Lecture Notes on Poisson Geometry

Myagmar Batbileg Sweitse van Leeuwen Cristian Ortiz

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Abstract

In these lectures we study symplectic realizations of Poisson manifolds. Using covariant geometry we are allowed to construct an explicit symplectic realization for any Poisson manifold. We relate symplectic realizations to the theory of symplectic groupoids. Using symplectic groupoids the existence of symplectic realizations translates into an integrability problem in Lie theory. Furthermore, we give some applications of symplectic groupoids to reduction procedures.

1 Symplectic Realizations

Given a Poisson manifold (M, π), we may consider the following two questions:

1. Can we find a symplectic manifold (S, ω) and a surjective submersion $\mu : (S, \omega) \to (M, \pi)$ which is a Poisson map?
2. If M is a differentiable manifold and $\mu : (S, \omega) \to M$ a differentiable map from a symplectic manifold (S, ω) to M, can we find a Poisson structure on M such that μ becomes a Poisson map?

As we have seen, the second question has an affirmative answer, provided by the following result:

Theorem 1.1. *Let (S, ω) be a symplectic manifold and $\mu : S \to M$ a surjective submersion with connected fibers. Consider the foliation $\mathcal{F}$ of S defined by the fibers of μ and take $\mathcal{F}^\perp$ the symplectic orthogonal of $\mathcal{F}$. The following conditions are equivalent:*

i) There exists a Poisson structure on M such that μ is a Poisson map.

ii) $\mathcal{F}^\perp$ is an integrable distribution in the sense of Frobenius.

The aim of this section is to give an affirmative answer to the first question. First of all, note that given a differentiable manifold M there is a canonical symplectic manifold associated to it, in fact the cotangent bundle T^*M carries a natural symplectic structure given by the exterior derivative of the tautological 1-form on T^*M. Hence, in order to answer the first question, it is natural to construct a symplectic realization of a Poisson manifold in terms of its cotangent bundle. We outline the main steps to construct symplectic realizations.

First, we construct a symplectic structure ω on an open neighborhood U of the zero section in T^*M, such that the symplectic orthogonal of the foliation of U by fibers of $\mathrm{pr} : U \to M$ is involutive. Now the previous theorem guarantees the existence of a Poisson structure π' on M such that pr becomes Poisson. Finally we prove that the Poisson structures π', π coincide. In order to do this, we need some covariant geometry.

Definition 1.2. *A* Poisson curve *in* (M, π) *is a curve* $a : I \subset \mathbb{R} \to \mathrm{T}^*M$ *which satisfies*

$$\pi^\sharp(a(t)) = \gamma_a'(t)$$

for all $t \in I$*, with* $\gamma_a = \mathrm{pr} \circ a : I \to M$ *the projected curve in* M*.*

When $I = [0, 1]$ we are allowed to talk about Poisson paths. Set $x_0 = \gamma_a(0)$ and $x_1 = \gamma_a(1)$. Define the set $P_{x_0,x_1}(M, \pi) = \{\text{Poisson paths from } x_0 \text{ to } x_1\}$. It is possible to define the concatenation of Poisson paths

$$P_{x_1,x_2}(M, \pi) \times P_{x_0,x_1}(M, \pi) \to P_{x_0,x_2}(M, \pi) : (a', a) \mapsto a' * a$$

Exercise 1.1: Prove that for each $x \in M$ the symplectic leaf through x coincides with the set of all points $y \in M$ that can be reached by a Poisson path starting at x.

Remark 1.3. The curve $a : I \to \mathrm{T}^*M$ in general is not completely determined by γ_a. However, if (M, π) is a symplectic manifold, then a is determined by γ_a via $a = i_{\gamma_a'}\omega$.

1.1 Poisson Paths and Contravariant Connections

Let (M, π) be a Poisson manifold and $p : E \to M$ a vector bundle over M.

Definition 1.4. *A* contravariant connection *on* E *is a bilinear map*

$$\nabla : \Omega^1(M) \times \Gamma(E) \to \Gamma(E) : (\alpha, s) \mapsto \nabla_\alpha s$$

satisfying the conditions:

i) $\nabla_{f\alpha}s = f\nabla_\alpha s$

ii) $\nabla_\alpha fs = f\nabla_\alpha s + (\mathcal{L}_{\pi^\sharp(\alpha)}f)s$

An example of a contravariant connection is given by the following construction: If ∇ is a connection on E, then $(\alpha, s) \to \nabla_{\pi^\sharp(\alpha)}(s)$ defines a contravariant connection on E.

Given $a \in P_{x_0,x_1}(M,\pi)$ and $u : [0,1] \to E$ a path above γ_a, we have an induced path $\nabla_a u : [0,1] \to E$ above γ_a. We use local coordinates to obtain a formula for the path defined above. If $\{e_1, ..., e_r\}$ is a local frame for E over $U \subset M$, ∇ is determined by

$$\nabla_\alpha e_i = \sum_{j=1}^{r} \alpha(X_i^j)e_j,$$

where (X_i^j) is a $r \times r$ matrix of vector fields on U. Assume that the path u in this coordinate system is given by

$$u(t) = \sum_{i=1}^{r} u^i(t)e_i,$$

then by the properties of a contravariant connection, we obtain

$$\nabla_a u(t) = \sum_{j=1}^{r} \frac{du^j}{dt}e_j + \sum_{j,i=1}^{r} \langle a(t), X_i^j \rangle u^i(t)e_j.$$

As in the Riemannian case, in contravariant geometry there exists the notion of parallel transport: For $a \in P_{x_0,x_1}(M,\pi)$ choose a point on the fiber $e_0 \in E_{x_o}$ and look at $u : [0,1] \to E$ above γ_a such that

$$\nabla_a u = 0$$
$$u(0) = e_0.$$

This problem has a unique solution $u = u_{a,e_0}$. This allows us to define the *parallel transport* $\mathcal{T}_a^t : E_{x_0} \to E_{\gamma_a(t)}$ in a natural way.

Remark 1.5. As $t = 1$, the parallel transport induces an "action"

$$P_{x_0,x_1}(M,\pi) \times E_{x_0} \to E_{x_1} : (a, e_0) \mapsto u_{a,e_0}(1), \tag{1.1}$$

with E_x the fiber of E over a point $x \in M$.

Exercise 1.2: Describe the parallel transport of the concatenation of Poisson paths? Write the connection ∇ in terms of parallel transport. Remember that in the Riemannian case it is possible to recover a connection via parallel transport.

1.2 Poisson Paths and Symplectic Realizations

Consider a symplectic manifold (S, ω) and a Poisson manifold (M, π).

Definition 1.6. *Let $a : I \to \mathrm{T}^*M$ be a Poisson curve. A* lift *of a along a Poisson map $\mu : (S, \omega) \to (M, \pi)$ is a curve $u : I \to S$ such that $\mu(u(t)) = \gamma_a(t)$ and $i_{u'}\omega = \mu^*(a(t))$.*

Remark 1.7. Look at the special case of a Poisson curve $a(t) = \mathrm{d}f_{\gamma_a(t)}$ induced by a smooth function $f \in C^\infty(M)$. If $u : I \to S$ is a lift of a along a Poisson map $\mu : S \to M$, then $\mathrm{d}\mu(u(t)) : T_{u(t)}S \to T_{\gamma_a(t)}M$, so we can consider the cotangent path $\mu^*(a(t))$ on S. The second property of the lift u becomes $i_{u'}\omega = d\mu^*f$. Therefore the derivative of the lift of u at $t \in I$ is just the Hamiltonian vector field of the induced map μ^*f, i.e.

$$u'(t) = X_{\mu^*f}(u(t))$$

It is important to highlight that lifts along Poisson maps are not always globally defined. However, local lifts are always well defined: For every $t_0 \in I$ and $y_0 \in S$ such that $\mu(y_0) = \gamma_a(t_0)$, there exists a $u : J \to S$ defined on some open interval $J \subset I$ containing t_0 with $u(t_0) = y_0$ and u is a lift of $a|_J$ along μ. Moreover, given a Poisson path $a \in P_{x_0,x_1}(M, \pi)$ and $y_0 \in S$ such that $\mu(y_0) = x_0$, there exists a maximal lift of a along μ starting at x_0, $u : I_{max} \to S$ with $0 \in I_{max}$. Therefore local existence and local uniqueness of lifts always hold. The natural question now is: Find a condition on the Poisson map such that lifts along it will be globally defined.

Definition 1.8. *A Poisson map $\mu : (S, \omega) \to (M, \pi)$ is called* complete *if for every $f \in C^\infty(M)$ the induced hamiltonian vector field $X_{\mu^*f} \in \mathfrak{X}(S)$ is complete.*

Exercise 1.3: Show that a Poisson map μ is complete if and only if for every Poisson path $a \in P_{x_0,x_1}(M, \pi)$ and $y_0 \in \mu^{-1}(x_0)$ the local lift of a through y_0 is defined on the entire interval $[0, 1]$.

Note that for a complete Poisson map $\mu : (S, \omega) \to (M, \pi)$ we can define a natural "action" conform (**??**) by

$$P_{x_0,x_1}(M, \pi) \times \mu^{-1(}x_0) \to \mu^{-1}(x_1) : (a, y_0) \mapsto u_{a,y_0}(1)$$

Exercise 1.4: Prove that $(a, \cdot) : \mu^{-1}(x_0) \to \mu^{-1}(x_1)$ is a diffeomorphism.

1.3 Geodesic Flow

Given a Poisson manifold (M, π) with a Riemannian metric $(\cdot, \cdot)$ on its cotangent bundle T^*M we can look at contravariant connections ∇ with the properties:

i) ∇ is *compatible* with the metric, in the sense that

$$(\nabla_\alpha \beta, \gamma) + (\beta, \nabla_\alpha \gamma) = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta, \gamma)$$

ii) ∇ is *torsion free*

$$\nabla_\alpha \beta - \nabla_\beta \alpha = [\alpha, \beta]_\pi$$

for all $\alpha, \beta, \gamma \in \Omega^1(M)$.

As in the Riemannian case, there exists a unique contravariant connection on (M, π) satisfying the properties above. This connection will be called *contravariant Levi-Civita connection*.

Now we look for a local expression for the contravariant Levi-Civita connection. If $(x_1, ..., x_n)$ is a coordinate chart, ∇ is determined by smooth functions Γ^k_{ij} where $1 \leq i, j \leq k$, defined as follows

$$\nabla_{\mathrm{d}x_i} \mathrm{d}x_j = \sum_{k=1}^{n} \Gamma^k_{ij} \mathrm{d}x_k$$

Definition 1.9. *A* Poisson geodesic *with respect to ∇ is a Poisson curve a with the property that $\nabla_a a = 0$.*

Locally, assume that the Poisson curve a can be written as

$$a(t) = \sum_{i=1}^{n} a^i(t)(dx_i)_{\gamma_a(t)}.$$

Then the geodesic condition becomes

$$\frac{\mathrm{d}a^k(t)}{\mathrm{d}t} = -\sum_{i,j=1}^{n} \Gamma^k_{ij}(\gamma_a(t))\ a^i(t)\ a^j(t). \tag{1.2}$$

Writing out

$$\pi = \sum_{i,j=1}^{n} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

the condition that a is a Poisson curve becomes

$$\frac{\mathrm{d}\gamma^i_a(t)}{\mathrm{d}t} = \sum_{j=1}^{n} \pi_{ij}(\gamma_a(t))\ a^j(t). \tag{1.3}$$

Hence, equations (**??**) and (**??**) are the local equations of a Poisson geodesic.

Remark 1.10. The Poisson geodesic equations have a unique maximal solution provided we prescribe $(a(0), \gamma_a(0))$. So, for $x \in M$ and $\xi \in T_x^*M$ we find a maximal geodesic $a_\xi : I \to \mathrm{T}^*M$ such that $a_\xi(0) = \xi$.

Now we proceed to define the geodesic flow in contravariant geometry. In order to do that, consider $\mathcal{D} = \{(\xi, t) \in \mathrm{T}^*M \times \mathbb{R} \mid a_\xi(t) \text{ is defined}\}$. This set is called the *domain* of the *geodesic flow* $\Phi^t(\xi) = a_\xi(t)$. The *geodesic vector field* $\mathcal{V} \in \mathfrak{X}(\mathrm{T}^*M)$ is obtained by glueing

$$\mathcal{V}_{x,y} = \sum_{i,j=1}^{n} \pi_{ij}(x) y^j \frac{\partial}{\partial x_i} - \sum_{i,j,k=1}^{n} \Gamma_{ij}^k(x) y^i \frac{\partial}{\partial y_k},$$

where $(x, y) \in \mathbb{R}^{2n}$. By construction we have $\mathcal{D} = \mathcal{D}(\mathcal{V})$ and Φ^t is the flow of the geodesic vector field $\mathcal{V}$.

Proposition 1.11. *Let $r \in \mathbb{R}^*$ and consider the fiberwise multiplication $m_r : \mathrm{T}^*M \to \mathrm{T}^*M$. Then*

1. *$(\xi, t) \in \mathcal{D}$ if and only if $(m_r(\xi), \frac{1}{r}t) \in \mathcal{D}$.*
2. *$\Phi^{rt}(\xi) = m_{\frac{1}{r}}(\Phi^t(m_r(\xi)))$ for every $(\xi, rt) \in \mathcal{D}$.*

Proof. If (a, γ) is a solution of equations (**??**) and (**??**) with $\gamma(0) = x$ and $a(0) = v$, then

$$\hat{\gamma}(t) = \gamma(rt)$$
$$\hat{a}(t) = ra(rt)$$

defines a solution of (1.1) and (1.2) with $\hat{\gamma}(0) = x$ and $\hat{a}(0) = rv$. □

Note that the second property in the proposition above says that Φ^t and Φ^{rt} are conjugated by the fiberwise multiplication m_r. That is, if we know Φ^1, we could construct Φ^t.

Corollary 1.12. *The set $\mathcal{D}_1 = \{\xi \in \mathrm{T}^*M \mid (\xi, 1) \in \mathcal{D}\} \subset \mathrm{T}^*M$ is an open neighborhood of the zero section and the following holds:*

1. *For every $\xi \in \mathcal{D}_1$ there exists a unique geodesic $a \in P_{x_0,x_1}(M, \pi)$ with $a(0) = \xi$.*
2. *$(\xi, t) \in \mathcal{D}$ if and only if $m_t(\xi) \in \mathcal{D}_1$.*
3. *$\Phi^t(\xi) = m_{\frac{1}{t}}(\Phi^1(m_t(\xi)))$ for every $(\xi, t) \in \mathcal{D}$.*

Alluding to the definition in the Riemannian case, we introduce the notion of *contravariant exponential map* defined by

$$\exp : \mathrm{pr} \circ \Phi^1 : \mathrm{T}^*M \to M$$

Recall that in the Riemannian case the exponential map is a diffeomorphism on a neighborhood of $0 \in T_xM$. In the contravariant setting the picture is quite different:

Proposition 1.13. *For every $x \in M$ consider the exponential at x defined by $\exp_x := \exp|_{\mathrm{T}_x^*M} : T_x^*M \to M$. Then the following holds:*

$$(\mathrm{d}\exp_x)_{0_x} = \pi^\sharp.$$

In particular, in the symplectic case, $\exp_x$ maps a neighborhood of zero in T_x^*M diffeomorphically into a neighborhood of x in M.

Proof. The proof is just a computation. Note that $\xi = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} t\xi$ is mapped to $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(t\xi)$ via $(\mathrm{d}\exp_x)_{0_x}$. On the other hand

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(t\xi) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{pr} \circ \Phi^t(\xi) \\ &= \pi^\sharp(\Phi^0(\xi)) \\ &= \pi^\sharp(\xi).\end{aligned}$$

□

Finally, we give an affirmative answer to the first main question proposed in the beginning of this section:

Theorem 1.14. *Every Poisson manifold (M, π) admits a symplectic realization. More precisely, let ∇ be a contravariant connection on T^*M and consider its geodesic flow $\Phi^t : \mathrm{T}^*M \to \mathrm{T}^*M$. Then*

$$\Omega = \int_0^1 (\Phi^t)^* \omega_{can} \, \mathrm{d}t,$$

when restricted to a small enough neighborhood $U \subset \mathrm{T}^*M$ of the zero section, becomes a symplectic structure and $\mathrm{pr}|_U : (U, \Omega) \to (M, \pi)$ is a Poisson map.

Proof. We highlight the main ideas of the proof. Look at Ω_{0_x} and write it as a matrix

$$\begin{pmatrix} \pi & I \\ -I & 0 \end{pmatrix}$$

This shows that Ω_{0_x} is nondegenerate, thus there is a neighborhood U where $\Omega|_U$ is nondegenerate. The derivative of the projection $(\mathrm{d}\,\mathrm{pr})_{0_x}$ sends the bivector induced by Ω_{0_x} into π_x. Now look for a Poisson structure π' on M such that $\mathrm{pr}|_U : U \to M$ is a Poisson map. By Theorem **??**, such π' exists if and only if $\mathcal{F}^\perp \subset \mathrm{T}^*U$ is involutive, where $\mathcal{F} = \ker(\mathrm{d\,pr})$. Actually $\mathcal{F}^\perp = \ker(\mathrm{d}\Phi^1)$ and this implies the integrability of $\mathcal{F}^\perp$. The rest of the proof is to check that $\pi' = \pi$. □

Example 1.15. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Consider the left translation map $L : \mathrm{T}^*G \to \mathfrak{g}^*$ defined by $L(\xi) = L_g^*(\xi) = L_{g^{-1}}(\xi)$ for every $\xi \in T_g^*G$. It is easy to check that L becomes a Poisson map when T^*G is equipped with the canonical symplectic structure and $\mathfrak{g}^*$ has the linear Poisson structure. Thus the cotangent bundle of a Lie group is a symplectic realization of the dual of its Lie algebra.

There exists another approach to symplectic realizations of Poisson manifolds, using the notion of symplectic groupoid. All the basic concepts related to Lie groupoids and symplectic groupoids will be introduced in the next section.

2 Symplectic groupoids

Motivation We have encountered the following two problems:

1. If we have a Poisson manifold (M, π), is there a symplectic manifold (S, ω) and a surjective submersion $\mu : S \to M$ such that this map is Poisson?

2. For a Poisson manifold (M, π), we can construct the Lie algebroid T^*M. Its Lie algebra structure is defined in terms of the Poisson bivector π via

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - \mathrm{d}(\pi(\alpha, \beta)).$$

   Given this Lie algebroid, what is the corresponding Lie groupoid?

It turns out that both of these questions are related to the topic of symplectic groupoids.

2.1 Lie groupoids and symplectic groupoids

Definition 2.1 (Lie groupoid)**.** *A* Lie groupoid *is a pair of smooth manifolds $(\mathcal{G}, M)$ and surjective submersions $s, t : \mathcal{G} \to M$ with:*

1. *A multiplication m on $\mathcal{G}_{(2)} = \mathcal{G} \times_M \mathcal{G} = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\}$, that is given by*

$$m : \mathcal{G} \times_M \mathcal{G} \to \mathcal{G} : (g, h) \mapsto g \cdot h;$$

Figure 1: The multiplication $m(g,h) = gh$ projected to M.

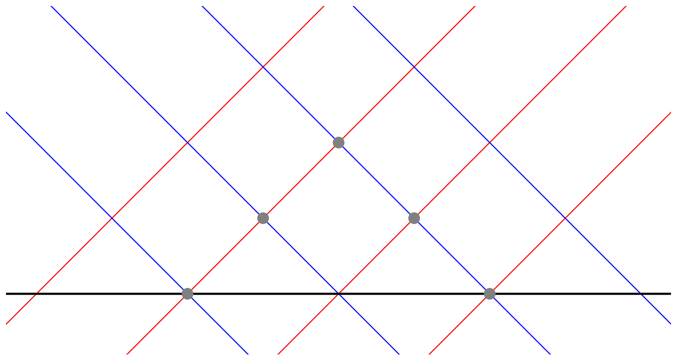

Figure 2: The multiplication $m(g,h) = gh$ in $\mathcal{G}$.

2. *An embedding* $\varepsilon : M \hookrightarrow \mathcal{G}$*; and*

3. *An inverse* $i : \mathcal{G} \to \mathcal{G}$.

The source map s *and target map* t *satisfy the following relations:*

$$\begin{aligned} s(g \cdot h) &= s(h) & t(g \cdot h) &= t(g) \\ i(g) \cdot g &= \varepsilon(s(g)) & g \cdot i(g) &= \varepsilon(t(g)) \\ g \cdot \varepsilon(s(g)) &= g & \varepsilon(t(g)) \cdot g &= g \\ (g \cdot h) \cdot k &= g \cdot (h \cdot k) & &\textit{(whenever defined).} \end{aligned}$$

From now on we will drop the dot denoting multiplication and assume that M is embedded in $\mathcal{G}$ (which is no restriction since $\varepsilon(M) \simeq M$), so that we can omit ε in the equations.

Definition 2.2 (Symplectic groupoid)**.** *A Lie groupoid* $\mathcal{G} \rightrightarrows M$ *equipped with a symplectic structure* $\omega \in \Omega^2(\mathcal{G})$ *is called a* symplectic groupoid *if*

$$\Gamma = \left\{ (g, h, m(g,h)) \,\middle|\, (g,h) \in \mathcal{G}_{(2)} \right\} \subseteq \mathcal{G} \times \mathcal{G} \times \overline{\mathcal{G}} \tag{2.1}$$

is a Lagrangian submanifold with respect to the symplectic form $\omega \oplus \omega \ominus \omega$ *on* $\mathcal{G} \times \mathcal{G} \times \overline{\mathcal{G}}$.

The statement that Γ is a Lagrangian submanifold means that it's dimension is $\frac{3}{2} \dim \mathcal{G}$, and it is isotropic, that is: the symplectic form $\omega \oplus \omega \ominus \omega$ restricted to $\mathrm{T}\Gamma \times \mathrm{T}\Gamma$ is zero.

The opposite sign of the symplectic form in $\overline{\mathcal{G}}$ arises in the same way as for symplectic transformations: Let $\Phi : (S, \omega_S) \to (X, \omega_X)$ be a symplectic transformation between two symplectic manifolds. Then $\omega_S = \Phi^*\omega_X$ and we see that $\Lambda := \{(s, \Phi(s)) \,|\, s \in S\}$ is a Lagrangian submanifold of $S \times X$ with respect to the symplectic form $\omega_S \ominus \omega_X \in \Omega^2(S \times X)$ by

$$\begin{aligned} \omega_S \ominus \omega_X((V, \mathrm{d}\Phi(V)), (V', \mathrm{d}\Phi(V'))) &= \omega_S(V, V') - \omega_X(\mathrm{d}\Phi(V), \mathrm{d}\Phi(V')) \\ &= \omega_S(V, V') - \Phi^*\omega_X(V, V') \\ &= \omega_S(V, V') - \omega_S(V, V') = 0, \end{aligned}$$

where $(V, \mathrm{d}\Phi(V))$ and $(V', \mathrm{d}\Phi(V'))$ are abitrary elements of $\mathrm{T}\Lambda$.

Proposition 2.3. *The subspace Γ as defined in equation (**??**) is isotropic if and only if*

$$m^*\omega = \mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\omega, \tag{2.2}$$

where $\mathrm{pr}_{1,2}$ denotes the projection $\mathrm{pr}_{1,2} : \mathcal{G}_{(2)} \to \mathcal{G}$ to the first resp. second component of $\mathcal{G}_{(2)}$.

Proof. Let $(V, W), (V', W') \in \mathrm{T}\mathcal{G}_{(2)}$, so that we have the corresponding elements $(V, W, \mathrm{d}m(V, W))$ and $(V', W', \mathrm{d}m(V', W'))$ in $\mathrm{T}\Gamma$, where we have omitted the basepoint. Then we have the equality

$$\begin{aligned} &\omega \oplus \omega \ominus \omega((V, W, \mathrm{d}m(V, W)), (V', W', \mathrm{d}m(V', W'))) \\ &\qquad = \omega(V, V') + \omega(W, W') - \omega(\mathrm{d}m(V, W), \mathrm{d}m(V', W')) \\ &\qquad = (\mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\omega - m^*\omega)\,((V, W), (V', W')) \end{aligned}$$

From this we immidiately see that the symplectic form $\omega \oplus \omega \ominus \omega$ on Γ vanishes precisely when $\mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\omega - m^*\omega = 0$, which proves the proposition. □

Remark 2.4. If Γ is isotropic, we can derive that it's dimension is $\frac{3}{2}\dim\mathcal{G}$. Hence Γ is Lagrangian and $\mathcal{G} \rightrightarrows M$ symplectic if we are in the situation of proposition **??**.

Remark 2.5. A 2-form $\omega \in \Omega^2(\mathcal{G})$ satisfying equation (**??**) is called *multiplicative.*

Exercise 2.1: Suppose $\mathcal{G}$ is a Lie group. Show that the only multiplicative 2-form is the zero form.

Some examples of symplectic groupoids

1. Let (M, ω) be a symplectic manifold. Then $M \times \overline{M} \rightrightarrows M$ with the symplectic form $\omega \ominus \omega$ is a symplectic groupoid. The fundamental groupoid of M is locally isomorphic to this one.

2. For any manifold M, $\mathrm{T}^*M \rightrightarrows M$ with the canonical symplectic form and source and target map $s, t = \mathrm{pr}_1 : \mathrm{T}^*M \to M$ the projection onto the base space is a symplectic groupoid. Multiplication in the groupoid then simplifies to fiberwise addition in the cotangent space.

3. For a Lie group G with Lie algebra $\mathfrak{g}$, the action groupoid $\mathrm{T}^*G \rightrightarrows \mathfrak{g}^*$ with the canonical symplectic form on the cotangent bundle is symplectic. We have $\mathcal{G} = \mathrm{T}^*G \simeq G \times \mathfrak{g}^*$, with source map $s : \mathcal{G} \to \mathfrak{g}^*$ projection onto the second component, and target map $t : \mathcal{G} \to \mathfrak{g}^* : (g, X) \mapsto gX$ the coadjoint action.

A groupoid like T^*G is called a *double groupoid*, because both $\mathrm{T}^*G \rightrightarrows \mathfrak{g}^*$ and $\mathrm{T}^*G \rightrightarrows G$ are groupoids. For any groupoid $\mathcal{G}$, $\mathrm{T}^*\mathcal{G}$ is a symplectic groupoid.

2.2 Properties of Symplectic groupoids

Let $\mathcal{G} \rightrightarrows M$ be a symplectic groupoid with symplectic form $\omega \in \Omega^2(\mathcal{G})$. This groupoid has the following properties:

Proposition 2.6. *$M \overset{\varepsilon}{\hookrightarrow} \mathcal{G}$ is a Lagrangian embedding, i.e. $\varepsilon(M)$ is a Lagrangian submanifold of $\mathcal{G}$.*

Proof. Let $x \in \varepsilon(M)$ and $X, Y \in \mathrm{T}_x\varepsilon(M)$. Then $(x, x, x) \in \Gamma$ because $m(x, x) = x$ and so $(X, X, X), (Y, Y, Y) \in \mathrm{T}_{(x,x,x)}\Gamma$. The fact that Γ is Lagrangian gives

$$
\begin{aligned}
0 &= \omega \oplus \omega \ominus \omega((X, X, X), (Y, Y, Y)) \\
&= \omega(X, Y) + \omega(X, Y) - \omega(X, Y) \\
&= \omega(X, Y).
\end{aligned}
\tag{2.3}
$$

Because $s, t : \mathcal{G} \to M$ are surjective submersions, the dimension of $\mathcal{G}_{(2)}$ is equal to $2 \dim \mathcal{G} - \dim M$. By construction, $\dim \Gamma = \dim \mathcal{G}_{(2)}$ and since Γ is Lagrangian in $\mathcal{G} \times \mathcal{G} \times \overline{\mathcal{G}}$, we obtain the equality $\frac{3}{2} \dim \mathcal{G} = 2 \dim \mathcal{G} - \dim M$ and we see that the dimension of M is half the dimension of $\mathcal{G}$.

From equation (**??**) it follows that $M \subset \mathcal{G}$ is isotropic and since it's dimension is half that of $\mathcal{G}$, it follows that M is Lagrangian. □

Proposition 2.7. *The inversion $i : \mathcal{G} \to \mathcal{G}$ is an antisymplectomorphism, i.e. $i^*\omega = -\omega$.*

Proof. Remember that $m(g, i(g)) = t(g)$, so we may choose a path $\gamma := (g, i(g), t(g)) \in \Gamma$. Let $(X, i_*(X), \mathrm{d}t(X)), (Y, i_*(Y), \mathrm{d}t(Y)) \in \mathrm{T}_\gamma\Gamma$. Using that Γ is isotropic, this gives

$$\begin{aligned} 0 &= \omega \oplus \omega \ominus \omega((X, i_*(X), \mathrm{d}t(X)), (Y, i_*(Y), \mathrm{d}t(Y)) \\ &= \omega(X, Y) + \omega(i_*(X), i_*(Y)) - \omega(\mathrm{d}t(X), \mathrm{d}t(Y)) \\ &= \omega(X, Y) + \omega(i_*(X), i_*(Y)) \\ &= (\omega + i^*\omega)(X, Y), \end{aligned}$$

where $\omega(\mathrm{d}t(X), \mathrm{d}t(Y))$ vanishes, because t is invariant under right multiplications. □

Proposition 2.8. *For any $g \in \mathcal{G}$,* $\ker \mathrm{d}_g s = (\ker \mathrm{d}_g t)^{\perp_\omega}$.

Proof. Another way of phrasing the proposition is that the tangent space of the s-fiber is (symplectically) orthogonal to the tangent space of the t-fiber at g. We will show this proposition by taking two paths in the fibers and show that the tangent vectors to these paths are orthogonal.

Let $c : (-\epsilon, \epsilon) \to \mathcal{G}$, $c(0) = c_0$, such that $s(c(\tau))$ is constant over time. By the groupoid property $i(c(\tau)) \cdot c(\tau) = s(c(\tau))$ it follows that

$$(i(c(\tau)), c(\tau), s(c_0))$$

is a path in Γ. It's tangent vector at $\tau = 0$ is given by $(-, X, 0)$, where X is the vector tangent to $c(\tau)$ at $\tau = 0$ and the first slot does not interest us.

Similarly take a path $d : (-\epsilon, \epsilon) \to \mathcal{G}$, $d(0) = d_0$ such that $t(d(\tau))$ is constant. Then $(i(d_0), d(\tau), i(d_0) \cdot d(\tau))$ is a path in Γ by construction and it's tangent vector at zero is given by $(0, Y, -)$, with Y the tangent vector of $d(\tau)$ at $\tau = 0$.

Suppose that the tangent vectors X, Y have the same basepoint, which is the case whenever $c_0 = d_0$. Then we infer from the isotropy of Γ that

$$\begin{aligned} 0 &= \omega \oplus \omega \ominus \omega((-, X, 0), (0, Y, -)) \\ &= \omega(-, 0) + \omega(X, Y) - \omega(0, -) \\ &= \omega(X, Y) \end{aligned}$$

Since the paths were chosen arbitrarily in the fibers, it follows that the tangent spaces thereof are orthogonal.

A property of symplectic orthogonal spaces is that $\dim F^{\perp_\omega} + \dim F = \dim E$ for any linear subspace F of E. From proposition **??** it follows that $\dim M =$

$\frac{1}{2}\dim\mathcal{G}$, whereas the kernels of s and t have dimension $\dim\mathcal{G} - \dim M = \frac{1}{2}\dim\mathcal{G}$. We find that $\dim\ker \mathrm{d}_g s = \dim(\ker \mathrm{d}_g t)^{\perp\omega}$, and since $\ker \mathrm{d}_g s \subseteq (\ker \mathrm{d}_g t)^{\perp\omega}$ and the kernels are closed linear spaces, this completes the proof. □

Corollary 2.9. $\{s^*f, t^*g\} = 0$ *for all* $f, g \in C^\infty(M)$.

Proof. For any surjective submersion $p : S \to M$ we have the identity

$$(\ker \mathrm{d}p)^{\perp\omega} = \left\{X_{p^*f} \,\middle|\, f \in C^\infty(M)\right\}.$$

By the previous proposition, it follows that $\ker \mathrm{d}s = \{X_{t^*f} \mid f \in C^\infty(M)\}$ and similarly for $\ker \mathrm{d}t$. By proposition **??** the elements of these spaces are symplectically orthogonal. The identity

$$\{s^*f, t^*g\} = \omega(X_{s^*f}, X_{t^*g}) = 0$$

then completes the proof. □

Proposition 2.10. *There is a unique Poisson structure on M such that $t : \mathcal{G} \to M$ is a Poisson map, up to isomorphisms. Moreover, $s : \mathcal{G} \to M$ is anti-Poisson.*

Proof. The symplectic orthogonal at any point $t^{-1}(m)$ over $m \in M$ is the s-fiber, which is integrable. Since t is a surjective submersion, the symplectic structure on $\mathcal{G}$ completely determines the Poisson structure on M. The second statement follows from the fact that $s = t \circ i$, where t is a Poisson map and i an antisymplectomorphism. □

This proof uses a theorem from a previous lecture. ★

2.3 Integrability in Poisson geometry

In this section we will study the following problem:

Given a Poisson manifold (M, π), can one find a symplectic groupoid $\mathcal{G} \rightrightarrows M$ such that the target map t is a Poisson map?

Lie has studied this question in the linear case, which resulted in Lie's third theorem. Instead of integrating a Lie algebra to a Lie group, he has solved this problem by constructing a cotangent bundle out of the dual of the Lie algebra.

Add reference to: Lie. ★

We have seen three examples (see section **??**, page **??**) where we could integrate a Poisson manifold M:

1. The Poisson manifold itself is a symplectic manifold: $\mathcal{G} = M \times \overline{M}$;
2. The Poisson bracket is zero: $\mathcal{G} = \mathrm{T}^*M$; and
3. The Poisson structure is a Lie-Poisson structure: $M = \mathfrak{g}^*$ and $\mathcal{G} = \mathrm{T}^*G$.

Proposition 2.11. *Let* (M, π) *be a Poisson manifold,* $\mathcal{G} \rightrightarrows M$ *a symplectic groupoid and* $A = \mathrm{Lie}(\mathcal{G})$ *a Lie algebroid over* M*. Then*

$$\sigma : A \to \mathrm{T}^*M : v \mapsto i_v\omega|_{\mathrm{T}M}$$

is an isomorphism of Lie algebroids.

Proof. We have $A = \ker \mathrm{d}s|_M$, so the sections of A are right invariant vector fields. Remember that $s : \mathcal{G} \to M$ is a surjective submersion, so that

$$\ker \mathrm{d}s \cap \mathrm{T}M = \{0\}.$$

If $\sigma(v) = 0$, then $v \in \mathrm{T}M^{\perp_\omega} = \mathrm{T}M$, and $v \in \ker \mathrm{d}s|_M$, so $v = 0$. It follows that σ is an isomorphism of vector bundles.

By the results of the following exercise, we may see u, v as right invariant vector fields and write $i_u\omega = t^*\alpha$, $i_v\omega = t^*\beta$. This gives

$$\begin{aligned}
\sigma([u,v]) &= i_{[u,v]}\omega \\
&= \mathcal{L}_u i_v\omega - i_v\mathcal{L}_u\omega \\
&= \mathcal{L}_u i_v\omega - i_v(i_u\mathrm{d}\omega + \mathrm{d}i_u\omega) \\
&= \mathcal{L}_u i_v\omega - i_v\mathrm{d}i_u\omega \\
&= \mathcal{L}_u i_v\omega - \mathcal{L}_v i_u\omega + \mathrm{d}i_v i_u\omega \\
&= \mathcal{L}_u(t^*\beta) - \mathcal{L}_v(t^*\alpha) + \mathrm{d}(\omega(u,v))
\end{aligned} \tag{2.4}$$

Since t is a Poisson map, $t^*\gamma = i_X\omega$ implies that $\mathrm{d}t(X) = \pi^\sharp(\gamma)$, so we have

$$\sigma([u,v]) = t^*\left(\mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - \mathrm{d}(\pi(\alpha,\beta))\right) = t^*[\alpha,\beta]_\pi. \tag{2.5}$$

This shows that σ is an isomorphism of Lie algebroids. □

Exercise 2.2: Show that $X \in \mathfrak{X}(\mathcal{G})$ is right invariant (and therefore can be identified with a section of A) if and only if $i_X\omega = t^*\alpha$ for some $\alpha \in \Omega^1(M)$.
Hint: Use that vectorfields are Hamiltonian with resepect to s^* along the fibers of t.

Applications of symplectic groupoids Symplectic groupoids showed up in the quantization of Poisson manifolds. We are mostly concerned with *symplectic reduction*. In the linear case, we have a Lie algebroid $\mathrm{T}^*G \rightrightarrows \mathfrak{g}^*$, and a moment map $J : (S, \omega) \to \mathfrak{g}^*$. Symplectic reduction gives orbits $J^{-1}(\mu)/G_\mu$, $\mu \in \mathfrak{g}^*$.

Add reference to: Marsden-Weinstein ★

In the nonlinear case, we have a symplectic groupoid $\mathcal{G} \rightrightarrows (M, \pi)$, a Poisson map $J : (S, \omega) \to (M, \pi)$ and a symplectic action of $\mathcal{G}$ on (S, ω). For $x \in M$, we may constider $J^{-1}(x)/\mathcal{G}_x$, which is symplectic by the isotropy of $\mathcal{G}$.

^ ^

Jst Jst

3 Construction of symplectic groupoids

In the previous section we have seen a different approach to symplectic realizations of Poisson manifolds via symplectic groupoids and how this problem translates into an integrability problem in Lie theory. In this section we are interested in the construction of symplectic groupoids and how to relate this kind of objects to reduction procedures.

Recall that a symplectic groupoid is a Lie groupoid $\mathcal{G}$ over a smooth manifold M, together with a symplectic form $\omega \in \Omega^2(\mathcal{G})$ which is multiplicative in the sense that the multiplication map $m : \mathcal{G}_{(2)} \to \mathcal{G}$ has a Lagrangian graph in $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$. We summarize the main properties of symplectic groupoids:

1. The identity section $\epsilon : M \to \mathcal{G}$ is a Lagrangian embedding and the inversion map $i : \mathcal{G} \to \mathcal{G}$ is an antisymplectomorphism.
2. $\ker(\mathrm{d}t) = \ker(\mathrm{d}s)^\perp$ that is, s-fibers and t-fibers are symplectic orthogonal to one another.
3. There exists a unique Poisson structure on M such that $t : \mathcal{G} \to M$ is a Poisson map. This implies that $s : \mathcal{G} \to M$ is an anti-Poisson map. Note that this says that $t : \mathcal{G} \to M$ is a symplectic realization.

We have seen that every Poisson manifold (M, π) has a naturally associated Lie algebroid, given by $(\mathrm{T}^*M, \pi^\sharp, [\cdot,\cdot]_\pi)$ where the bracket $[\cdot,\cdot]_\pi$ is the natural bracket on 1-forms. We reconsider the questions formulated in the previous section:

1. Given a Poisson manifold (M, π), is there a symplectic groupoid $(\mathcal{G}, \omega)$ over M inducing π as in item **??**?

2. Is there a Lie groupoid integrating $(\mathrm{T}^*M, \pi^\sharp, [\cdot, \cdot]_\pi)$?

The answer to the first question in general is no. There exist obstructions to solve this problem, for more details see [**?**]. If this question does have an affirmative answer, then the asnwer to the second question is yes. In fact, if $(\mathcal{G}, \omega)$ is a symplectic groupoid over M, we consider M as a Poisson manifold with the Poisson structure induced by $\mathcal{G}$. If we denote by A the Lie algebroid of $\mathcal{G}$, then by proposition **??** this algebroid is isomorphic to the algebroid T^*M and we are able to answer the second question.

3.1 Construction of symplectic groupoids

Now we outline how to construct symplectic groupoids. First of all, the simplest Lie algebroid we can consider is $\mathrm{T}M$, where M is a smooth manifold. This algebroid is integrated by the pair groupoid $M \times M$. If M is connected, we can consider the fundamental groupoid $\Pi(M)$. In general, given a Lie algebroid A over M, the fundamental groupoid is given by the quotient space $\Pi(A)$ of A-paths modulo A-homotopy. This quotient is not necessarily smooth, but if it is, this object is what we are looking for.

In the special case of a Poisson manifold (M, π), we would like to integrate the Lie algebroid $(\mathrm{T}^*M, \pi^\sharp, [\cdot, \cdot]_\pi)$. In this case, A-paths are exactly Poisson paths and we can look at $\Pi(A)$, the quotient space of Poisson paths modulo Poisson homotopy. Let us consider the space $P(M)$ of all paths in M, and look at $\mathrm{T}^*P(M)$ as a symplectic Banach manifold. This Banach manifold can be viewed as the set $\tilde{P}(\mathrm{T}^*(M))$ of all paths in T^*M and it contains the set of Poisson paths $P(\mathrm{T}^*M)$. Define $P_0\Omega^1(M) = \{\eta_t \in \Omega^1(M) \mid t \in [0, 1], \eta_0 = \eta_1 = 0\}$ and consider the map $J : \tilde{P}(\mathrm{T}^*(M)) \to (P_0\Omega^1(M))^*$ defined by

$$\langle J(a), \eta_t \rangle = \int_0^1 \left\langle \frac{\mathrm{d}}{\mathrm{d}t} p(a(t)) - \pi^\sharp(a(t)), \eta_t(p(a(t))) \right\rangle \mathrm{d}t. \tag{3.1}$$

Note that the zero level set of J coincides with the set of all Poisson paths in M. In this direction we state a theorem that provides a construction of symplectic groupoids out of T^*M:

Theorem 3.1 (Cattaneo-Felder)**.** *We have an induced Hamiltonian action of $P_0\Omega^1(M)$ on $\tilde{P}(\mathrm{T}^*M) \simeq \mathrm{T}^*(P(M))$ and the orbits correspond to cotangent homotopy. Moreover the Marsden-Weinstein quotient space*

$$\mathcal{G} = \tilde{P}(\mathrm{T}^*M) // P_0\Omega^1(M)$$

is a symplectic groupoid whenever it is smooth.

3.2 Symplectic reduction

Now we study the problem of symplectic reduction using symplectic groupoids.

Definition 3.2. *A Lie groupoid $\mathcal{G} \rightrightarrows P$ acts on a manifold M together with a map $J : M \to P$ if there exists a map $a : \mathcal{G} \times_J M \to M$ where $\mathcal{G} \times_J M = \{(g, y) \mid s(g) = J(y)\}$, satisfying the following properties:*

i) $J(gy) = t(g)$

ii) $(gh)y = g(hy)$

iii) $\epsilon(J(y))y = y$.

The map $J : M \to P$ in the definition above is called a *moment map* of the action of $\mathcal{G}$ on M. A smooth manifold with an action of a Lie groupoid $\mathcal{G}$ is called a $\mathcal{G}$-space.

Example 3.3. If $\mathcal{G}$ is Lie groupoid over P, then $\mathcal{G}$ acts on itself by left multiplication, with moment map $t : \mathcal{G} \to P$.

Example 3.4. If $\mathcal{G}$ is a Lie groupoid over P, then $\mathcal{G}$ acts on P with moment map given by $\mathrm{Id} : P \to P$.

Given a $\mathcal{G}$-space M, we have a moment map $J : M \to P$ and we have seen in the example above that P is a $\mathcal{G}$-space in a natural way. This follows immediately from the definitions that the moment map J is equivariant.

Example 3.5. Consider a Lie group G acting on a manifold P and take $\mathcal{G}$ the transformation groupoid associated to this action. Then $\mathcal{G}$-spaces are exactly G-spaces together with an equivariant map $J : M \to P$.

With this notion of action, we look at a very special case of Lie groupoids actions, that is given by symplectic groupoids. Consider a symplectic manifold M which is a $\mathcal{G}$-space with moment map $J : M \to P$.

Definition 3.6. *The action of $\mathcal{G}$ on M is called symplectic if the graph*

$$\Gamma = \{(g, y, gy) \mid s(g) = J(y)\}$$

is a Lagrangian submanifold of $\mathcal{G} \times M \times \bar{M}$.

In this case we say that M is a symplectic $\mathcal{G}$-space. As in the usual case of G-actions on a symplectic manifold, given a symplectic $\mathcal{G}$-space, the moment map $J : M \to P$ is a Poisson map, where P has the Poisson structure induced by $\mathcal{G}$. In the context of symplectic actions of Lie groupoids, we have the following result analogus to the usual reduction procedure for symplectic actions of Lie groups:

Theorem 3.7 (Mikami-Weinstein)**.** *Let M be a symplectic $\mathcal{G}$-space with moment map $J : M \to P$. If $x \in P$ is a regular value for J, then there exists a unique symplectic structure ω_x on $J^{-1}(x)/\mathcal{G}_x$ such that*

$$p^*\omega_x = i^*\omega,$$

where $p : J^{-1}(x) \to J^{-1}(x)/\mathcal{G}_x$ is the canonical projection and $i : J^{-1}(x) \to M$ is the inclusion map.