(M)$.*

*Then $(M, \pi_0)$ and $(M, \pi_1)$ are Poisson diffeomorphic.*

PROOF: Obviously, if we consider the problem leafwise, we obtain the symplectic case. On each leaf $S$ we have $\{\omega^0_S + B_t|_S\}_t \in [0, 1]$ such that if $t = 0$, then we obtain $\omega^0_S$ and if $t = 1$, then we obtain $\omega^1_S$.
$N'$ is the gauge transformation of $(N, \pi)$ with respect to $B = \phi^*\omega_{can}$. Use the family $B_t = tB$. Note that $dB_t/dt = B$ is exact because $\omega_{can}$ is exact. □

# Lecture 6

## 5 Main questions and examples

Let us state the two questions which formulate the problem of the symplectic realization of a Poisson manifold.

**Question 1:** Given a Poisson manifold $(M, \pi)$, can one find a symplectic manifold $(S, \omega)$ and a Poisson map $\mu : (S, \omega) \longrightarrow (M, \pi)$, such that $\mu$ becomes surjective submersion?

**Definition 5.1** *In this case we call $(S, \omega)$ a symplectic realization of $(M, \pi)$.*

**Question 2:** Given two manifolds $S, M$, where $S$ is symplectic and $\mu : S \longrightarrow M$ is a smooth map, can one find Poisson structure $\pi$ on $M$, such that $\mu$ becomes Poisson?

**Example 5.2** Let $M$ be a manifold with the Poisson structure $\pi = 0$ and let $S = T^*M$ be endowed with the canonical symplectic form on the cotangent bundle. Consider $\mu = pr : T^*M \longrightarrow M$.

**Definition 5.3** *An integrable system on a $2n$-dimensional symplectic manifold $(S, \omega)$ consists of $f_1, ..., f_n \in C^\infty(S)$ which are independent, i.e.,*

$$\{f_i, f_j\} = 0 \ \forall i, j = 1, n.$$

**Example 5.4** This can be viewed as a map

$$f = (f_1, \ldots, f_n) : S \longrightarrow \mathbb{R}^n,$$

and equalities for the independence hold if and only if $f$ is a Poisson map.

**Example 5.5** Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $M = \mathfrak{g}^*$ with the linear Poisson structure $\pi_{\text{lin}}$. Let $S = T^*G$ with the usual canonical form on the cotangent bundle. We can identify $T^*G$ with $G \times \mathfrak{g}^*$ using right translations. Let $\mu$ be the second projection.

**Exercise:** Let $M = \mathbb{R}^3$ and $\pi = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. Show that:

- $M = \mathfrak{g}^*$ and $\pi = \pi_{\text{lin}}$, for some Lie algebra $\mathfrak{g}$
- Use this to find an explicit symplectic form on $\mathbb{R}^6$ such that $pr_1 : \mathbb{R}^6 \to \mathbb{R}^3$ becomes a Poisson map.

**Solution:** Let $\mathfrak{g} = \mathfrak{su}(2) \simeq \mathbb{R}^3$. If we use the coordinates $(x, y, z)$ on $\mathfrak{g}^*$, we know that

$$\{x, y\} = z, \{y, z\} = x, \{z, x\} = y,$$

where $\{,\}$ is the Poisson bracket, induced by $\pi_{\text{lin}}$. Identifying $(\mathfrak{g}^*)^* = \mathfrak{g}$, we get the same conditions for the Lie brackets of these coordinates viewed as elements in the Lie algebra. Recall that the Poisson bracket in coordinates is just

$$\{f, g\}(\mu) = \sum_{i,j,k} c_{ij}^k x_k(\mu) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

for $f, g \in C^\infty(\mathfrak{g}^*)$ and $\mu \in \mathfrak{g}^*$. Applying this to the coordinates, we obtain the components of the Poisson bivector. Hence

$$\pi = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Let the symplectic form be

$$\omega = \sum_{i,j} \omega_{ij} dx_i \wedge dx_j,$$

where $(x_i)_{i=1}^6$ are the coordinates in $\mathbb{R}^6$. We use that $pr_1$ is a Poisson map, so

$$\pi(df, dg) \circ pr_1 = \{f \circ pr_1, g \circ pr_1\}_\omega$$

for $f, g \in C^\infty(\mathbb{R}^3)$. Applying this to the coordinates, we obtain that

$$\omega = x_3 dx_1 \wedge dx_2 + x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1.$$

**Example 5.6** For an arbitrary Poisson manifold $(M, \pi)$ let $S$ be a symplectic leaf of $M$ and $\mu$ the inclusion map.

**Example 5.7** Let $(S, \omega)$ be a symplectic manifold and let a Lie group $G$ act on it in a symplectic fashion. If the action is free and proper then $M = S/G$ becomes Poisson and the projection $S \to S/G$ is a Poisson map.

**Example 5.8** Let $G$ be as before. Recall that a hamiltonian $G$-space is a symplectic manifold $(S, \omega)$ together with an action of $G$ and an equivariant map $\mu : S \to \mathfrak{g}^*$ such that

(i) $\omega$ is $G$-invariant,

(ii) $i_{\rho(v)}\omega = -d\mu_v$ for every $v \in \mathfrak{g}$,

where $\mu_v \in C^\infty(S)$ is such that $\mu_v(x) = \langle \mu(x), v \rangle$ for every $x \in S$ and $\rho : \mathfrak{g} \to X(S)$ is the induced infinitesimal action.

Under these conditions the map $\mu : S \to \mathfrak{g}^*$, called momentum map, is a Poisson map.

**Exercise:** Show that if $G$ is a connected Lie group, then $(i)$ follows from $(ii)$. In this case the momentum map $\mu$ is equivariant if and only if $\mathcal{L}_{\rho(u)}(\mu_v) = \mu_{[u,v]}$ for every $u, v \in \mathfrak{g}$.

**Solution:** Assume that $i_{\rho(v)}\omega = -d\mu_v$. Then $di_{\rho(v)}\omega = 0$ and $i_{\rho(v)}d\omega = 0$ because the form $\omega$ is closed. Then using Cartan's formula we obtain that $\mathcal{L}_{\rho(v)}\omega = 0$. This can happens if and only if

$$t \mapsto (e^{t\rho(v)})^*\omega$$

does not depend on $t$. Therefore, it preserves the form and since $G$ is connected, every element preserves the form $\omega$.

**Example 5.9** More generally, consider a Lie algebra $\mathfrak{g}$ and a hamiltonian $\mathfrak{g}$-space, which contains a symplectic manifold $(S, \omega)$, together with an infinitesimal action of $\mathfrak{g}$ and an equivariant map $\mu : S \to \mathfrak{g}^*$ such that

$$i_{\rho(v)}\omega = -d\mu_v \ \forall v \in \mathfrak{g}.$$

In this case the map $\mu$ is also Poisson.

**Exercise:** Let $\mu : (S, \omega) \to (\mathfrak{g}^*, \pi_{\text{lin}})$ be a Poisson map. Show that

(i) there exists a unique action of $\mathfrak{g}$ on $S$ which makes $(S, \omega)$ into a hamiltonian $\mathfrak{g}$-space with the momentum map $\mu$,

(ii) if $\mathfrak{g}$ is a Lie algebra of a connected Lie group $G$ and $S$ is a compact manifold, then there exists a unique action of $G$ on $S$ making $(S, \omega)$ into a hamiltonian $G$-space the momentum map $\mu$.

**Solution:** Let $u, v \in \mathfrak{g}$. We can view these elements as linear functions on $\mathfrak{g}^*$. Using that $\mu$ is a Poisson manifold we obtain

$$\{u, v\}_{\pi_{\text{lin}}}(\mu(x)) = \{u \circ \mu, v \circ \mu\}_\omega(x)$$

for every $x \in S$. We can rewrite the first part of the equation as $\langle \mu(x), [u, v] \rangle = \mu_{[u,v]}(x)$. The second part is

$$\begin{aligned} \omega(H_{u\circ\mu}, H_{v\circ\mu}) &= -i_{v\circ\mu}\omega(H_{u\circ\mu}) \\ &= d(v \circ \mu)(H_{u\circ\mu}) \\ &= H_{u\circ\mu}(v \circ \mu) = \mathcal{L}_{H_{u\circ\mu}}(v \circ \mu) = \mathcal{L}_{H_{\mu_u}}\mu_v \end{aligned}$$

at the point $x \in S$. We obtain the infinitesimal version of the equivariance of $\mu$ if and only if $\rho(v) = H_{\mu_v}$ for every $v \in \mathfrak{g}$. The fact that $i_\rho(v)\omega = -d\mu_v$ follows from the definition of a hamiltonian vector field.

Assume that $\mathfrak{g}$ is a Lie algebra of a connected Lie group $G$ and $S$ is compact. Hence every vector field is complete and so the flow of $\rho(v)$ is just the 1-PS generated by $v \in \mathfrak{g}$. Since $G$ is connected, the action is uniquely determined by the actions of the 1-PS subgroups. Moreover, we have $\mathfrak{L}_{\rho(u)}(\mu_v) = \mu_{[u,v]}$ and $i_{\rho(v)}\omega = -d\mu_v$. Using the exercise above, we deduce that $(S, \omega)$ is a hamiltonian $G$-space.

# 6 Main theorems

**Theorem 6.1** *Let $(S, \omega)$ be a symplectic manifold and $\mu : S \longrightarrow M$ a surjective submersion with connected fibers. Let $\mathcal{F} \subseteq TS$ be the foliation of $S$ by the fibers of $\mu$ ($\mathcal{F}_x = \ker(d\mu)_x \ \forall x \in S$)*

*and* $\mathcal{F}^{\perp} \subset TS$ *be the symplectic orthogonal of* $\mathcal{F}$ ($\mathcal{F}_x^{\perp} = \{v \in T_xS : \omega_x(v,w) = 0, \forall w \in \mathcal{F}_x\}$)*. Then the following are equivalent*

1. *there exists a Poisson structure* $\pi$ *on* $M$ *such that* $\mu$ *is a Poisson map,*

2. $\mathcal{F}^{\perp}$ *is a foliation on* $S$ ($[X,Y] \in \Gamma\mathcal{F}^{\perp}$, $\forall X, Y \in \Gamma\mathcal{F}^{\perp}$)*.*

PROOF: The map $\mu^* : C^{\infty}(M) \to C^{\infty}(S)$, where $f \mapsto f \circ \mu$, is an injective map and induces an isomorphism between $C^{\infty}(M)$ and $\mu^*(C^{\infty}(M)) =: C^{\infty}_{\mathcal{F}-bas}(S)$, which is

$$\{f \in C^{\infty}(S) | \mathfrak{L}_V(f) = 0 \forall V \in \Gamma\mathcal{F}\}.$$

Hence (1) is equivalent to the fact that $C^{\infty}_{\mathcal{F}-bas}(S) \subseteq C^{\infty}(S)$ is closed under the bracket. Since $f \in C^{\infty}_{\mathcal{F}-bas}(S) \Leftrightarrow X_f \in \Gamma(\mathcal{F}^{\perp})$, condition (1) is equivalent to $[X,Y] \in \Gamma(\mathcal{F}^{\perp})$ for every hamiltonian vector field $X, Y \in \Gamma(\mathcal{F}^{\perp})$. The foliation $\mathcal{F}^{\perp}$ is controlled by a tensor $\mathcal{F}^{\perp} \times \mathcal{F}^{\perp} \longrightarrow TM/\mathcal{F}^{\perp}$ defined on the level of sections as $(X,Y) \mapsto class([X,Y])$. We use that for every $X_x \in \mathcal{F}_x^{\perp}$ there exists $f \in C^{\infty}(S)$ such that $X_x = X_f(x)$. □

**Theorem 6.2** *Every Poisson manifold admits a symplectic realization. More precisely, let* $(M,\pi)$ *be a Poisson manifold. Let* $\nabla$ *be a Poisson connection on* $T^*M$*, and let* $\phi_t : T^*M \longrightarrow T^*M$ *be the induced geodesic flow. Then*

$$\Omega = \int_0^1 \phi_t^*(\omega_{can}) dt$$

*restricted to a sufficiently small neighborhood* $U \subseteq T^*M$ *of the zero section of the cotangent bundle is a symplectic structure and the projection* $pr|_U : U \longrightarrow M$ *is a Poisson map.*

To understand the notions introduced in this theorem and to prove the theorem we introduce the concepts of contravariant geometry of a Poisson manifold. The idea of this contravariancy is to replace the Lie algebra structure of the tangent bundle $(TM, [.,.])$ by the natural Lie algebra structure of the cotangent bundle $(T^*M, [.,.]_\pi)$. These are related by $\pi^{\#}$ which preserves the brackets. The Lie bracket on $T^*M$ is defined in the following way

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^{\#}(\alpha)}\beta - \mathcal{L}_{\pi^{\#}(\beta)}\alpha - d(\pi(\alpha,\beta))..$$

# 7 Appendix

Let $M$ be a smooth manifold, and $p : E \longrightarrow M$ a vector bundle over $M$.

**Definition 7.1** *A connection on* $E$ *is a bilinear map*

$$\nabla : X(M) \times \Gamma(E) \longrightarrow \Gamma(E)$$

$$(X, s) \longmapsto \nabla_X(s)$$

*such that the following conditions hold:*

1. $\nabla_{fX}(s) = f\nabla_X(s)$

2. $\nabla_X(fs) = f\nabla_X(s) + \mathcal{L}_X(f)s$

*for every $X \in X(M)$, $f \in C^\infty(M)$ and $s \in \Gamma(E)$.*

Locally, if we have $\{e_1, \ldots, e_m\}$ as a frame for $E$, we can write

$$\nabla_X(e_i) = \sum_{j=1}^{m} \theta^{ij}(X) e_j \text{ for } i = \overline{1, m},$$

where $\theta^{ij}(X)$ are smooth functions on an open subset of $M$ where the trivialization lives. Thus $\nabla$ is locally determined by a set $\{\theta^{ij}\}$ of one-forms. Let $s = \sum_{j=1}^{m} s^i e_i$. Then

$$\nabla_X(s) = \sum_{i,j} \theta^{ij}(X) s^i e_j + \sum_{i=1}^{m} \mathcal{L}_X(s^i) e_i,$$

which at a point $x$ depends only on $X(x)$ and $s(\gamma(t))$, where $t \in (-\epsilon, \epsilon)$ and $\gamma$ is the integral curve of the vector field $X$ through $x$.
Using this observation, if we consider the paths $\gamma : [0,1] \longrightarrow M$ and the paths $u : [0,1] \longrightarrow E$ above $\gamma$ ($u(t) \in E_{\gamma(t)}$), we can define the curve

$$\nabla_{\frac{d\gamma}{dt}}(u) : [0,1] \longrightarrow E,$$

above $\gamma$. In local coordinates it is defined as

$$\nabla_{\frac{d\gamma}{dt}}(u) = \sum_{i,j} \theta^{ij}(\frac{d\gamma}{dt}) u^j e_i + \sum_i \frac{du_i}{dt} e_i.$$

Hence given $\gamma$, it makes sense to talk about paths $u : [0,1] \longrightarrow E$ above $\gamma$ which are parallel with respect to the connection $\nabla$ if

$$\nabla_{\frac{d\gamma}{dt}}(u) = 0 \ \forall t.$$

**Proposition 7.2** *Let $\gamma : [0,1] \longrightarrow M$ be a path. Let $\gamma(0) = x_0$ and $u_0 \in E_{x_0}$. There exists a unique path $u : [0,1] \longrightarrow E$ above $\gamma$, parallel with respect to $\nabla$, such that $u(0) = u_0$.*

Using this proposition we can define parallel transport with respect to $\gamma$ as a map

$$T_\gamma : E_{x_0} \longrightarrow E_{x_1}$$

$$u(0) \mapsto u(1),$$

where $x_0, x_1$ are the endpoints of $\gamma$ and $u$ which are chosen as in the proposition.