Final Exam, Group Theory, January 29, 2008

Exercise 1 Show that a group $(G, \cdot)$ is abelian if and only if $f : G \longrightarrow G$, $f(x) = x^{-1}$ is a group homomorphism. (*1p*)

Exercise 2 On $\mathbb{R}^2$ we consider the operation $\circ$ defined by

$$(a, b) \circ (a', b') = (aa' + bb', ab' + a'b),$$

and we define

$$G = \{(a, b) \in \mathbb{R}^2 : a^2 \neq b^2\}, \quad N = \{(a, b) \in \mathbb{R}^2 : a^2 - b^2 = 1\}.$$

1. Show that $(G, \circ)$ is a group, but $(\mathbb{R}^2, \circ)$ is not. (*2p*)
2. Show that N is a subgroup of $(G, \circ)$ and G/N is isomorphic to $\mathbb{R}^*$. (*0.5p*)
3. Show that $(G, \circ)$ and $(N, \circ)$ are not isomorphic. (*0.5p*)

Exercise 3

1. Find the order and the signature of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 3 & 1 & 5 & 6 & 7 & 4 & 9 & 10 & 8 & 11 & 2 \end{pmatrix}$$

(*1p*)

2. Start from the table that appears on the left hand side of the picture. At each step you are allowed to move a number which is neighbor to the empty square, into the empty square (neighbor to the empty square means that the small square that contains it has a common side with the empty square). You are allowed to perform such operations as many times as you want. Prove that you can never arrive at the configuration that appears on the right hand side of the picture. (*1p*)

1	2	3	4
5	6	7	8
9	10	11	

	3	1	5
6	7	4	9
10	8	11	2

Figure 1:

Exercise 4 Recall that the dihedral group D_{10} can be written as

$$D_{10} = \{e, r, r^2, \dots, r^9, s, rs, r^2s, \dots, r^9s\},$$

$$r^{10} = e, s^2 = e, rsr = s.$$

1. Does D_{10} contain an element of order 5? But of order 4? (*1p*)
2. Is $\{e, r, s, rs\}$ a subgroup of D_{10}? Is it normal? (*1p*)
3. Is $\{e, r^5, s, r^5s\}$ a subgroup of D_{10}? Is it normal? (*0.5p*)
4. Show that $\{e, r^2, r^4, r^6, r^8\}$ is a normal subgroup of D_{10} and compute the resulting quotient group. (*0.5p*)

Exercise 5 Classify all groups with $2009 = 7^2 \times 41$ elements. (*1p*)

Notes:

1. WRITE YOU NAME (AND STUDENT NUMBER) CLEARLY ON EACH OF THE PAPER SHEETS YOU USE.
2. You can give your solutions in English or Dutch.
3. You are expected to explain each of your answers.
4. You are allowed to use the book, the lecture notes, and material that you have written down at the werkcollege.
5. Advise: if you cannot answer a question in a decent amount of time, postpone it for later and make sure you answer all the questions that you do know how to answer.