Vector fields on spheres

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Abstract

In this thesis we will discuss the problem of finding the maximal number of continuous pointwise linear independent vector fields on spheres in Euclidean space and give a more or less self sustaining determination of this number. We will discuss and prove both the results on the lower limit by Hurwitz and Radon and the results on the upper limit by Adams aswell as the prerequisits to set up the machinery to prove them.

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Contents

1 Introduction

In mathematics, when one introduces a certain object, one often asks the question how much there are. How many finite fields are there (up to isomorphisms one for every power of a prime), how many topological spaces are there (a lot), how many compact surfaces are there (two sequences of countably many ones), et cetera. Sometimes these questions get answered pretty easily (the number of prime numbers), some get answered but are more difficult to do so (classification of normal subgroups), and some are not solved up until now (classification of topological manifolds, anyone?). In this thesis we will look at vector fields on the n-dimensional sphere S^n. These vector fields live on the tangent bundle of S^n, and hence exhibit some kind of linear structure.

Using this linear structure, we can check if a number of vector fields on S^n are linearly independent at a certain point, or linearly independent at every point. We can embed the n-sphere into $\mathbb{R}^{n+1}$ and we get the interesting observation that the tangent space at x of S^n really is the n-dimensional subspace of $\mathbb{R}^{n+1}$ consisting of all vectors perpindicular to x. In particular every vector field on S^n can be represented by a continuous function $v : S^n \to \mathbb{R}^{n+1}$ such that $\langle x, v(x) \rangle = 0$. In this way the linear structure of vector fields becom even more apparent. Now the question 'how much' arises, namely the following: what is the maximum number of pointwise linear independent vector fields on S^n?

Having established an educated guess for the answer, the problem really has two parts: first we prove that there are in fact that much pointwise linear independent vector fields, and second we prove that there cannot be more. The educated guess turns out to be that there are $\rho(n) - 1$ linear independent vector fields on S^{n-1}. Here, writing $m = (2a+1)2^{b+4c}$ with a, b and c integers such that $0 \leq b < 4$, $\rho(m)$ is defined as $\rho(m) = 8c + 2^b$. Using this educated guess, the first part of the question is relatively easy and it simply uses more or less linear algebra and has been proven by Hurwitz and Radon in the 1930's. When you remember that topology as a well-defined field of mathematics is born at the beginning of the 20th century, this is fairly early. The answer to the second part of the problem took some more time and was fully proven by J.F. Adams in 1961. We will give a more or less self-sustaining solution to the whole problem, discussing some background and underlying definitions of topology and geometry, though sometimes we will have to concede that some arguments we just have to assume.
We will also exhibit several corollaries of this problem, for instance we can also determine all the spheres which are parallelizable, and on which $\mathbb{R}^n$ there exist a division algebra. The last question heuristically means whether there are more algebraic structures like the real numbers, complex numbers, quaternions and octonians (spoiler: there aren't).

The reader is assumed to have some knowledge of basic (differential) topology. Insights in the more involved parts of differential topology and group theory, and in the basics of algebraic topology can be found in the appendices.

2 Rough overview of the thesis

The thesis will be split in two parts: determining a lower limit, and determining an upper limit. Since the first part does not need any specific definitions, we'll get right to it in chapter 3. There we will start by looking at some small cases (S^1, S^3) and, using these small cases, we will make connections to other problems. Naturally we will also prove there that there are at least $\rho(n) - 1$ independent vector fields on S^{n-1}.
The proof uses the definition of vector fields where we explicitely embed them into $\mathbb{R}^n$, i.e. we identify a vector field with a continuous function $v : S^{n-1} \to \mathbb{R}^n$ such that $\langle x, v(x) \rangle = 0$ for all x. With this identification we will construct a specific family of matrices which will induce the desired vector fields. The structure of this familiy of matrices will come from a set of properties that will make sure that the vector fields that they induce will be pointwise linear independent. The fact that we can make a family of $\rho(n) - 1$ matrices will be a corollary of a result from algebra, the Hurwitz-Radon Theorem.

The second part (and largest) part of this thesis is devoted to proving there are no $\rho(n)$ vector fields on S^{n-1}. This proof will be split in two: first we show that having $\rho(n)$ vector fields on S^{n-1} is equivalent to a more technical property of cohomology and then afterwards we prove that this property is not true. The technical result here reads as follows:

Theorem 8.1. For any $m \geq 1$ the space $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is not co-reducible.

What co-reducible (and reducible) is will be defined later on, but in this specific case we will have to prove that a certain map cannot have degree 1. We will prove Theorem 8.1 in Chapter 8, while Chapter 4 will be used to prove that it implies that there a not $\rho(n)$ vector fields on S^{n-1}.

The argument that links vector fields on spheres and Theorem 8.1 has 4 bullet points:

i) Establishing a fiber bundle X over S^{n-1} which has a cross section if and only if there are $\rho(n)$ independent vector fields on S^{n-1}

ii) Proving that a certain space Y is reducible if X has a cross section

iii) Defining a certain a duality such that spaces X_1 and X_2 which are duals of each other satisfy that X_1 is reducible if and only if X_2 is coreducible

iv) Proving that Y and $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ are duals with respect to that duality for some m

The manifold from point i) will turn out to be the Stiefel-manifold $V_{\rho(n)+1}(\mathbb{R}^n)$:

Definition 4.1. The Stiefel manifold is defined by $V_k(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times k} : A^t A = \mathbb{1}\}$, the set of the frames of k orthonormal vectors in $\mathbb{R}^n$.

This manifold codes the correct information, since one can see it as a fiber bundle over S^{n-1} that has a cross-section if and only if there are $\rho(n)$ pointwise linear independent vector fields on S^{n-1}. Now with a result in homotopy theory (Proposition 4.22) we will prove the following theorem, which takes care of ii):

Proposition 4.23. If $2\rho(n) + 2 < n$ and the projection $q : V_{\rho(n)+1}(\mathbb{R}^n) \to S^{n-1}$ has a cross section, then there is a map $S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$ which composed with the quotient map

$\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-2} = S^{n-1}$ is a map of degree 1, in particular $\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$ is reducible.

The duality which we will use will be called S-duality. Unfortunately a lot of constructions in Chapter 4 will only work for high enough dimensions, so we will extensively take the suspension of our space (and maybe take the suspension of that, maybe even more than two times) to increase the dimension. We now take care of point iii) by

Theorem 4.12. Let X and X' be finite CW-complexes that are S-dual to each other. Then X is S-reducible if and only if X' is S-coreducable.

The S here is an unfortunate result of the fact that this dual only works if the dimension of the space is high enough. It follows that for we can get rid of the S if the space is nice enough, as stated in theorem 4.8. Finally point iv) gets taken care of by:

Theorem 4.21. Let r denote the order of $J(\xi_{k-1})$ in $J(\mathbb{R}\mathrm{P}^{k-1})$. And let p such that $rp > n+1$ then $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{n-k}$ is an S-dual of $\mathbb{R}\mathrm{P}^{rp-n+k-2}/\mathbb{R}\mathrm{P}^{rp-n-2}$.

The J that pops up here deals with some kind of stable trivial properties of vector bundles, and it turns out (in chapter 5) that this J group will be finite. The theorem implies that for every n we can find a m with $\rho(n) = \rho(m)$ such that $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is S-dual to $\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$ (more or less). Now tracing back the bullet points, combined with Theorem 8.1 will result in the conclusion that there cannot be $\rho(n)$ pointwise linear independent vector fields on S^{n-1}.

What rests is proving theorem 8.1. We're going to do this with K-Theory, in which we will pick the information of vector bundles over a space X and turn it into rings $K_{\mathbb{K}}(X)$. We'll define this in chapter 5 and also show that maps $f : X \to Y$ act nicely on these rings. In particular since co-reducibility of $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ has something to do with maps between spheres, the K-Theory of spheres and the K-Theory of $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$, this will come as a great help to prove theorem 8.1.
To enhence the tools we have with K-Theory we will use linear properties of vector bundles to construct ring homomorphisms on K-Theory. These ring homomorphisms will come from group representations and will also act nicely with respect to maps $f : X \to Y$. All these properties we will derive in chapter 7. In that chapter we will also construct a specific ring homomorphism Θ which will be used throughout the thesis.
The general idea is that in chapter 8 we will calculate the complex K-Theory of $\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^k$ and from that first the complex K-Theory of $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^k$ and from that the real K-Theory of $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^k$. We'll also calculate what Θ will do on all this K-Theories and use this to do prove theorem 8.1. To ease the calculation of Θ we will also discuss characteristic classes of vector bundles, in particular Stiefel-Whitney classes and Chern classes, this discussion will take place in chapter 6.

3 On determining a lower limit

In this section we'll start with looking at some small cases (S^1 and S^3). With making a short stop at whether we can find nowhere vanishing vector fields at all on S^n for n odd respectively even, we finish this section by proving that we can find at least $\rho(n) - 1$ vector fields on S^{n-1}.

3.1 Vector field on the circle and S^3

Since the tangent space of an n-dimensional smooth manifold is n-dimensional at every point, we see that there are at most n independent vector fields on an n-manifold. In particular we see that there can be at most one linear independent vector field on S^1 and at most 3 on S^3. On the circle one can easily write one down as:

$$v_1 : S^1 \to \mathbb{R}^2 : (x, y) \mapsto (-y, x)$$

When sketching this vector field we see that this is the vector field which traces the circle counter-clockwise at constant speed.

On S^3 one can also exhaust the whole tangent space with pointwise linear independent vector fields since we have the following three vector fields:

$$\begin{aligned} v_1 : S^3 \to \mathbb{R}^4 : &\quad (x_1, x_2, x_3, x_4) \mapsto (-x_2, x_1, -x_4, x_3) \\ v_2 : S^3 \to \mathbb{R}^4 : &\quad (x_1, x_2, x_3, x_4) \mapsto (-x_3, x_4, x_1, -x_2) \\ v_3 : S^3 \to \mathbb{R}^4 : &\quad (x_1, x_2, x_3, x_4) \mapsto (-x_4, -x_3, x_2, x_1) \end{aligned}$$

These vector fields on S^1 and S^3 share an interesting property of these spheres which they only share with S^7 (and only even in part with S^7), namely that they stem from the algebraic structure of these spheres. When identifying $(x, y) \in S^1$ with $x + iy \in \mathbb{C}$ (and hence S^1 with the unit circle in $\mathbb{C}$) and $(x_1, x_2, x_3, x_4) \in S^3$ with $x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H}$ (and hence identifying S^3 with the Hamilton quaternions of unit length) we see that in both cases v_1 represents multiplying with i and v_2 and v_3 represent multiplying with j and k respectively.

Seeing that we can find the maximal number of independent vector fields on S^1 and S^3, we can ask ourselves for which n we can find precisely n independent vector fields on S^n. For this we have some definitions:

Definition 3.1. A smooth manifold M is called *parallelizable* iff the tangent bundle TM is trivial, i.e. TM is isomorphic to $M \times \mathbb{R}^n$.

Definition 3.2. A *Lie-group* is a smooth manifold G together with a group operation $m : G \times G \to G$ such that both m and the map $i : G \to G : g \mapsto g^{-1}$ are smooth.

Intertwining these defintions with vector fields we get the following results:

Proposition 3.3. A smooth manifold M of dimension n is parallelizable if and only if there exist n pointwise linear independent vector fields.

Proposition 3.4. Any Lie-group is parallelizable.

Proof of Proposition 3.3. Suppose M is parallelizable, then $TM \simeq M \times \mathbb{R}^n$. In that case we can define $v_i : M \to M \times \mathbb{R}^n$: $x \mapsto (x, e_i)$. It is clear that these maps are linear independent at each point $m \in M$, and induce independent vector fields on M, by composing with the isomorphism $M \times \mathbb{R}^n \to TM$.
Next suppose that there are n independent vector fields $\{v_1, ..., v_n\}$ on M, in that case we construct an isomorphism from TM to $M \times \mathbb{R}^n$ by setting $v_i(m) \simeq (m, e_i)$ for every $m \in M$ and $i \in \{1, ..., n\}$. □

Proof of Proposition 3.4. We denote the Lie-group by G. We define for any $g \in G$: $L_g : G \to G : h \mapsto gh$. By the group structure this map is bijective, and by the smooth structure it is a diffeomorphism, since it is smooth by assumption and has smooth inverse $L_{g^{-1}}$.
Now we want to make an isomorphism between $G \times T_eG$ and TG. From L_g we immediately get a isomorphism between T_eG and T_gG, namely the differential of L_g at e. We use all these maps to patch it together:

$$\Phi : G \times T_eG \to TG : (g, v) \mapsto d(L_g)_e v$$

This Φ establishes an isomorphism between $G \times T_eG$ and TG, so we see that G is parallelizable. $\square$

Now looking at the definition of $\rho(n)$ we can see that the only values of n for which $\rho(n) = n$ (and which are interesting in this setting) are $n = 2$, $n = 4$ and $n = 8$. These represent S^1, S^3 and S^7 respectively, which in turn represent the elements of unit length of the complex numbers, Hamilton quaternions and octonians respectively. Now since octonians do not form a group (they are are not associative), the following result is an important correlary from the result by Adams:

Theorem 3.5. The only values of n for which S^n is parallelizable are $n = 1$, $n = 3$ and $n = 7$. Furthermore the only S^n which are Lie groups are S^1 and S^3.

Having constructed some vector fields on small examples we can try to use these to make vector fields on higher dimensional spheres. The following nice construction helps in doing this:

Proposition 3.6. If S^{n-1} has k orthonormal vector fields then S^{qn-1} has k orthonormal vector fields for every integer $q \geq 1$.

Proof. Let $\{v_1, ..., v_k\}$ be k maps from S^{n-1} to $\mathbb{R}^n$ such that $\langle v_i(x), x\rangle = 0$ and $\langle v_i(x), v_j(x)\rangle = \delta_{ij}$ for all i, j. Now we can group the coordinates of a vector $y \in S^{qn-1}$ in groups of n numbers, that is we have $y = (a_1x_1, ..., a_qx_q)$ such that $x_i \in S^{n-1}$ and $\sum_i a_i^2 = 1$. Now define functions $S^{qn-1} \to \mathbb{R}^{qn}$ by the following recipe:

$$v_i' : S^{qn-1} \to \mathbb{R}^{qn} : (a_1x_1, ..., a_qx_q) \mapsto (a_1v_i(x_1), ..., a_qv_i(x_q))$$

Now we get that $\langle y, v_i'(y)\rangle = \sum_j a_j^2 \langle x_j, v_i(x_j)\rangle = 0$ and also $\langle v_i'(y), v_j'(y)\rangle = \sum_m a_m^2 \langle v_i(x_m), v_j(x_m)\rangle = \sum_m a_m^2 \delta_{ij} = \delta_{ij}$, so we see that $\{v_1', ..., v_k'\}$ are k orthonormal vector fields on S^{qn-1}. $\square$

This for instance results in the existence of at least three vector fields on S^{4k-1} for any integer k. For S^7 we already had 7 by the octoninans, but for S^{11} we use this as a nice example.
Since the vector fields v_1, v_2 and v_3 are linear maps associated with matrices A_1, A_2 and A_3, from the defintion we see that v_1', v_2', v_3' are just the linear maps associated with $A_1 \oplus A_1 \oplus A_1$, $A_2 \oplus A_2 \oplus A_2$ and $A_3 \oplus A_3 \oplus A_3$ respectively. We get the following three vector fields on S^{11} which indeed follow to be orthogonal:

$$\begin{aligned}
v_1' : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) &\mapsto (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7, -x_{10}, x_9, -x_{12}, x_{11}) \\
v_2' : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) &\mapsto (-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6, -x_{11}, x_{12}, x_9, -x_{10}) \\
v_3' : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) &\mapsto (-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5, -x_{12}, -x_{11}, x_{10}, x_9)
\end{aligned}$$

We'll look at these maps again when in a few sections we exhibit another way to make three orthogonal vector fields on S^{11}.

3.2 Existence of nowhere vanishing vector fields

An important first step to tackle the ultimate problem of finding the maximal number of independent vector fields is to ask whether there exists at least one pointwise linear independent vector field, i.e. a nowhere vanishing vector field on S^n. If n is odd, writing $n = 2k - 1$, the sphere S^n lives in $2k$-Euclidean space and hence we can easily write down one nowhere vanishing vector field:

$$v : S^{2k-1} \to \mathbb{R}^{2k} : (x_1, x_2, ..., x_{2k-1}, x_{2k}) \mapsto (-x_2, x_1, ..., -x_{2k}, x_{2k-1})$$

So we see that for all odd-dimensional spheres we have a nowhere vanishing vector field, so we see that there are atleast 1 pointwise linear independent vector fields.

Now if $n = 2k$ and hence even there is a result which is possibly best known by it's name, namely the "Hairy Ball Theorem", which states that every vector field on S^{2k} has a point where it vanishes:

Proposition 3.7. If n is even then there exist no nowhere vanishing vector fields on S^n.

This theorem has many proofs, all using different approaches (combinatorics, differential forms, homotopy theory), but in my opinion the most intuitive argument (by Milnor [2]) is by using basic analysis.
First of all we note that by the Stone-Weierstrass Theorem we can approximate a non-vanishing continuous vector field by a non-vanishing continuously differentiable vector field, so proving that the latter doesn't exist implies that the first does not as well.

Proposition 3.7.1. If n is even then there exists no nowhere vanishing continuously differentiable vector field on S^n.

Proof of Proposition 3.7.1. Suppose we have a nowhere vanishing continuously differentiable vector field $v : S^n \to \mathbb{R}^{n+1}$, without loss of generality we can set $||v(x)|| = 1$ for all x (otherwise we normalize, which is a smooth operation since $||v(x)||$ is never 0). If we set

$$A = \{y \in \mathbb{R}^{n+1} | 1 - \epsilon \leq ||y|| \leq 1 + \epsilon\}$$

for some $0 < \epsilon < 1$ (hence A is compact), we can expand v onto A by continuing linear over the rays (i.e. setting $v(x) = ||x|| v(x/||x||)$). Now we use v to define a pertubation of the inclusion of A into $\mathbb{R}^{n+1}$:

$$f_t(x) = x + tv(x)$$

Now since $\langle x, v(x) \rangle = 0$ for all $x \in A$, we get that $||f_t(x)|| = ||x|| \sqrt{1 + t^2}$. Now we have the following claims:

Claim 1. For $t > 0$ small enough f_t is one-to-one and the volume of $f_t(A)$ is polynomial in t.

Claim 2. For $t > 0$ small enough the unit sphere gets mapped into the sphere with radius $\sqrt{1 + t^2}$.

Assuming both claims are true we get a contradiction. By the second claim we get that A gets mapped into $\{\sqrt{1 + t^2}(1 - \epsilon) \leq ||x|| \leq \sqrt{1 + t^2}(1 + \epsilon)\}$, which has volume $\sqrt{1 + t^2}^{n+1} \mathrm{vol}(A)$. Since $\sqrt{1 + t^2}^{n+1}$ is not a polynomial when n is even, we get a contradiction with claim 1, hence v has a zero.

Proof of Claim 1. Since A is compact and v is continuously differentiable, we get that there is a $c > 0$ such that $||v(x) - v(y)|| \leq c||x - y||$ (since all the partial derivates are continuous, they attain a maximum on A, and taking the maximum of these maxima we get c). Now let $0 < t < \frac{1}{c}$, and suppose that $f_t(x) = f_t(y)$. We get that $x - y = t(v(y) - v(x))$ and hence $||x - y|| = |t|||v(y) - v(x)|| \leq |t|c||x - y||$ since $0 < |t|c < 1$ we get that $x = y$, so f_t is injective.

Now we can use the change of variables theorem to get:

$$\mathrm{vol}(f_t(A)) = \int_{f_t(A)} 1\mathrm{d}y = \int_A |\det(Df_t(x))|\mathrm{d}x$$

Now since $Df_t(x) = \mathbb{1} + tDv(x)$ and the fact that the determinant of a finite dimensional matrix is a polynomial in its entries we get that $\det(Df_t(x)) = 1 + t\sigma_1(x) + ... + t^k\sigma_k(x)$. Here all the σ_i's are polynomials of all the partial derivatives of v and hence continuous in x. Now choosing t small enough we have that $\det(Df_t(x)) > 0$ for all $x \in A$ and we get:

$$\mathrm{vol}(f_t(A)) = \int_A |\det(Df_t(x))|\mathrm{d}x = \int_A \det(Df_t(x))\mathrm{d}x = \int_A \sum_{i=0}^{k} t^i\sigma_i(x)\mathrm{d}x = \sum_{i=0}^{k} t^k \left[\int_A \sigma_i(x)\mathrm{d}x\right]$$

Proof of Claim 2. When t is small enough, such that the determinant of $Df_t(x)$ always stricly positive and hence $Df_t(x)$ is always non-singular (see the proof of Claim 1), we get with the Inverse Function Theorem that f_t maps opens to opens. That means that when we restrict f_t as a map from S^n to $B^{n+1}_{\sqrt{1+t^2}}$ that we have that $f_t(S^n)$ is open in the larger sphere. On the other hand S^n is compact so we get that the image is also compact and hence closed in the larger sphere. Since the codomain is connected, we see that since the image is both open and closed in the codomain, is has to be the whole codomain, so $f_t(S^n) = B^{n+1}_{\sqrt{1+t^2}}$. □

Having proven that Hairy Ball Theorem we can now conclude that there exists a nowhere vanishing vector field on S^n if and only if n is odd.
Strickly speaking we suffice with this proof since we only talk abou spheres, however there is another, more deep, result from which we can conclude the same and can also be used for more general spaces. This result will be the subject of the next section.

3.3 Existence of nowhere vanishing vector fields II: the Poincare-Hopf Theorem

The Poincare-Hopf Theorem gives a strict requirement for when there is a nowhere vanishing vector field on a manifold, namely via the Euler characteristic. It follows that a compact manifold has a nowhere vanishing vector field if and only its Euler characteristic is zero. We will show this by using a triangulation of a given space, but any compact manifold is triangulizable, so this method checks out. We start by defining the Euler-characteristic for a CW-complex M:

Definition 3.8. The *Euler characteristic* if defined by $\chi(M) = \sum_i (-1)^i \#i$-cells.

Remark: Strictly speaking the χ is defined using the homology groups of M, but it turns out that the result is the same, in particular $\chi(M)$ is independent of the specific CW-decomposition of M.

Example 3.9. Since the sphere S^n has a 0-cell and a n-cell we have $\chi(S^{2n+1}) = 0$ and $\chi(S^{2n}) = 2$. The oriented close surface of genus g has a 0-cell, a 2-cell and $2g$ 1-cells, so $\chi(\Sigma_g) = 2 - 2g$.

The projective space $\mathbb{R}\mathrm{P}^n$ as a cell in every dimension up to and including n, so $\chi(\mathbb{R}\mathrm{P}^n) = 0$ if n is odd and $\chi(\mathbb{R}\mathrm{P}^n) = 1$ if n is even.

Now the Poincare-Hopf Theorem states the following:

Theorem 3.10. Let M be a compact orientable manifold, then there exists a nowhere vanishing vector field on M if and only if $\chi(M) = 0$.

It is immediately clear that this implies the Hairy Ball Theorem.

In this section we will give a rough proof of this theorem. A more precise proof can be found in several books on smooth manifolds (i.e. Guillaume & Pollack), or in [3]. First we define the mapping degree of a map between orientable manifolds. For this we take a map $f : M \to N$ with M and N compact orientable manifolds of the same dimension and $y \in N$ a regular value for f. In that case we know that $f^{-1}(y)$ is a discrete set of points on a compact manifold and hence finite and we then define:

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \mathrm{sign}(df)_x$$

Here $\mathrm{sign}(df)_x$ must be understood as $\mathrm{sign}(\det(df_x))$ with respect to the positively oriented basis. The following lemma will be frequently used to show that degrees are invariant under extensions:

Lemma 3.11 (Extension Lemma)**.** Let M be a $n+1$-dimensional manifold with boundary and N be a n-dimensional manifold, furthermore let $f : \partial M \to N$ be a map and $y \in N$ a regular value of f. If f has an extension $F : M \to N$ then $\deg(f, y) = 0$.

Using this lemma we can for instance prove that the degree of two homotopic maps g_0 and g_1 with common regular value agree. Here we use that the map from $M \times \{0, 1\} \to N$ defined as $g(\cdot, 0) = g_0$ and $g(\cdot, 1) = g_1$ has an extension to $M \times [0, 1]$ in the homotopy map between g_0 and g_1 and hence $\deg(g, y) = 0$. But since $\partial(M \times [0, 1]) = M \times \{1\} - M \times \{0\}$ we see that $\deg(g, y) = \deg(g_1, y) - \deg(g_0, y) = 0$.
Now this notion allows us to define the mapping degree even if y is not a regular value of f: we find a g homotopic to f which has a regular value at y and compute the degree there. All in all we get a function $\deg(f, \cdot) : N \to \mathbb{Z}$. Now when looking at neighbourhoods around regular values (which lay dense in N) one can prove that this function is locally constant, and hence constant if M and N are connected and in that case we set $\deg(f) = \deg(f, y)$.

Next we define the index of a zero of a vector field, which we will first do in $\mathbb{R}^n$. Suppose we have a vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ for which $V(0) = 0$ and we have a closed disk D around 0 such that the only zero of V on D is 0. Then we have a function $\partial D \to S^{n-1} : x \to V(x)/|V(x)|$. We define index of the zero then as the degree of this map. It follows from the Extension Lemma that the degree does not depend on the chosen disk.
If we now have a vector field with discrete zeroes, then we choose charts around those zeroes and use the same construction (the answer is independent of the choice of charts). We will denote the index of a zero x of a vector field V by $\mathrm{Ind}_x(V)$.

Next suppose that we have a disk D containing some zeroes $x_1, ..., x_k$ of V. Punching open disks D_i such that $D_i \cap \{x_1, .., x_k\} = \{x_i\}$ from D, we see that we can extend $V(x)/|V(x)|$ from $\partial(D - \cup_i D_i)$ to the whole of $D - \cup_i D_i$, and hence from the extension lemma we see that the degree of $V(x)/|V(x)|$ on ∂D is equal to the sum of the indices of the zeroes. We'll use this to prove one of the important constructions on vector fields, namely:

Lemma 3.12. Let x be a zero of a vector field V and U a neighbourhood of x in M containing no other zero than x. Let D be a closed disk in U whose interior contains x. There exists a vector field V_1 that equals V outside of D with only finitely many zeroes in D, all of which are non-degenerate. Any such V_1 satisfies $\sum_{x\in V^{-1}(0)} \mathrm{Ind}_x(V) = \sum_{x\in V_1^{-1}(0)} \mathrm{Ind}_x(V_1)$.

Proof. From partitions of unity we get that there is a bump function $\rho : M \to \mathbb{R}$ such that $\rho|_{M-D} = 0$ and ρ is 1 in a neighbourhood around x. Without loss of generality we can assume U to be a chart domain. Let $a \in \mathbb{R}^n$ and in coordinate representation we define $V_1(z) = V(z) + \rho(z)a$ on U and $V_1(z) = V(z)$ outside U. If a is small enough then $V_1(z)$ can only be zero when $\rho = 1$, and if we now choose a such that $-a$ is a regular value of V in coordinate representation, then we have that the zeroes of V_1 will be non-degenerate and still of finite number. Now since ρ has to be 0 on ∂D we have that V and V_1 agree on the boundary and we have:

$$\sum_{x\in V^{-1}(0)\cap D} \mathrm{Ind}_x V = \deg((V/|V|)_{\partial D}) = \deg((V_1/|V_1|)_{\partial D}) = \sum_{x\in V^{-1}(0)\cap D} \mathrm{Ind}_x V_1$$

Since V and V_1 are the same outside of U, the indeces of the zeroes outside U are also the same and hence the sum of all the indeces is also the same. □

Doing this trick around all the degenerate zeroes we have the following corollary:

Corollary 3.13. If V is a vector field with only finitely many zeroes then there is a vector field W with only finitely many zeroes which are all non-degenerate such that $\sum_{x\in V^{-1}(0)} \mathrm{Ind}_x V = \sum_{x\in W^{-1}(0)} \mathrm{Ind}_x W$.

Another finding in local coordinates is that if we have a non-degenerate zero x that $\mathrm{Ind}_x V = \mathrm{sign}((dV)_x)$, so in the last corollary we have $\sum_{x\in W^{-1}(0)} \mathrm{Ind}_x W = \sum_{x\in W^{-1}(0)} \mathrm{sign}((dV)_x)$.

Now since we saw that degrees do only depend on homotopy classes we see that if we group zeroes of two vector fields V and W in a smart away together in disks D that the degree of $V/|V|$ and $W/|W|$ will be the same on ∂D (using the linear structure of TM we see that two V and W are homotopic via the homotopy $tV + (1-t)W$), so we see that under weak assumptions that $\sum_{x\in V^{-1}(0)} \mathrm{Ind}_x V$ does not depend on the vector field V, as long as it has a finite number of zeroes. Next up we'll construct a vector field such that the sum equals the Euler characteristic of M, and then this fact shows that if we have a vector field with no zeroes then $\chi(M) = 0$.

Given a trianugulation T (which is finite since M is compact) we construct the vector field V_T in the following way. Given a simplex Δ^n in M we set that V_T has a zero in the center of Δ^n and the flow lines flow from the center of Δ^n to the centers of the Δ^{n-1}-simplices that make up the boundary of Δ^n and we extend the vector field in between, see the picture below.

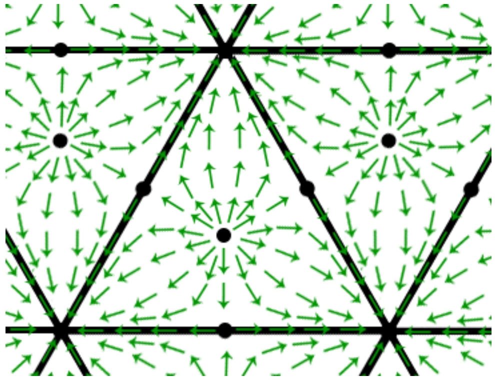

Figure 1: The vector field V_T

Now from the picture it's clear that the zero in the center of a simplex Δ^i is a saddle which sends flow lines out in a i-dimensional hyperplane and recieves it from the other higher dimensional simplices. Hence a intuitive argument from analysis shows that the index of the zero is $(-1)^i$. Now we get:

$$\sum_{x \in V_T^{-1}(0)} \mathrm{Ind}_x V_T = \sum_i \sum_{x \text{ center of } i\text{-simplex}} (-1)^i = \sum_i (-1)^i \# i\text{-cells} = \chi(M)$$

Hence for any vector field with a finite number of zeroes we have $\sum_{x \in V^{-1}(0)} \mathrm{Ind}_x V = \chi(M)$.
As promised, from this fact it follows that if M admits a nowhere vanishing vector field than $\chi(M) = 0$ since, of course, for that vector field the sum over zeroes is the empty sum so it is equal to zero.
On the other hand if $\chi(M) = 0$ we know that V_T has the same number of zeroes with index $+1$ as zeroes with index -1. Now we can homotope V_T such that the zeroes are paired together such that that every zero of index $+1$ is in a chart domain together with 1 zero of index -1 and no other zeroes and vice versa. Now we can do the opposite of the splitting lemma and merge the zeroes, effectively cancelling each other and we're left with no zeroes at all. More stricly we see that if we take a disk around a pair of such zeroes, we have a degree 0 map on the boundary of that disk and we can extend that map onto the whole disk, leaving no zeroes behind.

So we see that it is necessary and sufficient to have $\chi(M) = 0$ for a compact manifold M to have a nowhere vanishing vector field.
Since $\chi(S^n) = 1 + (-1)^n$, we see that this also proves that there is a nowhere vanishing vector field on S^n if and only if n is odd.
Now we continue with trying to up this lower limit if n is odd.

3.4 Lower limit of number of independent vector fields on S^n and the Hurwitz-Radon Theorem

Having proven that there is always at least one nowhere vanishing vector field on S^n when n is odd, we can now ask ourselves how many independent ones we can find. To end up at the best positive result for this, we make a small detour to algebra. It follows that the number of linear independent vector fields on S^{n-1} is closely related to bilinear maps $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$.
First we choose orthonormal bases $\{e_1, ..., e_n\} \subset \mathbb{R}^n$ and $\{e_1, ..., e_k\} \subset \mathbb{R}^k$. These give rise to inner

products $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ and $\mathbb{R}^k$, which in turn give rise to norm maps N both $\mathbb{R}^n \to \mathbb{R}$ and $\mathbb{R}^k \to \mathbb{R}$ set by $N(x) = \sqrt{\langle x, x \rangle}$. Now suppose that we have a bilinear map $\star : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$, which satisfies

$$N(x \star y) = N(x)N(y)$$

and

$$x \star e_1 = x.$$

Using the properties we set on our maps (in particular the result that $N(e_i + e_j)^2 = 2$ if $i \neq j$) and the properties of inner-product spaces (namely $\langle x, y \rangle = N(x+y)^2 - N(x)^2 - N(y)^2$) we obtain that for $i \neq j$:

$$\begin{aligned} \langle x \star e_i, x \star e_j \rangle &= N(x \star e_i + x \star e_j)^2 - N(x \star e_i)^2 - N(x \star e_j)^2 \\ &= N(e_i + e_j)^2 N(x)^2 - N(x)^2 N(e_i)^2 - N(x)^2 N(e_j)^2 \\ &= (2 - 1 - 1)N(x)^2 = 0 \end{aligned}$$

In particular for $i = 1$ we get that $\langle x, x \star e_j \rangle = 0$. Now defining $v_i : S^{n-1} \to S^{n-1}$ by $v_i(x) = x \star e_i$ for $i = 2, ..., k$ we see that we get $k - 1$ pointwise independent smooth vector fields on S^{n-1}.
On the other hand if we have $k - 1$ independent vector fields, we can make them orthonormal and hence construct from it in much the same manner a bilinear form $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ upholding the same rules as before. So we see that there is a very natural correspondence between bilinear forms of this sort and vector fields on spheres.

An other equivalent question from algebra is for which triples (i, j, k) we can find formulas of the form

$$(x_1^2 + ... + x_i^2)(y_1^2 + ... + y_j^2) = (z_1^2 + ... + z_k^2)$$

where $z_1, ..., z_k$ are bilinear in $\vec{x}$ and $\vec{y}$. An example of this is the formula

$$(x_1^2 + x_2^2)(y_1^2 + y_2)^2 = (x_1 y_1 - x_2 y_2)^2 + (x_2 y_1 + x_1 y_2)^2$$

which comes from the multiplication of complex numbers. Since these sorts of equations boil down to $N(x)^2 N(y)^2 = N(z)^2$ with z bilinear in x and y we can make from such an equation a bilinear form $\mathbb{R}^i \times \mathbb{R}^j \to \mathbb{R}^k$. It is clear that such a bilinear function satisfying $N(x)N(y) = N(x \star y)$ gives rise to such a multipication of squares, so we again get a correspondence. Now the last representation of the problem is called the Hurwitz-Radon Theorem which deals with finding such tuples (i, j, k) and states that $(n, \rho(n), n)$ is such a triple, hence there are $\rho(n) - 1$ independent vector fields on S^{n-1}. Since all these deal with linear vector fields and hence matrices, we prove this by using the fourth way to represent the problem: families of matrices. The goal here is to find matrices such that multiplication with these matrices induce pointwise linear independent vector fields on S^{n-1}.

The first property that we want to set on our matrices is that $\langle x, Ax \rangle = 0$ for all x, since then the map $x \mapsto Ax$ is a vector field on S^{n-1}. The following lemma identifies these matrices as the skew-symmetric matrices:

Lemma 3.14. For every matrix A we have $\langle x, Ax \rangle = 0$ for all x if and only if $A + A^t = 0$.

Proof. Suppose that $\langle x, Ax \rangle = 0$ for all x. Now for arbitrary x, y we have:

$$\langle x, (A + A^t)y \rangle = \langle x, Ay \rangle + \langle Ax, y \rangle = \langle x, Ax \rangle + \langle x, Ay \rangle + \langle y, Ax \rangle + \langle y, Ay \rangle = \langle (x + y), A(x + y) \rangle = 0$$

Since this is for all x, y it follows that $A + A^t = 0$. On the other hand if $A + A^t = 0$ we get for all x:

$$2\langle x, Ax\rangle = \langle x, Ax\rangle + \langle Ax, x\rangle = \langle x, (A + A^t)x\rangle = 0 \qquad \square$$

We want to use these skew-symmetric matrices to make $\rho(n) - 1$ pointwise linear independent vector fields on S^{n-1}. That means that first of all that we also require them to be nowhere vanishing. This can be made sure of if we set that the matrices we look at are also orthogonal. The independent part can be made sure of by wanting all vectors to form an orthonormal system at every point, that is, if $\{A_1, ..., A_k\}$ is our family of matrices to make the vector fields, that $\{A_1x, ..., A_kx\}$ form an orthonormal system for every $x \in S^{n-1}$. In particular this means that for $i \neq j$ that $\langle A_ix, A_jx\rangle = \langle x, A_i^tA_jx\rangle = 0$ for every x so by the virtue of Lemma 3.14 we have $A_i^tA_j + A_j^tA_i = 0$, which by skew-symmetry of A_i and A_j is equivalent to saying $A_iA_j = -A_jA_i$. Now putting all these together we can put a precise definition of the families of matrices we want:

Definition 3.15. A *Hurwitz-Radon family* of matrices (*H-R family*) is a set of $n \times n$-matrices $\{A_1, ..., A_k\}$ such that:

i) $A_i = -A_i^t$ for all i (skew-symmetry)

ii) $A_i^tA_i = -A_i^2 = \mathbb{1}$ for all i (orthogonality)

iii) $A_iA_j = -A_jA_i$ for all $i \neq j$ (mutual orthogonality)

Now having made the definition we can formuate the final theorem which will give the biggest prositive result to the problem:

Theorem 3.16. For every $n \geq 2$ there exists a H-R family of $\rho(n) - 1$ $n \times n$-matrices.

By the previous discussion this theorem has the corollary that there are *at least* $\rho(n) - 1$ pointwise linear independent smooth vector fields on S^{n-1}:

Corollary 3.17. For every $n \geq 2$ there exist $\rho(n) - 1$ continuous vector fields on S^{n-1} which are pointwise linear independent.

Proof of Corollary 3.17. For fixed n we take the H-R family $\{A_1, ..., A_{\rho(n)-1}\}$ we get by virtue of theorem 3.16. Now set $v_i : S^{n-1} \to \mathbb{R}^n$ to be $v_i(x) = A_ix$. This family of vector fields are smooth and hence continuous and pointwise linear independent since $\{v_1(x), ..., v_{\rho(n)-1}(x)\}$ form an orthonormal system for all x by the definition of the A_i's. $\square$

The following construction for the proof of Theorem 3.16 is by Geramita and Pullman [4]

Proof of Theorem 3.16. First we define the following 2×2-matrices:

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we have the following base cases:

Claim 1.

i) $\{R\}$ is a H-R family of $\rho(2) - 1$ 2×2-matrices

ii) $\{R \otimes \mathbb{1}_2, P \otimes R, Q \otimes R\}$ is H-R family of $\rho(4) - 1$ 4×4-matrices

iii) $\{\mathbb{1}_2 \otimes R \otimes \mathbb{1}_2,\ \mathbb{1}_2 \otimes P \otimes R,\ Q \otimes Q \otimes R,\ P \otimes Q \otimes R,\ R \otimes P \otimes Q,\ R \otimes P \otimes P,\ R \otimes Q \otimes \mathbb{1}_2\}$ is a H-R family of $\rho(8) - 1$ 8×8-matrices

From these base cases we can inductively grow our families using these steps:

Claim 2. Suppose $\{A_1, .., A_s\}$ is a H-R family of $n \times n$-matrices, then:

i) $\{R \otimes \mathbb{1}_n\} \cup \{Q \otimes A_i | i = 1, ..., s\}$ is a H-R family of $s + 1$ $2n \times 2n$-matrices

ii) If $\{B_1, ..., B_7\}$ is the H-R family of 8×8-matrices of Claim 1.iii) then $\{R \otimes \mathbb{1}_{8n}\} \cup \{P \otimes \mathbb{1}_8 \otimes A_i | i = 1, ..., s\} \cup \{Q \otimes B_j \otimes \mathbb{1}_n | j = 1, ..., 7\}$ is a H-R family of $16n \times 16n$-matrices of order $s + 8$.

iii) $\{A_i \otimes \mathbb{1}_b | i = 1, ..., s\}$ is a H-R family of $nb \times nb$-matrices of order s

Both claims follow repeatedly using the properties of R, P and Q together with $(A \otimes B)(C \otimes D) = (AB) \otimes (CD)$ and $(A \otimes B)^t = A^t \otimes B^t$.

That we can now build a H-R family of $\rho(n) - 1$ $n \times n$-matrices follows from the following arguments. First we note that Claim 1 takes responsibility for $n = 2, 4, 8$ and then combining Claims 1.iii) and 2.i) we get $n = 16$ ($\rho(16) = \rho(8) + 1$). Now we have $\rho(16k) = \rho(k) + 8$ so repeatedly using Claim 2.ii) gives us families of $\rho(n) - 1$ $n \times n$-matrices for all $n = 2^t$. Now for b odd we have $\rho(kb) = \rho(k)$ so at last we can use claim 2.iii) to finish the deal. □

This construction is somewhat mysterious, but it follows from puzzeling with R, Q and P. For instance when looking at property $i)$ for a H-R family we see that R is antisymmetric and Q, P and $\mathbb{1}_2$ are symmetric, so R acts as 'odd' and Q and P act as 'even'. Since we want something 'odd' we must have an odd number of R's (or add an even number of R's in the follow-up claim). The same thing works for property $ii)$ since $-R^2 = Q^2 = P^2 = \mathbb{1}_2^2 = \mathbb{1}_2$.
The most important part however is checking part $iii)$. Here we use $RQ = -QR$, $RP = -PR$ and $PQ = -QP$ whereas all the other pairs (e.g. RR or $Q\mathbb{1}_2$) are commutative. Again we want to effectively pick up one minus-sign. So making this construction boils down to trying a lot of combinations of tensor products and finding the right one (or let somebody else find it).

The most interesting cases are the cases which we cannot solve by guessing ($n = 2, 4, 8$, which are the multiplication tables for complex numbers, quaternions and octonians respectively), but these are either quite empty (when n is odd), not very interesting ($n = 6$ which is just $R \otimes \mathbb{1}_3$, equivalently for just taking an odd multiple of $n = 2, 4, 8$), or for very big n (since the most interesting part of the construction is in going to $n = 16$ and multiplying with 16). The downside of very big n is that the matrices will become big, but also that since $\rho(n)$ goes about logarithmically, the matrices will become very empty with mostly 0's, since the number of entries grows quadratically. We can however compare the cases we've guessed from the algebraic structure with those from the construction of the theorem. For $n = 2$ we see that they argree, but this ain't that hard since the tangent space of S^1 is one dimensional. For $n = 4$ we recall the results of the quaternionic strucutre and get the following result:

$$v_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$Q \otimes R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad R \otimes \mathbb{1}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad P \otimes R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

(NB: Here we identify the functions v_i with the matrix that they come from)
So we see that they do not exactly coincide, but they are the same up to some minus signs. So we see that they induced two different though very similar multiplication rules on $\mathbb{R}^4$.

What we also can do is compare the results of this process for S^{11} with the three vector fields we found earlier, recall them to be:

$$\begin{aligned}
v_1' &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \mapsto (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7, -x_{10}, x_9, -x_{12}, x_{11}) \\
v_2' &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \mapsto (-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6, -x_{11}, x_{12}, x_9, -x_{10}) \\
v_3' &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \mapsto (-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5, -x_{12}, -x_{11}, x_{10}, x_9)
\end{aligned}$$

If we calculate the vector fields that we get with this tensor product, it follows that we get:

$$\begin{aligned}
v_1 &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \mapsto (x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, -x_1, -x_2, -x_3, -x_4, -x_5, -x_6) \\
v_2 &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \mapsto (x_{10}, x_{11}, x_{12}, -x_7, -x_8, -x_9, x_4, x_5, x_6, -x_1, -x_2, -x_3) \\
v_3 &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \mapsto (x_4, x_5, x_6, -x_1, -x_2, -x_3, -x_{10}, -x_{11}, -x_{12}, x_7, x_8, x_9)
\end{aligned}$$

The differences are interesting, since v_1 and v_2 really scramble up the coordinates while v_1', v_2' and v_3' are more grouped. This is really the difference between taking direct sums and tensor product of matrices: recall that v_i' is the linear map from $A_i \oplus A_i \oplus A_i$ (with A_i the result from S^3) while v_i is the map $A_i \otimes \mathbb{1}_3$. The natural question that arises now since we have two sets of vector fields is whether they are independent of each other, i.e. if we can make more that three pointwise independent vector fields from combining the two sets. It turns out that we cannot do this. For instance if we plug in e_1 we get:

$$v_3(e_1) = -e_4 = -v_3'(e_1)$$

So we see that adding v_3 to the set $\{v_1', v_2', v_3'\}$ adds an independence in a point. Similarly we get:

$$v_2(e_5) = e_8 = v_3'(e_5)$$

$$v_1(e_8) = -e_5 = v_3'(e_8)$$

So we see that adding v_1, v_2 or v_3 to the set $\{v_1', v_2', v_3'\}$ results in an independence in a point, which is the desired result since $\rho(12) - 1 = 3$, so finding a fourth pointwise independent vector field would be a problem.

Just on a sidenote we see that we make these H-R families of a lot of tensor products of R, P and Q. Now these matrices also pop-up in quantum mechanics in the discussion of spin systems.

Now since an m-coupled spin system has normalized states made up out of tensor products of $|\uparrow\rangle_i$ or $|\downarrow\rangle_i$ for all indices $i = 1, ..., m$, it effectively lives on $S^{(2^m-1)}$. Now since multiplying with an odd number does not change much, in all the effective cases (S^{n-1} with n a power of two) the vector fields constructed in this manner coincide with coupled spin-operators, which are one of the most elementary examples of quantum mechanical systems.

4 Proving an upper limit for the number of vector fields

In this section we'll talk about how to prove that there cannot be $\rho(n)$ independent vector fields on S^{n-1}. We'll see that this is equivalent to another problem (Theorem 8.1), and the rest of the thesis will be devoted to proving this equivalent problem. First we'll start with some defintions.

4.1 Stiefel manifold

The Stiefel manifold will be of particular interest since it turns out to be a fiber bundle that has a cross-section exactly when there are a certain number of vector fields on a sphere (see Theorem 4.2). This gives the link between multiple maps (a number of vector fields) and only one map between manifolds.

Since we work with continuous vector fields which can be formed into continuous orthonormal frames by Gram-Schmidt, it is natural for us to look at the manifold of orthonormal frames, that is:

Definition 4.1. The *Stiefel manifold* is defined by $V_k(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times k} : A^tA = \mathbb{1}\}$, the set of the frames of k orthonormal vectors in $\mathbb{R}^n$.

One can realise $V_k(\mathbb{R}^n)$ as the quotient $O(n)/O(n-k)$, more or less by deleting the last $n-k$ rows from elements of $O(n)$. More specific we know that any two orthonormal k-frames in $\mathbb{R}^n$ can be deformed in each other via an orthonormal transformation, so we see that $O(n)$ acts on $V_k(\mathbb{R}^n)$. Seeing $O(n-k)$ as the subgroup of $O(n)$ that acts on the $n-k$ vectors perpindicular to a given k-frame, we see that $O(n-k)$ acts trivially on $V_k(\mathbb{R}^n)$ and hence $V_k(\mathbb{R}^n) \simeq O(n)/O(n-k)$. Since $V_1(\mathbb{R}^n)$ is the set of vectors in $\mathbb{R}^n$ of length 1, it follows $V_1(\mathbb{R}^n) = S^{n-1}$. Denoting $A \in V_k(\mathbb{R}^n)$ by $A = (v_1, ..., v_k)$ we can define $p : V_k(\mathbb{R}^n) \to V_l(\mathbb{R}^n)$ by $(v_1, ..., v_k) \mapsto (v_1, ..., v_l)$ (for $l \leq k$), in particular if $l = 1$ we see that $V_k(\mathbb{R}^n)$ is a fiber bundle over S^{n-1}. Some specific cases of this are $V_2(\mathbb{R}^n)$ which consists of tuples of two orthonormal vectors, hence $V_2(\mathbb{R}^n) = TS^{n-1}$ and $V_n(\mathbb{R}^n)$ which is by definition equal to $O(n)$.

The Stiefel manifold is connected to vector fields in the following natural way:

Theorem 4.2. There are k orthogonal vector fields on S^{n-1} if and only if $V_{k+1}(\mathbb{R}^n)$ has a cross-section.

Proof. If we have orthogonal vector fields $v_1, ..., v_k$ then for every $x \in S^{n-1}$ we have that $\{x, v_1(x), ..., v_k(x)\}$ is a orthonormal set. Now defining $s : S^{n-1} \to V_{k+1}(\mathbb{R}^n)$ by $s(x) = (x, v_1(x), ..., v_k(x))$ we see that this is a well-defined function for which $ps = \mathrm{id}_{S^{n-1}}$, so $V_{k+1}(\mathbb{R}^n)$ has a cross-section.

On the other hand if we have a cross-section $s : S^{n-1} \to V_{k+1}(\mathbb{R}^n)$, we set functions $v_1, ..., v_k$

from S^{n-1} to $\mathbb{R}^n$ such that $s(x) = (x, v_1(x), ..., v_k(x))$ (just project the i'th coloumn of s to a vector of $\mathbb{R}^n$, the first coloumn is fixed by the fact that s is a cross-section). Now that means that $\langle v_i, v_j \rangle = \delta_{ij}$ and $\langle x, v_i \rangle = 0$ for all i so we get that all the v_i's are vector fields on S^{n-1} and furthermore $\{v_1, ..., v_k\}$ is an orthonormal set of vector fields. $\square$

4.2 Reducible and co-reducible spaces

In this section we'll discuss the defintion of reducible and co-reducible spaces. This will give us the possibility to formulate a theorem that is equivalent to proving that there are no $\rho(n)$ vector fields on S^{n-1} (Theorem 8.1).

Definition 4.3. A space X is *reducible* if there is a $n \in \mathbb{N}$ and $f : S^n \to X$ such that $f_* : \widetilde{H}_i(S^n) \to \widetilde{H}_i(X)$ is an isomorphism for all $i \geq n$.

Definition 4.4. A space X is *co-reducible* if there is a $n \in \mathbb{N}$ and $g : X \to S^n$ such that $g^* : \widetilde{H}^i(S^n) \to \widetilde{H}^i(X)$ is an isomorphism for all $i \leq n$.

In short it means that (co)-reducible spaces act like a sphere for all intents and purposes in either low or high dimensions. For certain CW-complexes we can refine this defintion to make it more intuitive:

Lemma 4.5. Let X be a n-dimensional CW-complex with only 1 n-cell and let v be the quotient map $X \to X/X^{n-1} = S^n$. Then X is reducible if and only if there is a map $f : S^n \to X$ such that $vf \simeq \mathbb{1}_{S^n}$ (i.e. has degree 1).

Proof. Suppose X is reducible, since $H_n(X) = \mathbb{Z}$ (since there is only 1 n-cell), and $H_k(X) = 0$ for $k > n$ we know that the map f that induces an isomorphism f_* must be a map $f : S^n \to X$. Now we have the following diagram / exact sequence:

$$\begin{tikzcd}
 & \widetilde{H}_n(S^n) \arrow[d, "f_*"', "\cong"] \arrow[dr, "(vf)_*"] & & \\
0 = \widetilde{H}_n(X_{n-1}) \arrow[r] & \widetilde{H}_n(X) \arrow[r, "v_*"] & \widetilde{H}_n(S^n) \arrow[r] & \widetilde{H}_{n-1}(X_{n-1})
\end{tikzcd}$$

At once we see that v_* is injective and hence it is not the zero-map (since $\widetilde{H}_n(X) \neq 0$). Now $\widetilde{H}_{n-1}(X_{n-1})$ is a free group (in fact it is $\mathbb{Z}^k$ with k the number of $n-1$-cells), which means that the kernel of the map $\mathbb{Z} = \widetilde{H}_n(S^n) \to \widetilde{H}_{n-1}(X_{n-1})$ and hence the image of v_* can only be 0 or $\mathbb{Z}$. Since v_* is not the zero-map this means that the image is non-trivial and hence has to be $\mathbb{Z}$, so we see that v_* is an isomorphism, and we see that $(vf)_*$ is an isomorphism and hence vf is homotopic to the identity.
On the other hand if such a f exists we can chase the same diagram and see that f_* is an isomorphism of the H_n's, and since X is n-dimensional we immediately have $H_k(X) = 0$ for $k > n$, so f_* certainly is an isomorphim of the H_k's for $k > n$. $\square$

Lemma 4.6. Let X be a CW-complex such that $X^n = S^n$ (i.e. X can be decomposed as having among others 1 0-cell, no cells of dimension $1, ..., n-1$ and 1 n-cell) and let $u : S^n \to X$ be the inclusion. Then X is co-reducible if and only if there is a map $g : X \to S^n$ such that $gu \simeq \mathbb{1}_{S^n}$ (i.e. has degree 1).

Proof. The proof uses the same arguments as in the previous lemma, only now with induced maps on *co*homology. □

The equivalent problem to the non-existence of $\rho(n)$ vector fields on S^{n-1} now becomes:

Theorem 8.1. For any $m \geq 1$ the space $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is not co-reducible.

Seeing that $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ satisfies the requirements for Lemma 4.6, we see that we want to prove that there is no map $f : \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1} \to S^m$ such that the composition:

$$S^m = \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{m-1} \xrightarrow{i} \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1} \xrightarrow{f} S^m$$

is homotopic to the identity.

The rest of this chapter is devoted to showing that this is indeed an equivalent problem.

4.3 S-maps and S-duality

Since we're going to talk about reducible and co-reducible spaces, we want to establish some kind of duality between spaces such that one is "sort of" reducible if and only if its dual is "sort of" coreducible (the "sort of" being the point of talking in this section). This is done by developing the notion of S-maps, and it will follow that under not too strong assumptions if X and X' are S-duals to each other then X is S-reducible if and only if S-coreducible. Furthermore we will see that S-(co)reducibility and (co)reducibility will coincide if enough homotopy groups of the space under consideration are trivial.

We denote by $[X, Y]$ the set of homotopy classes of maps between two spaces X and Y. Since we can suspend a map $f : X \to Y$ to a map $Sf : SX \to SY$, and since this operation behaves nicely with respect to homotopy classes we find maps $[X, Y] \to [SX, SY]$. Now if $[SX, SY]$ would be equal to $[X, Y]$ for any X, Y we would spare ourselves some work, but unfortunately it isn't. However we can look at the sequence

$$[X, Y] \to [SX, SY] \to [S^2X, S^2Y] \to ...$$

From the fact that $[S^kX, S^kY] \simeq [X, \Omega^k S^k Y]$ and that maps to loopspaces exhibit a canonical group structure (juxtaposition of paths), which is abelian for $k \geq 2$, we have that the maps $[S^kX, S^kY] \to [S^{k+1}X, S^{k+1}Y]$ become homomorphisms between abelian groups when $k \geq 2$, now we get:

Definition 4.7 ([11, Def 2.1, p.215])**.** We denote by $\{X, Y\}$ the direct limit of the sequence of abelian group $[S^2X, S^2Y] \to [S^3X, S^3Y] \to ...$, an element of $\{X, Y\}$ is called an *S-map* from X to Y.

Since starting at either $[X, Y]$ or $[SX, SY]$ does not matter here, we see that indeed $\{X, Y\} = \{SX, SY\}$, so we found our invariant under supsending. It follows that one can actually make sure that $[X, Y] = \{X, Y\}$ under suitable assumptions:

Theorem 4.8 ([11, Thm 2.2, p.206])**.** If X is a CW-complex of dimension n and Y is an r-connected space, such that $n - 1 < 2r - 1$ then the natural function $[X, Y] \to \{X, Y\}$ is a bijection.

Recall that here r-connected means that $\pi_i(X) = 0$ for all $0 \leq i \leq r$.
In short this means that if we blow up our spaces big enough (i.e. making sure that all the interesting stuff happens in the higher dimension parts), eventually we'll get to a bijection between $[X, Y]$ and $\{X, Y\}$.

Now we define the duality what we're looking for. Note that if we have a function $u : X \wedge X' \to S^n$ and a map $f : Z \to X'$ we get a map $u_Z(f) : X \wedge Z \to S^n$ defined by $u_Z(f)(x, z) = u(x, f(z))$ (which is well-defined under suitable choices of base-points). On the other hand we can do the same construction with a map $g : Z \to X$, which we will call $u^Z(g)$. Since these operations behave nicely with respect to homotopy-classes (and hence S-classes), we get two homomorphisms $u_Z : \{Z, X'\} \to \{X \wedge Z, S^n\}$ and $u^Z : \{Z, X\} \to \{Z \wedge X', S^n\}$. The usual case discussed is when $Z = S^k$, in which case we will call u_Z and u^Z, u_k and u^k respectively. Now X and X' share nice properties if these homomorphisms are isomorphisms, so we have the following definition:

Definition 4.9. An *n-duality* is a map $u : X \wedge X' \to S^n$ such that u_k and u^k are isomorphisms for all k. If such an u exists, we call X the *n-dual* of X'. Furthermore we call X an *S-dual* of X' is some suspension of X is n-dual to some suspension of X' for some n.

Since taking arbitrary suspensions will come back more times, we make a small linguistic definition:

Definition 4.10. Two spaces X and Y are of the same *S-type* if $S^m X$ and $S^n Y$ are homotopy equivalent for some $m, n \in \mathbb{N}$.

The main motivation for the defintion of n-duality is the following example, which was the first example of n-dual spaces:

Theorem 4.11. [12] Let X be a finite subcompex of S^n and let X' be a subset of $S^n - X$ such that X' is homotopy equivalent to $S^n - X$. Then X and X' are n-duals.

Now from playing with diagrams one can now come to the following conclusion:

Theorem 4.12. Let X and X' be finite CW-complexes that are S-dual to each other. Then X is S-reducible if and only if X' is S-coreducable.

Proof. This is a result of diagram chasing the result of a very involved statement in cohomology theory. See [11, Thm 8.4, p.216] □

This framework allows us to make statements about the coreducibility of spaces given that some other space is reducible. The only thing that we want is a tool to compute S-duals. For this we need a new type of space, which we will introduce in the next section.

4.4 Thom spaces

In this section we will describe a certain kind of spaces made out of vector bundles, which turn out to be nice since they give a link between properties of vector bundles and the S-duality of certain spaces. We'll use these spaces afterwards to calculate S-duals of spaces $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^k$.

For this section X is a differentiable manifold and ξ is a vector bundle over X. Now since every fiber of ξ is a vector space, we intuitively can take the sphere or disk of every fiber. One can make this precise by making a smooth function from $E(\xi)$, the total space of ξ, to $\mathbb{R}$ which restricts to a (non-trivial) metric on every fiber. This metric can be constructed from taking every trivialization (or an atlas of trivialization), taking the (canonical) metric on the trivialization, and sum all of those via a partition of unity. The exact construction is beyond the scope of this thesis but we can assure ourselves that choosing different atlases will yield homeomorhic results, so we define:

Definition 4.13. Let ξ be a vector bundle of rank n over X. Then the *disk bundle* $D(\xi)$ is the fiber bundle over X that satisfies that every fiber is homeomorphic to D^n and that $D(\xi)$ is a subbundle of ξ (seeing ξ as a fiber bundle). The *sphere bundle* is defined by $S(\xi) = \partial D(\xi)$ and the *Thom space* $T(\xi)$ is defined by $T(\xi) = D(\xi)/S(\xi)$.

Remark: If we have a bundle metric on ξ, the sphere and disk bundle are just all element in ξ with length equal to 1 and less or equal then 1 respectively. In this sense one could also define $S(\xi)$ just like $D(\xi)$ but then setting all fibers homeormophic to S^{n-1}. At last it follows that while the exact form of $D(\xi)$ is dependent on choices, the homeomorphism class is not.

Example 4.14 (Trivial bundle over X)**.** If $\xi = X \times \mathbb{R}^n$, then $D(\xi) = X \times D^n$ and $S(\xi) = X \times S^{n-1}$.

From standard analysis we know that the open ball in n dimensions is homeomorphic to $\mathbb{R}^n$, or more suggestive: $D^n - S^{n-1} \simeq \mathbb{R}^n$. Now in the same way we have that if X is compact that $D(\xi) - S(\xi) \simeq E(\xi)$ and hence the one-point compactifications of both spaces are homeomorphic. Now since the one-point compactification of $D(\xi) - S(\xi)$ is just $D(\xi)/S(\xi)$. We get the following proposition:

Proposition 4.15. If X is compact and ξ is a real vector bundle over X then $T(\xi)$ is homeomorphic to the one-point compactification of $E(\xi)$.

This also shows why isomorphic vector bundles ξ and ξ' have the same Thom space, since in particular the isomorphism is a homeomorphism between $E(\xi)$ and $E(\xi')$, so the one-point compactifications are also homeomorphic.
Recalling that $(X \times Y)^+ = X^+ \wedge Y^+$ we use the previous proposition to get the following lemma:

Lemma 4.16. For ξ and η two real vector bundles over compact spaces we have $T(\xi \times \eta) = T(\xi) \wedge T(\eta)$.

Now denoting θ^n as the trivial n-dimensional bundle over a space X, we have that $\xi \oplus \theta^n$ over X is isomorphic to $\xi \times \mathbb{R}^n$ over $X \times \{p\}$, so we get the following important proposition in the context of S-types:

Proposition 4.17. The Thom space $T(\xi \oplus \theta^n)$ is homeomorphic to $S^n(T(\xi))$.

Proof. From the fact that $\xi \oplus \theta^n$ is isomorphic to $\xi \times \mathbb{R}^n$ we have that $T(\xi \oplus \theta^n)$ is homeomorphic to $T(\xi \times \mathbb{R}^n)$ which is equal to $T(\xi) \wedge T(\mathbb{R}^n)$ by the previous lemma. Now $T(\mathbb{R}^n)$ is just $(\mathbb{R}^n)^+ = S^n$ so we have $T(\xi \oplus \theta^n) \cong T(\xi) \wedge S^n$ which is homeomorphic to $S^n(T(\xi))$. □

In particular we see that throwing a trivial bundle to our vector bundle does not change the S-type of the Thom space. Furthermore we have that if X and $T(\xi)$ are S-duals if and only if X and $T(\xi \oplus \theta^n)$ are S-duals for some $n \in \mathbb{N}$.

Now the point of establishing Thom spaces is that they come in quite handy in determining S-duals, which follows from the following theorem:

Theorem 4.18 ([11, Thm 3.7, p.208])**.** If ξ and η are two vector bundles over a closed differentiable manifold such that $\xi \oplus \eta \oplus \tau(X)$ is trivial over X (where $\tau(X)$ is the tangent bundle of X), then $T(\xi)$ and $T(\eta)$ are S-duals.

Proof. This theorem will follow from the following claim:

Claim. Let ν be a normal bundle of a compact manifold X. Then the Thom space $T(\nu)$ is an S-dual of $X/\partial X$.

Proof of the Claim. We will not prove this, the interested reader can look at [11, Thm 3.6,p.208]

Now let X, ξ and η as in the assumptions of the theorem. We can now look at $D(\xi)$ which on it's own is a compact differentiable manifold, such that $\partial D(\xi) = S(\xi)$ and hence $D(\xi)/\partial D(\xi) = T(\xi)$. Since paths on $D(\xi)$ are paths that run through X and the linear spaces attached at the same time we see that the tangent bundle to $D(\xi)$ is $\pi^*(\xi \oplus \tau(X))$. Since $\xi \oplus \eta \oplus \tau(X)$ is trivial, we intuitively see that η is normal to $\xi \oplus \tau(X)$ and hence $\pi^*(\eta)$ is a normal bundle of $D(\xi)$. Now since $\pi : D(\xi) \to X$ is a homotopy equivalence (in very much the same way that a disk and a point are homotopy equivalent) we see that $T(\pi^*(\eta))$ and $T(\eta)$ are homotopy equivalent and then from the claim it follows that $T(\eta)$ and $T(\xi)$ are S-duals. $\square$

Since we now have a tool to calculate S-duals we will use this in next section to determine S-duals of spaces of the kind $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^k$.

4.5 Calculations on projective spaces

Using the link between Thom spaces and S-duals we will show that spaces of the form $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^k$ are actually Thom spaces and then use the results of the preceding section to calculate S-duals of these spaces.

We see $\mathbb{R}\mathrm{P}^k$ as the quotient of S^k under $x \sim -x$. We denote by ξ_k the line bundle over $\mathbb{R}\mathrm{P}^k$ defined by the quotient of $S^k \times \mathbb{R}$ by $(x, y) \sim (-x, -y)$. This bundle is also called the tautological line bundle, which can be seen by the other definition of ξ_k. Since $\mathbb{R}\mathrm{P}^k$ is the set of lines in $\mathbb{R}^{k+1}$ through the origin, one can look at the set:

$$\{(l, v); v \in l\} \subset \mathbb{R}\mathrm{P}^k \times \mathbb{R}^{k+1}.$$

This has an obvious projection to $\mathbb{R}\mathrm{P}^k$ and this makes it into a line bundle over $\mathbb{R}\mathrm{P}^k$. It turns out that this is exactly the same bundle as ξ_k, so intuitively speaking ξ_k sticks to every line $l \in \mathbb{R}\mathrm{P}^k$ the line l itself as linear space.

Now we can set $m\xi_k = \xi_k \oplus \cdots \oplus \xi_k$ (m times) and get that $E(m\xi_k)$ is homeomorphic to $S^k \times \mathbb{R}^m$ under the quotient $(x, y) \sim (-x, -y)$. In the same manner we have $D(m\xi_k) = S^k \times D^m$ and $S(m\xi_k) = S^k \times S^{m-1}$ modulo the same relation. In particular we see that $[x, y] \in D(m\xi_k)$ (the equivalence class of $(x, y) \in S^k \times D^m$) then $[x, y] \in S(m\xi_k)$ if and only if $||y|| = 1$. Now we have the following oberservation:

Proposition 4.19. The Thom space $T(m\xi_k)$ and the truncated projetive space $\mathbb{R}\mathrm{P}^{m+k}/\mathbb{R}\mathrm{P}^{m-1}$ are homeomorphic.

Proof. We will define a map from $S^k \times D^m \to S^{k+m}$ that factors through both quotients into a map $D(m\xi_k) \to \mathbb{R}\mathrm{P}^{k+m}$ such that $D(m\xi_k) - S(m\xi_k) \to \mathbb{R}\mathrm{P}^{k+m} - \mathbb{R}\mathrm{P}^{m-1}$ is a homeomorphism.
We define $f : S^k \times D^m \to S^{k+m}$ by $f(x, y) = (y, (1 - ||y||^2)x)$ (here we see S^{k+m} as a subset of $\mathbb{R}^m \times \mathbb{R}^{k+1}$ such that $(x, y) \in S^{k+m}$ if and only if $||x||^2 + ||y||^2 = 1$). Now we see that $f(S^k \times S^{m-1}) = S^{m-1}$ and $f^{-1}(S^{m-1}) = S^k \times S^{m-1}$. Also we see that $f(-x, -y) = -f(x, y)$, so taking the quotient with the anti-podal equivalence relation on both sides we get an induced map $g : D(m\xi_k) \to \mathbb{R}\mathrm{P}^{k+m}$. Working back through the quotients we see that $g(S(m\xi_k)) = \mathbb{R}\mathrm{P}^{m-1}$ and $g^{-1}(\mathbb{R}\mathrm{P}^{m-1}) = S(m\xi_k)$. Now we see that since f is a smooth and invertible function for $||y|| \neq 1$ we get that $f : S^k \times \mathring{D}^m \to S^{k+m} - S^{m-1}$ is a homeomorphism. Since the quotients on both sides factor very nicely through f on this subdomain (i.e. the quotient on the domain links things if and only if the image get linked by the quotient on the codomain), we see that $g : D(m\xi_k) - S(m\xi_k) \to \mathbb{R}\mathrm{P}^{k+m} - \mathbb{R}\mathrm{P}^{m-1}$ is also a homeomorphism.
So we see that $D(m\xi_k) - S(m\xi_k) \cong \mathbb{R}\mathrm{P}^{k+m} - \mathbb{R}\mathrm{P}^{m-1}$, so in particular we get that their one-point compactifications must also be homeomorphic, so we get that $T(m\xi_k) \cong \mathbb{R}\mathrm{P}^{k+m}/\mathbb{R}\mathrm{P}^{m-1}$. □

Now since we can establish spaces of the form $\mathbb{R}\mathrm{P}^{n+k}/\mathbb{R}\mathrm{P}^k$ as operations on vector bundles over $\mathbb{R}\mathrm{P}^{k-1}$, in the light of Lemma 4.18 we want to determine $\tau(\mathbb{R}\mathrm{P}^{k-1})$. We have the following result, which is a paraphrase of [11, 4.8, p.17]:

Proposition 4.20. The vector bundles $\tau(\mathbb{R}\mathrm{P}^{k-1}) \oplus \theta_1$ and $k\xi_{k-1}$ are isomorphic.

Using this we will work towards proving a result on S-dualities between projective spaces. This result will mention a certain group, namely the group $J(\mathbb{R}\mathrm{P}^{k-1})$. This group will be thorougly discussed in chapter 5, but the relevant results are:

Theorem 5.7. For every finite CW-complex X, $J(X)$ is a finite group.

Theorem 5.8. Let ξ and η be two vector bundles over X. If $S(\xi)$ and $S(\eta)$ have the same homotopy type, $T(\xi)$ and $T(\eta)$ are homotopy equivalent. In particular if $J(\xi) = J(\eta)$ then $T(\xi)$ and $T(\eta)$ have the same S-type.

The result on S-duality of projective spaces now reads:

Theorem 4.21. Let r denote the order of $J(\xi_{k-1})$ in $J(\mathbb{R}\mathrm{P}^{k-1})$. And let p such that $rp > n + 1$ then $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{n-k}$ is an S-dual of $\mathbb{R}\mathrm{P}^{rp-n+k-2}/\mathbb{R}\mathrm{P}^{rp-n-2}$.

Proof. Since $J(rp\xi_{k-1}) = J(0)$ we know that for the trivial bundle θ^{rp} and $r\xi_{k-1}$ induces the same Thom-space up to S-type and hence we know that since we want to establish an S-duality between spaces we can treat $rp\xi_{k-1}$ like it's trivial. Now we can fill in Theorem 4.18 with $\xi = (n - k + 1)\xi_{k-1} \oplus \theta_1$ and $\eta = (rp - n - 1)\xi_{k-1}$. We get:

$$\xi \oplus \eta \oplus \tau(\mathbb{R}\mathrm{P}^{k-1}) = (n - k + 1)\xi_{k-1} \oplus (rp - n - 1)\xi_{k-1} \oplus \theta_1 \oplus \tau(\mathbb{R}\mathrm{P}^{k-1}) = rp\xi_{k-1},$$

hence $\xi \oplus \eta \oplus \tau(\mathbb{R}\mathrm{P}^{k-1})$ is trivial for all intents and purposes and $T(\xi)$ and $T(\eta)$ are S-duals of each other. So plugging in η and ξ we have $T(\xi) = T((n-k+1)\xi_{k-1} \oplus \theta_1)$ and $T(\eta) = T((rp-n-1)\xi_{k-1})$.

By the remark made before about taking the direct sum with trivial bundles, we have that from the fact that $T(\xi)$ and $T(\eta)$ are S-duals we see that $T((n-k+1)\xi_{k-1})$ and $T((rp-n-1)\xi_{k-1})$ are S-duals.
If we now plug in the results into proposition 4.19 we get that $T((n-k+1)\xi_{k-1}) \cong \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{n-k}$ and $T((rp-n-1)\xi_{k-1}) \cong \mathbb{R}\mathrm{P}^{rp-n+k-2}/\mathbb{R}\mathrm{P}^{rp-n-2}$, hence we get the desired result. $\square$

4.6 The final argument

In this section we will piece together all the steps into a proof that having no $\rho(n)$ vector fields on S^{n-1} is equivalent to Theorem 8.1. First we recall the nice construction that results in a free increase of the dimension from the previous chapter:

Proposition 3.6. If S^{n-1} has k orthonormal vector fields then S^{qn-1} has k orthonormal vector fields for every integer $q \geq 1$.

Next we're going to work towards a statement on reducibility of a shunted projective space (spaces of the form $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^k$). First we construct a map θ from $\mathbb{R}\mathrm{P}^{n-1}$ to $O(n)$, namely the map that sends $[x]$ ($x \in \mathbb{R}^n - \{0\}$) into the matrix that represents reflexion through the hyperplane perpindicular to x (note that this yields the same for $[x]$ as for $[\lambda x]$ and hence is well-defined). Explicitely we get: $\theta([x])y = y - 2\frac{y \cdot x}{x \cdot x}x$. This map is compatible with the inclusions of $\mathbb{R}\mathrm{P}^{n-k-1}$ in $\mathbb{R}\mathrm{P}^{n-1}$ and $O(n-k)$ in $O(n)$:

$$\begin{tikzcd}
\mathbb{R}\mathrm{P}^{n-k-1} \arrow[r, "\theta"] \arrow[d] & O(n-k) \arrow[d] \\
\mathbb{R}\mathrm{P}^{n-1} \arrow[r, "\theta"] \arrow[d] & O(n) \arrow[d] \\
\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1} \arrow[r] & O(n)/O(n-k) \ = V_k(\mathbb{R}^n)
\end{tikzcd}$$

so we get a map θ from $\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1}$ to $O(n)/O(n-k) = V_k(\mathbb{R}^n)$.

Now it follows that this map θ induces isomorphisms between the lower dimensional homotopy groups of the Stiefel manifold and the truncated projective space:

Proposition 4.22. [11, Prop 9.2, p.217] For $i < 2n-2k-1$ the induced homomorphism $\theta_* : \pi_i(\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-k-1}) \to \pi_i(V_k(\mathbb{R}^n)$ is an isomorphism.

Now from this it follows that we have that $\pi_{n-1}(\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2})$ and $\pi_{n-1}(V_{\rho(n)+1}(\mathbb{R}^n))$ are isomorphic via θ_* when n is such that $n-1 < 2n-2(\rho(n)+1)-1$ i.e. if $2\rho(n)+2 < n$ (which only fails for $n = 1,2,3,4,6,8,16$). Using this we get the following proposition:

Proposition 4.23. If $2\rho(n)+2 < n$ and the projection $q : V_{\rho(n)+1}(\mathbb{R}^n) \to S^{n-1}$ has a cross section s, then there is a map $f : S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$ which composed with the quotient map $p : \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-2} = S^{n-1}$ is a map of degree 1, in particular $\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$ is reducible.

$$\begin{tikzcd}
S^{n-1} \arrow[r, "s"] \arrow[dr, "f"'] & V_{\rho(n)+1}(\mathbb{R}^n) \arrow[r, "q"] & S^{n-1} \\
& \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2} \arrow[u, "\theta"] \arrow[ur, "p"'] &
\end{tikzcd}$$

Proof. Suppose we have a cross-section $s : S^{n-1} \to V_{\rho(n)+1}$, i.e. $qs = 1\!\!1_{S^{n-1}}$. Then by the previous remark we have a function $f : S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$ such that $\theta_*[f] = [\theta \circ \ f] = [s]$, i.e. $\theta \circ f \simeq s$ So $q \circ \theta \circ f \simeq q \circ s = 1\!\!1_{S^{n-1}}$ and hence $\deg(q \circ \theta \circ f) = 1$. Now if $q \circ \theta$ is in fact the quotient map, then we know that f is the desired map such that the composition has degree 1.

$$\begin{tikzcd}
\mathbb{R}\mathrm{P}^{n-1} \arrow[r, "\pi_1"] \arrow[d, "\theta"] & \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2} \arrow[r, "p"] \arrow[d, "\theta"] & S^{n-1} \\
O(n) \arrow[r, "\pi_2"] & V_{\rho(n)+1}(\mathbb{R}^n) \arrow[ur, "q"] &
\end{tikzcd}$$

Claim. The function $q \circ \theta$ and the quotient map $p : \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2} \to S^{n-1}$ coincide.

Proof of the claim: Since $\theta : \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2} \to V_{\rho(n)+1}(\mathbb{R}^n)$ is defined as the induced map from the map $\theta : \mathbb{R}\mathrm{P}^{n-1} \to O(n)$ under the quotients on both sides, we can go over step up in the diagram and prove equality there. So if we denote π_1 as the quotient $\mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-\rho(n)-2}$ and π_2 as the quotient $O(n) \to V_{\rho(n)+1}(\mathbb{R}^n)$, we want to prove that $q \circ \pi_2 \circ \theta = p \circ \pi_1$. Now $p \circ \pi_1$ is just the quotient from $\mathbb{R}\mathrm{P}^{n-1}$ to $\mathbb{R}\mathrm{P}^{n-1}/\mathbb{R}\mathrm{P}^{n-2} = S^{n-1}$, which is defined by $(p \circ \pi_1)[x] = (1 - 2x_1^2, -2x_1x_2, ..., -2x_1x_n)$ (where $[x]$ is represented by $x \in S^{n-1}$). Now on the other hand $(q \circ \pi_2 \circ \theta)[x]$ is the vector $\mathrm{refl}_x e_1$, where refl_x is reflection through the hyperplane perpindicular to x, which is given by $e_1 - \frac{2x_1}{||x||^2}x$ which is $e_1 - 2x_1 x$ if we denote $[x]$ by some $x \in S^{n-1}$, and hence we see that $q \circ \pi_2 \circ \theta = p \circ \pi_1$, and hence $q \circ \theta = p$ and the claim is proven. $\square$

We'll use this proposition in the following theorem to obtain the final result:

Theorem 4.24. If there are $\rho(n)$ orthonormal vector fields on S^{n-1} then there is an integer $m \geq 1$ with $\rho(m) = \rho(n)$ such that $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is coreducible.

Proof. If there are $\rho(n)$ orthonormal vector fields on S^{n-1} then by proposition 3.6 there are $\rho(n)$ vector fields on S^{qn-1}, so then by theorem 4.2 we have a cross-section of $V_{\rho(n)+1}(\mathbb{R}^{qn})$. If we now let q such that $qn > 2\rho(qn) + 2$ we have by proposition 4.23 that $\mathbb{R}\mathrm{P}^{qn-1}/\mathbb{R}\mathrm{P}^{qn-\rho(qn)-2}$ is reducible. If we choose q to be odd we have $\rho(qn) = \rho(n)$ so we get that $\mathbb{R}\mathrm{P}^{qn-1}/\mathbb{R}\mathrm{P}^{qn-\rho(n)-2}$ is reducible.
Now let r be the order of $J(r\xi_{\rho(n)})$ in $J(\mathbb{R}\mathrm{P}^{\rho(n)-1})$. Then we fill in theorem 4.21 with $'n' = qn - 1$ and $'k' = \rho(n) + 1$, so we want p such that $rp > qn$, so $m = rp - qn \geq 1$. If we let p be divisible by $2n$, then we have that m is an odd multiple of n (since q is odd) and hence $\rho(n) = \rho(m)$. Now theorem 4.21 gives us that $\mathbb{R}\mathrm{P}^{m+\rho(n)}/\mathbb{R}\mathrm{P}^{m-1} = \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is an S-dual of $\mathbb{R}\mathrm{P}^{qn-1}/\mathbb{R}\mathrm{P}^{qn-\rho(n)-2}$ and hence we get that $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is S-coreducible.

Now in the light of theorem 4.8 we get that $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is $m-1$-connected, so if we make m

large enough (though still such that m is an odd multiple of n), we get that $\mathbb{RP}^{m+\rho(m)}/\mathbb{RP}^{m-1}$ is S-coreducible if and only if it is coreducible, since then maps from arbitrary suspensions and maps from the space itself are interchangible. So we see that $\mathbb{RP}^{m+\rho(m)}/\mathbb{RP}^{m-1}$ is coreducible for some m such that $\rho(m) = \rho(n)$. $\square$

Now since using Gram-Schmidt, we can make a family of pointwise indepent vector fields into a family of orthonormal vector fields, and hence we have now reduced the problem of the upper limit to proving theorem 8.1. The rest of the thesis will discuss various elements which will be needed to set up the machinery to prove it, and of course we will prove it.

5 K-Theory

We're going to prove Theorem 8.1 with K-Theory, a way to turn spaces X into rings $K_{\mathbb{K}}(X)$ by looking at the number of different $\mathbb{K}$-vector bundles over X. In this section we will discuss the basics of K-theory.

5.1 Definition of K-Theory

In this section we'll cover the basic definition of K-theory and also set it up as a cohomology theory (a short description of cohomology theories can be found in the appendix).
First we recall the following construction for semi-groups:

Definition 5.1. For an abelian semigroup A we define $K(A)$ to be the abelian group $A \times A/\Delta$ where Δ is the subsemigroup consisting of all elements of the form (a, a).

Remark: Since Δ and $A \times A$ are only semigroups, a priori we only know that $K(A)$ is merely a semigroup but we have $((a, b)+\Delta)+((b, a)+\Delta) = (a+b, a+b)+\Delta = 0+\Delta$, so we have an inverse so $K(A)$ is a group. Also we note that $K(A)$ is the smallest group such that we can inject A in it via a semigroup homomorphism. In particular we have that whenever it is defined, the group operation of A conincides with the group operation of $K(A)$.

Since effectively we only add inverses for elements that don't already have them we also know that for every semigroup homomorphism $A \to G$ (where G is a group) there is a unique group homomorphism $K(A) \to G$.

Let X be some manifold. On the set of vector bundles over X we have the direct sum operation $\oplus$. Since $E \oplus F \simeq E' \oplus F$ if $E \simeq F$, $\oplus$ also induces an operation on $\mathrm{Vect}_{\mathbb{K}}(X)$. Here $\mathrm{Vect}_{\mathbb{K}}(X)$ is the set of isomorphism classes of $\mathbb{K}$-vector bundles over X (for $\mathbb{K}$ some field, most times either $\mathbb{R}$ or $\mathbb{C}$). In fact $\oplus$ induces a semigroup structure on $\mathrm{Vect}_{\mathbb{K}}(X)$.

Definition 5.2. For X a manifold we define $K(X)$ to be $K(\mathrm{Vect}_{\mathbb{C}}(X))$ and $KO(X)$ to be $K(\mathrm{Vect}_{\mathbb{R}}(X))$.

Remark: When we make general statements on K-Theory, independent of which field we use, we will also use the notation $K_{\mathbb{K}}(X)$ for $K(\mathrm{Vect}_{\mathbb{K}}(X))$.

Now a map $f : X \to Y$ gives us a map $f^* : \mathrm{Vect}_{\mathbb{K}}(Y) \to \mathrm{Vect}_{\mathbb{K}}(X)$ by taking a vector bundle $E \in \mathrm{Vect}_{\mathbb{K}}(Y)$ and setting the fiber $f^*(E)_x$ to be equal to the fiber $E_{f(x)}$. Since f^* is a semigroup homomorphism, it extends to a group homomorphism $f^* : K(Y) \to K(X)$ in the complex case and $f^* : KO(Y) \to KO(X)$ in the real case.

Next we define the most used formulation of K-theory namely reduced K-theory. Having established the induced map between $K(X)$ and $K(Y)$ we define:

Definition 5.3. For a pointed space (X, x_0) we set $\widetilde{K}_{\mathbb{K}}(X)$ to be the kernel of the map $i^* : K_{\mathbb{K}}(X) \to K_{\mathbb{K}}(\{x_0\})$ where $i : \{x_0\} \to X$ is the inclusion.

Remark: Since vector bundles over a point are identified by just their rank (since all are of the form $\mathbb{K}^k$), we see that $K(\{x_0\}) = \mathbb{Z}$ and $K_{\mathbb{K}}(X) = \widetilde{K}_{\mathbb{K}}(X) \oplus \mathbb{Z}$.

Since we see that the functors K and KO induce maps that go the other way around (i.e. the induced maps of a map $X \to Y$ go from $K_{\mathbb{K}}(Y)$ to $K_{\mathbb{K}}(X)$, in terms of category theory: K and KO are contravariant functors), we strive to make both into generalized cohomology theories. The first step is to define for positive n: $K^{-n}(X) = K(S^n X)$ and $KO^{-n}(X) = KO(S^n X)$. This may seem arbitrary, but it follows that in this way we can actually make the exact sequence of the pair work, and also define K^{+n}, but more of that in a minute. The next parts will be devoted to give an overview of how one makes K-Theory into a generalized cohomology theory.

5.2 Bott periodicity

To make the functors $\widetilde{K}$ and $\widetilde{KO}$ into a cohomology theory, we also want to define K^n for positive n's. A priori we don't know how to do this since the minus first suspension is not really something. The solution comes in the form of Bott periodicity. Heuristically it says that the isomorphisms classes of the lower order K-groups are actually isomorphic to those that come before them, in a stringent periodic pattern. Reversing this pattern we can construct *higher* order K-groups, i.e. K^n for $n > 0$. The results are the following:

Theorem 5.4. [5, Thm 2.4.9.,p77] For any space X the rings $K(X)$ and $K(S^2 X)$ are isomorphic. In particular up to isomorphism class we have $K^{-n} = K^{-n+2}$.

Theorem 5.5. For any space X the rings $KO(X)$ and $KO(S^8 X)$ are isomorphic. In particular up to isomorphism class we have $KO^{-n} = KO^{-n+8}$.

These theorems follow from the results that $K(S^2 \times X) \simeq K(X) \oplus K(S^2)$ and $KO(S^8 \times X) \simeq KO(X) \oplus KO(S^8)$ (c.f. [5, Cor 2.2.3, p.46]). Also it follows that $\widetilde{K}(X \times Y) \simeq \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)$ (here $X \wedge Y$ is the smash product of X and Y, cf. the appendix), and combining both oberservations one gets the desired results.

5.3 K-Theory as a cohomology theory

From Bott periodicitiy it follows that $K^{-n+2} = K^{-n}$. Reversing this periodicity, we can define K^n for $n > 0$. Also we define relative K-Theory by setting $K^n(X, Y) = \widetilde{K}^n(X/Y)$. Having established the higher (and lower) order groups of the K-Theory of a space and the relative groups we can look at the other elements of the axioms for a cohomology theory. Excision and additivity (see the appendix on cohomology theories) are more or less clear, excision since for open sets U we have

$X/A \simeq (X-U)/(A-U)$ and additivity since vector bundles on seperate components of a space really do not communicate with each other. The exact sequence of the pair is more involved, but follows from [5, Prop 2.4.4.,p.71]. The fact that complex K-Theory has period 2 means that the exact sequence of the pair really comes down to the following cyclic exact sequence:

$$\begin{array}{ccccc} \widetilde{K}^1(X) & \rightarrow & \widetilde{K}^1(A) & \rightarrow & \widetilde{K}^1(X/A) \\ \uparrow & & & & \downarrow \\ \widetilde{K}^0(X/A) & \leftarrow & \widetilde{K}^0(A) & \leftarrow & \widetilde{K}^0(X) \end{array}$$

Using all these we see that complex K-Theory is actually a generalized cohomology theory. It is uniquely determined on CW-complexes by the axioms and by the following values on spheres:

n mod 2	0	1
$\widetilde{K}(S^n)$	$\mathbb{Z}$	0

All these steps also work for real K-Theory, only that there we have a periodicity of 8. In that case, the values on spheres are a bit messier and they are given by:

n mod 8	0	1	2	3	4	5	6	7
$\widetilde{KO}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

5.4 Another twist on K-Theory: the J-group

In this section we will discuss another functor that has to do with vector bundles: the J-group. It is closely related to K-Theory, but will have some nice features that will come in very handy in the last part of this thesis. The J-group really goes around structures induced by a vector bundle ξ: the sphere bundle $S(\xi)$, the disk bundle $D(\xi)$ and the Thom space $T(\xi)$ as discussed in section 4.4. The nice feature of the J-group is that when two vector bundles have the same value in the J-group, their Thom spaces have the same S-types. The definition of the J-group now goes as follows:

Just like in K-Theory we first define a equivalence relation on vector bundles: two (real) vector bundles ξ and η over X are stable equivalent if and only if $\xi \oplus \theta^m$ and $\eta \oplus \theta^n$ are isomorphic for some $m, n \in \mathbb{N}$. Now from [11, Thm 3.8,p.105] it follows that we can identify the equivalence classes with elements from $\widetilde{KO}(X)$.
Doing more or less doing the same trick on the induced sphere bundles we construct the J-group. That is:

Definition 5.6. We define $J(X)$ as the set of equivalence classes of $\text{Vect}_{\mathbb{R}}(X)$ under the relation that $\xi \sim \eta$ if and only if $S(\xi \oplus \theta^m)$ and $S(\eta \oplus \theta^n)$ are fibre homotopic for some m, n. We denote the J-class of ξ by $J(\xi)$.

Obviously we have the surjection $\widetilde{KO}(X) \to J(X)$ and this induces a well-defined group structure on $J(X)$. This also shows that $J(S^k)$ is finite for $k \not\equiv 0$ mod 4, since then it is either the trivial group or $\mathbb{Z}_2$. As mentioned before we will use (or more precisely: have used in chapter 4) the J-group to show that for every vector bundle ξ there is a number r such that $T(r\xi)$ is trivial.

That such an r exists follows from the following theorem, which will be the talking point for the remainder of this section:

Theorem 5.7. For every finite CW-complex X, $J(X)$ is a finite group.

Proof. For $X = S^{4n}$ this follows from discussion on the J-homomorphism from stable homotopy theory. This homomorphism is related to a construction on maps on spheres and gives a map $J : \pi_r(SO(n)) \to \pi_{r+n}(S^n)$. It follows that we can identify $\widetilde{KO}(S^{r+1})$ with some $\pi_r(SO(n))$ and then $J(S^{r+1})$ becomes isomorphic to the image of J (hence the name $J(X)$). That this is image is indeed finite has been a source of a lot of research and [7, Thm 3.7, p.144] proves that it is indeed finite, although in the last remaining case (i.e. looking at S^{4t}) it grows very rapidly, for instance $J(S^8) = \mathbb{Z}_{240}$.

So we know that $J(S^k)$ is always finite. To use this to show that it is also finite for finite CW-complexes we note that any finite CW-complex is just beginning with some points and taking the mapping cone of some map from S^n to what you already have a certain amount of times. In particular let say we have some space X and add a n-cell. That means we have a attachment map $f : S^{n-1} \to X$ and then we get a map $S^{n-1} \to X \to C(f)$ where $C(f)$ is the mapping cone of f (which is X with the n-cell attached). It gives rise to the so-called Puppe-sequence:

$$S^{n-1} \xrightarrow{f} X \xrightarrow{i} C(f) \to S^n \xrightarrow{Sf} SX \xrightarrow{Si} SC(f) \to ...$$

Now a so-called *topological half-exact functor* is a functor F from a category of spaces (all topological spaces, CW-complexes, etc.) to an abelian category (most often abelian groups), such that the Puppe sequence turns into an exact sequence:

$$F(S^{n-1}) \to F(X) \to F(C(f)) \to F(S^n) \to ...$$

It turns out that J is such a functor and hence we get the above exact sequence with the J groups. Now supposing that $J(X)$ is finite we see that since there are only finite groups in the sequence, that $J(C(f))$ is also finite. So indeed we see that when we do this procedure a finite amount of times it follows that J stays finite so indeed for a finite CW-complex X we have that $J(X)$ is a finite group. $\square$

Having shown that $J(X)$ is finite for finite CW-complexes, we show an important theorem to use this in the discussion on Thom-spaces:

Proposition 5.8. Let ξ and η be two vector bundles over X. If $S(\xi)$ and $S(\eta)$ have the same homotopy type, $T(\xi)$ and $T(\eta)$ are homotopy equivalent. In particular if $J(\xi) = J(\eta)$ then $T(\xi)$ and $T(\eta)$ have the same S-type.

Proof. The first statement follows from the fact that a homotopy equivalence can be extended to a homotopy equivalence between $D(\xi)$ and $D(\eta)$ and hence to the quotients, $T(\xi)$ and $T(\eta)$. The second statement follows from the first since $S^m(T(\xi)) = T(\xi \oplus \theta^m)$ and $S^n(T(\eta)) = T(\eta \oplus \theta^n)$ are homotopy equivalent since $S(\eta \oplus \theta^n)$ and $S(\xi \oplus \theta^m)$ are supposed to be homotopy equivalent. $\square$

Now since $J(X)$ is finite for all finite CW-complexes, for all $\xi \in \mathrm{Vect}_{\mathbb{R}}(X)$ there is an r such that $J(r\xi) = J(0)$, so in particular it follows that $T(k\xi)$ and $T(\theta^m)$ have the same S-type. This puts an end to the gap we have left in the proofs in chapter 4.

6 Stiefel-Whitney classes and Chern classes

In this section we will give some background on the algebraic structure of cohomology, in particular the total cohomology ring H^* and use this to discuss some characteristic classes of vector bundles. These classes will be used throughout this thesis to ease arguments which prove or disprove equalities in K-theory. In this section we will use notation and definitions (singular simplices, cochains, etc) which can be found in the appendix.

6.1 Ring properties of cohomology

We recall that if we have a singular simplex $\sigma : \Delta^k \to X$ we can find a induced simplex $\Delta^l \to X$ (for $l \leq k$) by looking at $\sigma|_{[v_1,\ldots,v_l]}$. This will induce a product map on the cochains with coefficients in a group G (that is $C^\bullet(X;G)$) as follows:

Definition 6.1. For $\phi \in C^k(X;G)$ and $\psi \in C^l(X;G)$ the *cup product* $\phi \smile \psi \in C^{k+l}(X;G)$ is defined by:

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|_{[v_1,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,v_{k+l}]})$$

The fact that we can take this product on the cochains to a product in cohomology follows from this lemma:

Lemma 6.2. For $\phi \in C^k(X;G)$ and $\psi \in C^l(X;G)$ we have $\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi$.

Proof. Since ϕ lives in $C^k(X;G)$, ψ in $C^l(X;G)$ and $\phi \smile \psi$ in $C^{k+l}(X;G)$, we need to plug in a $k+l+1$ simplex in $\delta(\phi \smile \psi)$, a $k+1$ simplex in $\delta\phi$ and a $l+1$ simplex in $\delta\psi$. So for $\sigma : \Delta^{k+l+1} \to X$ we have the following results:

$$\begin{aligned}
(\delta\phi \smile \psi)(\sigma) &= \sum_{i=0}^{k+1} (-1)^i \phi(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}) = \sum_{i=0}^{k+1} (-1)^i (\phi \smile \psi)(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}) \\
(-1)^k(\phi \smile \delta\psi)(\sigma) &= \sum_{i=k}^{k+l+1} (-1)^i \phi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}) = \sum_{i=k}^{k+l+1} (-1)^i (\phi \smile \psi)(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]})
\end{aligned}$$

Now we see that there is some overlap in the sums, namely we do $i = k$ and $i = k+1$ in both. But when we look good at the $i = k+1$ case of the first sum and the $i = k$ sum of the second, we see that they cancel each other, so we get:

$$\begin{aligned}
(\delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi)(\sigma) &= \sum_{i=0}^{k+l+1} (-1)^i (\phi \smile \psi)(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}) \\
&= (\phi \smile \psi)\left(\sum_{i=0}^{k+l+1} (-1)^i \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right) \\
&= (\phi \smile \psi)(\partial\sigma) \\
&= \delta(\phi \smile \psi)(\sigma)
\end{aligned}$$

So indeed we see that $\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi$. $\square$

We now see that a product of two cocycles is again a cocycle, since $\delta(\phi \smile \psi) = \phi \smile 0 \pm 0 \smile \psi = 0$. Furthermore the product of a cocycle and a coboundary (or the other way around) is again a coboundary since $\phi \smile \delta\psi = \pm\delta(\phi \smile \psi)$ if ϕ is a coboundary (i.e. if $\delta\phi = 0$) and $\delta\phi \smile \psi = \delta(\phi \smile \psi)$ if ψ is a coboundary. All in all we get:

Definition 6.3. The *cupproduct* on cohomology is the induced map of the cupproduct on cochains, that is we define:

$$\smile : H^k(X;G) \times H^l(X;G) \to H^{k+l}(X;G)$$

by $\{\phi\} \smile \{\psi\} = \{\phi \smile \psi\}$.

It is clear that this product acts cute with respect to the group-structure of cohomology (in that is satisfies the properties of a ring), so we can define the following ring:

Definition 6.4. The *cohomology ring* of a space X (with coefficients in G) is the ring

$$H^*(X;G) = (\oplus_n H^n(X;G), +, \cdot).$$

We denote elements with formal sums $\sum_i \alpha_i$ with $\alpha_i \in H^i(X;G)$, so that the group-structure of cohomology gives $(\sum_i \alpha_i) + (\sum_i \beta_i) = \sum_i (\alpha_i + \beta_i)$. The product is defined as: $(\sum_i \alpha_i) \cdot (\sum_i \beta_i) = \sum_k \sum_{l=0}^k (\alpha_l \smile \beta_{k-l})$.

We'll finish this section with an important example:

Proposition 6.5. [8, Thm 3.12, p.212] The total cohomology of the real projective space is given by $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \simeq \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \simeq \mathbb{Z}_2[\alpha]$ (with the generator $\alpha \in H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z})$. In the complex case we have $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \simeq \mathbb{Z}[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \simeq \mathbb{Z}[\alpha]$ (for some generator $\alpha \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$).

6.2 Stiefel-Whitney classes

Having esthablished the structure of the total cohomology ring, we can talk about Stiefel-Whitney class of a real vector bundle ξ over a space X. The idea is that we attach to every vector bundle a label which will be an element of $H^*(X;\mathbb{Z}_2)$, such that this labelling has some nice properties (in particular we can distinguish some non-isomorphic vector bundles). We will stick to this axiomatic approach and thoroughly believe that such a labelling exists. In fact it does, and the construction of it can be found in chapter 8 of the book by Milnor and Stasheff [9].

Theorem 6.6. There is a unique way to associate to any integer $i \geq 0$ and any real vector bundle ξ of any space X an element $w_i(\xi) \in H^i(X;\mathbb{Z}_2)$, such that the following axioms hold:

Axiom 1: $w_0(\xi) = 1 \in H^0(X;\mathbb{Z}_2)$ and $w_i(\xi) = 0$ if $i > n$ where n is the rank of ξ.

Axiom 2: If a map $f : X \to Y$ is covered by a bundle map from ξ to η (a vector bundle over Y) then $w_i(\xi) = f^* w_i(\eta)$.

Axiom 3: If ξ and η are vector bundles over the same base space then:

$$w_k(\xi \oplus \eta) = \sum_{i=0}^k w_i(\xi) \smile w_{k-i}(\eta)$$

Axiom 4: For the line bundle ξ_1 over $\mathbb{R}\mathrm{P}^1$ we have $w_1(\xi_1) \neq 0$.

Definition 6.7. We define the *i'th Stiefel-Whitney class* of a vector bundle ξ to be $w_i(\xi)$ and the *total Stiefel-Whitney class* of ξ to be

$$w(\xi) = 1 + w_1(\xi) + w_2(\xi) + ... \in H^*(X; \mathbb{Z}_2)$$

Remark: we can also formulate the theorem as that there is a unique map w from the real vector bundles over any space X to $H^*(X; \mathbb{Z}_2)$ as above. In that case the third axiom reduces to $w(\xi \oplus \eta) = w(\xi)w(\eta)$.

From the four axioms we can derive the following properties of the Stiefel-Whitney class:

Proposition 6.8. If ξ and η are isomorphic then $w_i(\xi) = w_i(\eta)$.

Proof. If ξ and η are isomorphic, the identity map $X \to X$ is covered by the isomorphism between ξ and η. In particular we get $w_i(\xi) = \mathbb{1}^*(w_i(\eta)) = w_i(\eta)$. □

Proposition 6.9. If ϵ is a trivial bundle then $w_i(\epsilon) = 0$ for $i > 0$.

Proof. If ϵ is trivial, there is a bundle map from a vector bundle over a point to ϵ. Since a point has no cohomology for $i > 0$, we get that the pullback must map to zero, so $w_i(\epsilon) = 0$. □

A direct consequence of the last proposition:

Proposition 6.10. If ϵ is trivial then $w_i(\epsilon \oplus \eta) = w_i(\eta)$.

Proposition 6.11. Let ξ be a vector bundle of rank n. If ξ has k cross-sections which are pointwise linear independent then:

$$w_{n-k+1}(\xi) = w_{n-k+2}(\xi) = ... = w_n(\xi) = 0$$

Proof. Using a Riemannian metric we can split ξ in a trivial k-bundle ϵ induced by the cross-sections and $\epsilon^\perp$ which has dimension $n-k$ (see [9, Thm 3.3, p.28]), now the preceding proposition says that $w_i(\xi) = w_i(\epsilon^\perp)$ and then the first axiom says that it is zero is $i > n - k$. □

6.3 Chern classes

Where the Stiefel-Whitney classes will work for real vector bundles, we'll use Chern classes for complex vector bundles. Just as before we will give an axiomatic description of these classes and use these with the confidence that it uniquely describes a labelling. The construction of Chern classes can be found in chapter 14 of Milnor and Stasheff [9]. Also we note that since complex bundles are somehow even dimensional real bundles all stuff now happens in the even dimensional cohomology-groups and the classes run up to twice the complex dimension of the bundles. Also since complex vector bundles are orientable, we now choose coefficients in $\mathbb{Z}$ instead of in $\mathbb{Z}_2$. We get the following definiton:

Theorem 6.12. There is a unique way to associate to any integer $i \geq 0$ and any complex vector bundle ξ over any space X an element $c_i(\xi) \in H^{2i}(X; \mathbb{Z})$ such that the following axioms hold:

Axiom 1: $c_0(\xi) = 1$ and $c_i(\xi) = 0$ for $i > n$ where n is the dimension of ξ.

Axiom 2: If a map $f : X \to Y$ is covered by a bundle map from ξ to η (a complex vector bundle over Y) then $c_i(\xi) = f^*(c_i(\eta))$.

Axiom 3: If η and ξ are both complex vector bundles over X then:

$$c_k(\xi \oplus \eta) = \sum_{i=0}^{k} c_i(\xi) \smile c_{k-i}(\eta)$$

Axiom 4: For the tautological complex line bundle λ_1 over $S^2 = \mathbb{C}\mathrm{P}^1$ the element $c_1(\lambda_1)$ is the generator of $H^2(S^2; \mathbb{Z}) = \mathbb{Z}$.

Definition 6.13. For a complex vector bundle η the *i'th Chern class* is defined as $c_i(\eta)$ while the *total Chern class* of η is defined by:

$$c(\eta) = 1 + c_1(\eta) + c_2(\eta) + ... \in H^*(X; \mathbb{Z})$$

Remark: just like with the Stiefel-Whitney classes one could also prove the unique existence of c. In that case the third axiom reduces to $c(\xi \oplus \theta) = c(\xi)c(\theta)$.

We note that the axioms for both classes are very much alike. For this reason the following proposition are proven by using the exact same steps as in the real case:

Proposition 6.14. If ξ and η are isomorphic then $c_i(\xi) = c_i(\eta)$.

Proposition 6.15. If ϵ is a trivial bundle then $c_i(\epsilon) = 0$ for $i > 0$.

Proposition 6.16. If ϵ is trivial then $c_i(\epsilon \oplus \eta) = c_i(\eta)$.

Now both in the real and complex case we have that if two vector bundles are isomorphic then they have the same Stiefel-Whitney classes or the Chern class. In particular w and c factor to maps $w : \mathrm{Vect}_{\mathbb{R}}(X) \to H^*(X; \mathbb{Z}_2)$ and $c : \mathrm{Vect}_{\mathbb{C}}(X) \to H^*(X; \mathbb{Z})$. Now we can also just look at just just the first class and restrict to line bundles. We get maps

$$w_1 : \mathrm{Vect}^1_{\mathbb{R}}(X) \to H^1(X; \mathbb{Z}_2)$$

and

$$c_1 : \mathrm{Vect}^1_{\mathbb{C}}(X) \to H^2(X; \mathbb{Z}).$$

The set of line bundles form a group if we put on it the tensor product as group operation. Since the cocycle-maps of line bundles are just real or complex functions which do not attain zero, we can always invert them and since tensor product of 1×1-matrices and normal multiplication coincide we see that we always can invert line bundles with respect to the tensor product. Now for line bundles ξ and η we also have the formulas $c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$ and $w_1(\xi \otimes \eta) = w_1(\xi) + w_1(\eta)$ (depending on whether they are complex or real of course), so we see that in this setting c_1 and w_1 are group homomorphisms. However it becomes a lot more fun due to this theorem:

Theorem 6.17. [6, Prop 3.10, p.86] If X has the homotopy type of a CW-complex, then c_1 and w_1 are isomorphisms between the isomorphism classes of real/complex line vector bundles and $H^1(X;\mathbb{Z}_2)/H^1(X;\mathbb{Z})$.

Remarks: In particular this theorem means that for every $\alpha \in H^1(X;\mathbb{Z}_2)$ we can find a real line bundle ξ with $w_1(\xi) = \alpha$ and for every $\beta \in H^2(X;\mathbb{Z})$ we can find a complex line bundle η with $c_1(\eta) = \beta$. Also this theorem implies that isomorphism classes of line bundles are characterised by their Stiefel-Whitney or Chern class.

6.4 Chern character

Having discussed the Chern classes we now introduce the notion of Chern character, an algebraic tool which will induce a ring homomorphism between K-theory of a space X and its homology. We will define it for line bundles, will invoke the desired algebraic structure on it to get a definition of direct sums of line bundles and finally will discuss an important tool in the discussion of vector bundles. It is understood that since we work with Chern classes here, all bundles are complex.

Definition 6.18. For line bundles ξ we define $\mathrm{ch}^q(\xi) \in H^{2q}(X;\mathbb{Q})$, the $2q$-dimensional part of the Chern character, as $\frac{c_1(\xi)^m}{m!}$.
Note that we want to divide by integers to do this, hence why switch to cohomology with coefficients in $\mathbb{Q}$. Just like before we define the *Chern character* of ξ by $\mathrm{ch}(\xi) = \sum_q \mathrm{ch}^q(\xi)$. In doing so we spot that in terms of formal power series we have $\mathrm{ch}(\xi) = e^{c_1(\xi)}$.

We strive to have $\mathrm{ch}(\xi \oplus \eta) = \mathrm{ch}(\xi) + \mathrm{ch}(\eta)$ and $\mathrm{ch}(\xi \otimes \eta) = \mathrm{ch}(\xi)\mathrm{ch}(\eta)$. In this light if $\xi = \eta_1 \oplus \cdots \oplus \eta_n$ with η_i a line bundle we have no other choice than to define:

Definition 6.19. If ξ is a sum of line bundles, that is $\xi = \eta_1 \oplus \cdots \oplus \eta_n$ with η_i line bundles we define:

$$\mathrm{ch}(\xi) = \mathrm{ch}(\eta_1) + \cdots + \mathrm{ch}(\eta_n) = e^{c_1(\eta_1)} + \cdots + e^{c_1(\eta_n)}$$

Now we have defined the Chern character for any direct sum of vector bundles. The next question is whether we can define it thoroughly for vector bundles which are not of that form. It turns out we can, although we cheat ourselves into the previous case with the so called 'Splitting Principle':

Theorem 6.20. For any complex vector bundle ξ over a paracompact space X there is a space Y, and a map $p : Y \to X$ such that:

i) The induces map on cohomology $p^* : H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ is injecitve

ii) The pullback bundle $p^*(\xi)$ splits in line bundles, that is $p^*(\xi) = \eta_1 \oplus \cdots \oplus \eta_n$ with η_i line bundles over Y.

In particular we can use this to define the Chern character for any vector bundle ξ over a paracompact space X (note that about all spaces discussed in this thesis are paracompact), in particular we set:

$$\mathrm{ch}(\xi) = (p^*)^{-1}(\mathrm{ch}(p^*(\xi)))$$

Now formally from we can describe the Chern character is a different way. Supposing that ξ is of rank k and we have elements $x_1, ..., x_k \in H^2(X;\mathbb{Z}_2)$, such that:

$$\begin{aligned}
c_1(\xi) &= \sum_i x_i \\
c_2(\xi) &= \sum_{i_1<i_2} x_{i_1}x_{i_2} \\
c_3(\xi) &= \sum_{i_1<i_2<i_3} x_{i_1}x_{i_2}x_{i_3} \\
&\vdots
\end{aligned}$$

In that case we can, writing the polynomial $\sum_i X_i^k$ as a polynomial $Q_k(\sigma_1, ..., \sigma_k)$ in the elementary symmetric polynomials, get the following result:

$$\mathrm{ch}(\xi) = \sum_k Q_k(c_1(\xi), ..., c_k(\xi))$$

When we write the first terms out we then get this:

$$\mathrm{ch}(\xi) = \mathrm{rk}(\xi) + c_1(\xi) + \frac{1}{2}(c_1(\xi)^2 - 2c_2(\xi)) + \frac{1}{6}(c_1(\xi)^3 - 3c_1(\xi)c_2(\xi) + 3c_3(\xi)) + ...$$

Practically it follows that up to pullbacks we can find these x_i's using the Splitting Principle and hence this formula checks out.
Now from the defintion with p^* it is easy to see that $\mathrm{ch}(\xi \oplus \eta) = \mathrm{ch}(\xi) + \mathrm{ch}(\eta)$ in particular since it is so for sums of line bundles and hence by the Splitting Principle for all bundles. On the other hand the identity $c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$ (and hence $\exp(c_1(\xi \otimes \eta)) = \exp(c_1(\xi))\exp(c_1(\eta)))$ for line bundles now extends to the identity $\mathrm{ch}(\xi \otimes \eta) = \mathrm{ch}(\xi)\mathrm{ch}(\eta)$ for any two vector bundles ξ and η. In particular we see that ch is a semi-ring homomorphism from $\mathrm{Vect}_{\mathbb{C}}(X)$ to $H^*(X;\mathbb{Q})$, and hence extends to a ring homomorphism $K(X) \to H^*(X;\mathbb{Q})$. The last important property of this character that we will use is the following:

Proposition 6.21. [10, Prop 2.3, p.345] If $H^*(X;\mathbb{Z})$ has no torsion, then ch is injective.

7 Group representiation and relation to K-theory

In this section we will discuss representations of groups and how they communicate with the K-theory. We will construct a certain kind of representations that when used on K-Theory bring to the surface a certain part of the structure of the K-Theory of projective spaces, which we will use to prove Theorem 8.1. We'll start with the definition of representations:

7.1 Group representations

Definition 7.1. A *representiation* of a group G on a vector space V (over a field $\mathbb{K}$) is a group homomorphism $\alpha : G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the group of invertible linear transformations of V. In the case that V is finite dimensional one can (and we will) identify $\mathrm{GL}(V)$ with $\mathrm{GL}(n, \mathbb{K})$, the space of invertible $n \times n$-matrices with entries in $\mathbb{K}$.

We will discuss the relation between representations and K-theory. Since K-theory talks about equivalence classes of isomorphic bundles, we will also introduce a notion of isomorphic representations:

Definition 7.2. Two representations $\alpha : G \to \mathrm{GL}(V)$ and $\beta : G \to \mathrm{GL}(W)$ are *isomorphic* if there is a vector space isomorphism $\lambda : V \to W$ such that for every $g \in G$ we have $\lambda \circ \alpha(g) \circ \lambda^{-1} = \beta(g)$.

If we look at representation as mapping into matrix groups, we see that two representations $\alpha : G \to \mathrm{GL}(n, \mathbb{K})$ and $\beta : G \to \mathrm{GL}(n', \mathbb{K}')$ are isomorphic if there is an invertible $n \times n'$-matrix A with entries in $\mathbb{K}$ (which then has to be equal to $\mathbb{K}'$) such that $\alpha(g) = A\beta(g)A^{-1}$ for all $g \in G$, in particular we get $n = n'$ and $\mathbb{K} = \mathbb{K}'$.

In the remainder we will only look at representations into finite dimensional vector spaces and for that we also fix our field $\mathbb{K}$. In particular every representation will be a map $G \to \mathrm{GL}(n, \mathbb{K})$ for some n.
On representations we have a semi-group structure, namely that if we have representations $\alpha : G \to \mathrm{GL}(V)$ and $\beta : G \to \mathrm{GL}(W)$ we can define $\alpha \oplus \beta : G \to \mathrm{GL}(V \oplus W)$ by $(\alpha \oplus \beta)(g) = (\alpha(g), \beta(g))$. Using the same construction as with K-theory we can construct the group-extension of this semi group (though only the representations into matrix groups):

Definition 7.3. We denote by $A'_{\mathbb{K}}(G)$ the set of all equivalence classes of representations $G \to \mathrm{GL}(n, \mathbb{K})$ for some $n \in \mathbb{N}$, and by $F'_{\mathbb{K}}(G)$ the free abelian group generated by the elements of $A'_{\mathbb{K}}(G)$. By $T'_{\mathbb{K}}(G)$ we denote the subgroup of $F'_{K}(G)$ generated by the elements $\{\alpha \oplus \beta\} - \{\alpha\} - \{\beta\}$ with α and β representations in $A'_{\mathbb{K}}(G)$. The group of *virtual representations* is defined as $K'_{\mathbb{K}}(G) = F'_{\mathbb{K}}(G)/T'_{\mathbb{K}}(G)$.
Since we have a semigroup-homomorphism of the semigroup of representations to $\mathbb{Z}$ (namely the degree-map that sends $f : G \to \mathrm{GL}(n, \mathbb{K})$ to n), which passes through all quotients, this extends to a group homomorphism $K'_{\mathbb{K}}(G) \to \mathbb{Z}$ which is called the virtual degree map.

7.2 Combining group representations with bundles

Now that we have defined representations, we want them to fit into the algebraic structure of vector bundles just like continuous maps. In particular we want to define various composition. Here we use the notation α for a represenation $\mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(n', \mathbb{K}')$, β for a respresentation $G \to \mathrm{GL}(n, \mathbb{K})$, f for a continuous map $Y \to Z$, g for a continuous map $X \to Y$ and ξ for a $\mathbb{K}$-vector bundle of rank n over Y. Now first we define the following, more easy, compositions:

i) the representation $\alpha \cdot \beta$ is just the composition of maps.

ii) the map $f \cdot g$ is also just the composition of maps.

iii) the vector bundle $\xi \cdot g$ we define by the pullback bundle that is $\xi \cdot g = g^*(\xi)$ (i.e. 'precomposing' the vector bundle with the map g, hence the notation).

iv) the vector bundle $\alpha \cdot \xi$ is the vector bundle defined by composing the cocycle-maps of ξ with the representation α (c.f. the appendix on vector bundles)

Next we define more involved compositions, which will be the most interesting in the arguments to come. We again denote by f a map, by η a vector bundle, by κ an element of $K_{\mathbb{K}}(X)$, by α and β

representations and by θ a virtual representation:

Lemma 7.4. It is possible to define compositions $\theta \cdot \alpha$, $\theta \cdot \eta$ and $\kappa \cdot f$ so that $\theta \cdot \alpha$ lies in the right $K'_{\mathbb{K}}(G)$ (i.e. if θ maps into $\mathbb{K}$-matrices $\theta \cdot \alpha$ should be an element of $K'_{\mathbb{K}}(G)$) and $\theta \cdot \eta$ and $\kappa \cdot f$ are elements of the right $K_{\mathbb{K}}(X)$ (that is, if θ maps into $\mathbb{K}$-matrices then $\theta \cdot \eta$ should be in $K_{\mathbb{K}}(X)$ and if $\kappa \in K_{\mathbb{K}}(X)$ then $\kappa \cdot f \in K_{\mathbb{K}}(X)$) such that the following holds:

i) Each composition is linear in the first factor

ii) They reduce to $\beta \cdot \alpha$, $\alpha \cdot \eta$ and $\eta \cdot f$ if we replace elements of the quotient groups with elements of the underlying semi-groups

iii) The following associativity formulas hold:

$$\begin{aligned} (\theta \cdot \beta) \cdot \alpha &= \theta \cdot (\beta \cdot \alpha) \\ (\theta \cdot \eta) \cdot f &= \theta \cdot (\eta \cdot f) \\ (\theta \cdot \alpha) \cdot \xi &= \theta \cdot (\alpha \cdot \xi) \\ (\kappa \cdot g) \cdot f &= \kappa \cdot (g \cdot f) \end{aligned}$$

iv) If $\alpha = \mathbb{1}$ then $\theta \cdot \alpha = \theta$ and if $f = \mathbb{1}$ then $\kappa \cdot f = \kappa$.

Proof. All three follow from the extension property of semi-group homomorphisms to group homomorphisms. In particular a represenetation $\alpha \in F'_{\mathbb{K}}(G)$ induces a semi-group homomorphism on $F'_{\mathbb{K}}(G)$ namely $R_\alpha : F'_{\mathbb{K}}(G) \to F'_{\mathbb{K}}(G)$ defined as $R_\alpha(\beta) = \beta \cdot \alpha$. This extends to a group homomorphism $\widetilde{R}_\alpha : K'_{\mathbb{K}}(G) \to K'_{\mathbb{K}}(G)$ and we set $\theta \cdot \alpha$ as $\widetilde{R}_\alpha(\theta)$. The same construction we do for $\theta \cdot \eta$ and for $\kappa \cdot f$ (the last one coming from $\mathrm{Vect}_{\mathbb{K}}(X)$). All the properties now follow from the properties of the underlying multiplication of the semi-groups. $\square$

Now we know how we can compose a vector bundle with a (virtual) representation of the appropriate dimension. What we want however is that we get some sort of (virtual) representation that we can put on *any* bundle (or element of the K-theory) without thinking about the rank of the bundle. This will be done by making a sequence $\{\theta_n\}_n$ of virtual representations of $\mathrm{GL}(n, \mathbb{K})$. We can't however take any sequence of representations, since we want the elements to communicate nicely such that the whole operation is linear. With this in mind we make the following definition. First let:

$$\mathrm{pr}_1 : \mathrm{GL}(m, \mathbb{K}) \times \mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(m, \mathbb{K})$$
$$\mathrm{pr}_2 : \mathrm{GL}(m, \mathbb{K}) \times \mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})$$

be projections onto the first and second entries. It is clear that these are representations.

Definition 7.5. A sequence $\Theta = \{\theta_n\}_n$ of virtual representations is *additive* if we have:

$$\theta_{m+n} \cdot (\mathrm{pr}_1 \oplus \mathrm{pr}_2) = (\theta_m \cdot \mathrm{pr}_1) + (\theta_n \cdot \mathrm{pr}_2)$$

In short an additive sequence preserves the structure of a direct sum, at least for the obvious projections. It follows that just having it for the projections will actually give this property for any construction one wants to do with such a sequence:

Lemma 7.6. If Θ is an additive sequence then for any two $\mathbb{K}$-bundles ξ and η over X of rank m and n respectively we have:

$$\theta_{m+n} \cdot (\xi \oplus \eta) = (\theta_m \cdot \xi) + (\theta_n \cdot \eta)$$

Proof. From ξ and η we can construct the bundle $\xi \times \eta$ with cocycle-maps mapping into the group $\mathrm{GL}(m, \mathbb{K}) \times \mathrm{GL}(n, \mathbb{K})$. Now from projection it is clear that $\mathrm{pr}_1 \cdot (\xi \times \eta) = \xi$ and $\mathrm{pr}_2 \cdot (\xi \times \eta) = \eta$ so that we have $(\mathrm{pr}_1 \oplus \mathrm{pr}_2) \cdot (\xi \times \eta) = \xi \oplus \eta$. Using the various arithmatic rules derived in the previous lemma's we now get:

$$\begin{array}{rcl}
\theta_{m+n} \cdot (\xi \oplus \eta) & = & \theta_{m+n} \cdot ((\mathrm{pr}_1 \oplus \mathrm{pr}_2) \cdot (\xi \times \eta)) \\
& = & (\theta_{m+n} \cdot (\mathrm{pr}_1 \oplus \mathrm{pr}_2)) \cdot (\xi \times \eta) \\
& = & (\theta_m \mathrm{pr}_1 + \theta_n \mathrm{pr}_2) \cdot (\xi \times \eta) \\
& = & \theta_m \cdot (\mathrm{pr}_1 \cdot (\xi \times \eta)) + \theta_n \cdot (\mathrm{pr}_2 \cdot (\xi \times \eta)) \\
& = & \theta_m \cdot \xi + \theta_n \cdot \eta
\end{array}$$

□

Much the same construction can be used to show that $\theta_{m+n} \cdot (\alpha \oplus \beta) = (\theta_m \cdot \alpha) + (\theta_n \cdot \beta)$ for any representations $\alpha : G \to \mathrm{GL}(m, \mathbb{K})$ and $\beta : G \to \mathrm{GL}(n, \mathbb{K})$.
This all results into the following lemma on composing additive sequences with itself and with elements of the K-theory. Here we use the notation Ψ, Φ and Θ for additive additive sequences, α for a representation $G \to \mathrm{GL}(n, \mathbb{K})$, ξ for a bundle of rank n, κ for an element of $K_{\mathbb{K}}(X)$ and f for a continuous map:

Lemma 7.7. There are compositions $\Phi \cdot \Theta$, $\Phi \cdot \alpha$ and $\Phi \cdot \kappa$ such that $\Phi \cdot \Theta$ is an additive sequence, $\Phi \cdot \alpha$ lies in $K'_{\mathbb{K}}(G)$ and $\Phi \cdot \kappa$ lies in $K_{\mathbb{K}}(X)$ and the following properties hold:

i) Each composition is bilinear in the two factors

ii) $(\Phi \cdot \Theta)_n = \Phi \cdot \theta_n$, when we see α as an element of $K'_{\mathbb{K}}(G)$ we have $\Phi \cdot \alpha = \phi_n \cdot \alpha$ and if we see ξ as an element of $K_{\mathbb{K}}(X)$ $\Phi \cdot \xi = \phi_n \cdot \xi$.

iii) The following associativity formulas hold:

$$\begin{array}{rcl}
(\Psi \cdot \Phi) \cdot \Theta & = & \Psi \cdot (\Phi \cdot \Theta) \\
(\Phi \cdot \kappa) \cdot f & = & \Phi \cdot (\kappa \cdot f) \\
(\Phi \cdot \Theta) \cdot \kappa & = & \Phi \cdot (\Theta \cdot \kappa)
\end{array}$$

iv) We denote by $\mathbb{1}$ the additive sequence with n'th entry the identity map $\mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})$ then:

$$\begin{array}{rcl}
\mathbb{1} \cdot \Theta & = & \Theta \\
\Phi \cdot \mathbb{1} & = & \Phi \\
\mathbb{1} \cdot \kappa & = & \kappa
\end{array}$$

Proof. We construct $\Phi \cdot \Theta$ from $\Phi \cdot \theta_n$ as imposed by ii), and just as in the previous lemma we construct $\Phi \cdot \theta_n$ as the extension of a semi-group homomorphism that sets $\Phi \cdot \alpha = \phi_n \cdot \alpha$. The same goes for constructing $\Phi \cdot \kappa$. These are bilinear since we assume Θ to be additive. Since all the relations hold in the semi-group setting, they also hold for the final result. □

In particular the last associativity equation shows that a sequence Θ of virtual representations induces a group homomorphism $\Theta : K_{\mathbb{K}}(X) \to K_{\mathbb{K}}(X)$ that is natural for continuous maps.

7.3 Relation between representations and K-theory and products

Now that we know that we can intertwine the additive structures of representations and K-Theory, we also want to intertwine the multiplicative structure.

On vector bundles we have the notion of tensor products, i.e. we have a semigroup homomorphism:

$$\otimes : \mathrm{Vect}_{\mathbb{K}}(X) \times \mathrm{Vect}_{\mathbb{K}}(X) \to \mathrm{Vect}_{\mathbb{K}}(X).$$

This map extends to a map

$$\otimes : K_{\mathbb{K}}(X) \times K_{\mathbb{K}}(X) \to K_{\mathbb{K}}(X)$$

for which we get:

Proposition 7.8. The triple $(K_{\mathbb{K}}(X), \oplus, \otimes)$ is a commutative ring with unit the trivial line bundle.

Also if we have two representations $\alpha : G \to \mathrm{GL}(V)$ and $\beta : G \to \mathrm{GL}(W)$ we get a representation $\alpha \otimes \beta : G \to \mathrm{GL}(V \otimes W)$ set by $(\alpha \otimes \beta)(g) = \alpha(g) \otimes \beta(g)$, this also makes $K'_{\mathbb{K}}(G)$ into a commutative ring with unit the representation $G \to \mathrm{GL}(1, \mathbb{K})$ $g \mapsto 1$.
Now we can do the same discussion as with the sum:

Definition 7.9. A sequence Θ of virtual representations is called *multiplicative* if we have:

$$\theta_{mn} \cdot (\mathrm{pr}_1 \otimes \mathrm{pr}_2) = (\theta_m \cdot \mathrm{pr}_1) \otimes (\theta_n \cdot \mathrm{pr}_2)$$

Just as with the direct sum we can now extend this multiplicative property to representations and bundles:

Lemma 7.10. If Θ is a multiplicative sequence then for any two representations $\alpha : G \to \mathrm{GL}(m, \mathbb{K})$ and $\beta : G \to \mathrm{GL}(n, \mathbb{K})$ then:

$$\theta_{mn} \cdot (\alpha \otimes \beta) = (\theta_m \cdot \alpha) \otimes (\theta_n \cdot \beta)$$

Furthermore if ξ and η are two bundles over X with rank m and n then:

$$\theta_{mn} \cdot (\xi \otimes \eta) = (\theta_m \cdot \xi) \otimes (\theta_n \cdot \eta)$$

Now if Θ is also an additive sequence, then we can linear extend computations from representations to virtual representations to sequences and from bundles to elements of the K-theory, so we get:

Lemma 7.11. If Θ is both additive and multiplicative then we have:

$$\begin{aligned} \Theta \cdot (\psi \otimes \phi) &= (\Theta \cdot \psi) \otimes (\Theta \cdot \phi) \\ \Theta \cdot (\kappa \otimes \lambda) &= (\Theta \cdot \kappa) \otimes (\Theta \cdot \lambda) \end{aligned}$$

All these definitions will function to make sense of the following proposition which will serve as the main tool of setting up the proof of Theorem 8.1

Proposition 7.12. An additive sequence sequence $\Theta = \{\theta_n\}$ with $\theta_n \in K'_{\mathbb{K}'}(\mathrm{GL}(n, \mathbb{K}))$ induces a group homomorphism:

$$\Theta : K_{\mathbb{K}}(X) \to K_{\mathbb{K}'}(X)$$

given by $\kappa \mapsto \Theta \cdot \kappa$. This map has the following properties:

i) It is natural with respect to continuous maps, that is the following diagram is commutative

$$\begin{tikzcd}
K_{\mathbb{K}}(X) \arrow[r, "\Theta"] & K_{\mathbb{K}'}(X) \\
K_{\mathbb{K}}(Y) \arrow[r, "\Theta"] \arrow[u, "f^*"] & K_{\mathbb{K}'}(Y) \arrow[u, "f^*"]
\end{tikzcd}$$

ii) If the sequence Θ is also multiplicative then Θ is a ring homomorphism (i.e. it preserves products)

iii) If θ_1 has virtual degree 1 then Θ maps the unit in $K_{\mathbb{K}}(X)$ into the unit of $K_{\mathbb{K}'}(X)$.

Proof. The first two items follow from the previous lemmas. For iii) we note that the unit in $K_{\mathbb{K}}(X)$ is the trivial bundle $X \times \mathbb{K}$. Since this map has trivial cocycles we get that any representation of degree 1 maps this into again trivial cocycle. Since this property extends to virtual representations of virtual degree 1, hence we see that if θ_1 has virtual degree 1 we get that $\theta_1(X \times \mathbb{K}) = X \times \mathbb{K}'$. So we see that indeed the unit gets mapped into the unit $\square$

We'll finish this part with some simple examples:

Example 7.13. Consider the following homomorphisms:

i) The injection of real matrices into complex matrices:

$$c_n : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{C})$$

ii) The injection of the complex matrices into the real matrices:

$$r_n : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})$$

Where an entry $a + bi$ gets mapped into a 2×2-block $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

iii) The conjugation map:

$$t_n : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$$

Where $t_n(M) = \overline{M}$ (i.e. if $M_{ij} = a + bi$ then $t_n(M)_{ij} = a - bi$).

These homomorphisms form sequences $c = \{c_n\}$, $r = \{r_n\}$ and $t = \{t_n\}$, which are all additive and apart from c are also multiplicative.
We see that if we first to c to some matrix $M \in \mathrm{GL}(n, \mathbb{R})$ we get a 'complex' matrix with only real entries. If we then apply r we get effectively that every entry becomes a diagonal 2×2-block

with itself on the diagonal, so we see that $rc(M) = M \oplus M$. On the other hand if we first do r and then c we see that an entry $a + bi$ gets mapped into a (complex) 2×2-block $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, which via diagonlization (with the matrix $O = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$) is seen to be equivalent to the 2×2-block $\begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix}$ which is the direct sum of the original entry and its conjugate, so we see that for any $M \in \mathrm{GL}(n, \mathbb{C})$ we get that $cr(M) \simeq M \oplus \overline{M}$.
Now using the last proposition we get group homomorphisms:

$$\begin{aligned} c: &\quad KO(X) \to K(X) \\ r: &\quad K(X) \to KO(X) \\ t: &\quad K(X) \to K(X) \end{aligned}$$

of which the first and the last are ring homomorphisms. Using the calculations in the matrix-setting we get that also in the K-theory-setting we get $rc = 2$ (i.e. $rc(\kappa) = \kappa + \kappa$) and $cr = 1 + t$. The first one is clear while the second follows from the fact that the conjugation-matrix which links $cr(M)$ and $M \oplus \overline{M}$ (the O discussed above) is independent of M, and hence the equivalence of the matrices gets transported to a equality in the K-theory-sense.

7.4 A special sequence of virtual representations

In this section we will discuss a special sequence of representations. This sequence, when transformed into a group homomorphism on K-theory will have certain nice properties as will become clear in proposition 7.15. We will use these properties fairly often in the parts to come. The setup of the sequences can be with any underlying field, but here we will restrict ourselves to $\mathbb{R}$ and $\mathbb{C}$. The required properties of the sequences read as follows:

Proposition 7.14. For every $k \in \mathbb{Z}$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ there is a sequence $\Psi^k_{\mathbb{K}}$ of virtual representatioins such that:

i) $\psi^k_{\mathbb{K},n}$ is a virtual representation of $\mathrm{GL}(n, \mathbb{K})$ over $\mathbb{K}$ with virtual degree n.

ii) The sequence $\Psi^k_{\mathbb{K}}$ is both additive and multiplicative.

iii) $\psi^k_{\mathbb{K},1}$ is the k'th power of the identity representation of $\mathrm{GL}(1, \mathbb{K})$ (where we use the tensor product on representations as multiplication)

iv) Recalling the sequence c of Example 7.13 we have $\Psi^k_{\mathbb{C}} \cdot c = c \cdot \psi^k_{\mathbb{R}}$.

v) $\Psi^k_{\mathbb{K}} \cdot \Psi^l_{\mathbb{K}} = \Psi^{kl}_{\mathbb{K}}$.

vi) Let G be a topological group and let θ be a virtual representation of G over $\mathbb{K}$ then the following formula holds for the character χ : $\chi(\Psi^k_{\mathbb{K}} \cdot \theta)g = \chi(\theta)g^k$.

vii) For any n $\psi^1_{\mathbb{K},n}$ is the identity representation, $\psi^0_{\mathbb{K},n}$ is the trivial representation of degree n, while $\psi^{-1}_{\mathbb{K},n}$ is the representation defined by $\psi^{-1}_{\mathbb{K},n}(M) = (M^T)^{-1}$.

Proof. We recall the r'th exterior product $E^r(V)$ of a vector space V, in particular for finite dimensional vector spaces. In that case, if we fix a basis $v_1, ..., v_n$ of V, we generate $E^r(V)$ by the elements

$v_{i_1} \wedge \cdots \wedge v_{i_r}$ for $1 \leq i_k \leq n$ imposing multilinearity and anti-symmetry. In particular a basis for $E^r(V)$ are the elements $v_{i_1} \wedge \cdots \wedge v_{i_r}$ for $i_1 < i_2 < \cdots < i_r$, and hence $E^r(V)$ is a $\frac{n!}{r!(n-r)!}$-dimensional vector space. Now if we have a linear map A from V to itself, we can take it over to a linear map $E^r A$ from $E^r(V)$ to itself, simply by imposing $E^r A(v_{i_1} \wedge \cdots \wedge v_{i_r}) = (Av_{i_1}) \wedge \cdots \wedge (Av_{i_r})$. In particular we get a group homomorphism (and hence representation): $E^r_{\mathbb{K}} : \mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}\left(\frac{n!}{r!(n-r)!}, \mathbb{K}\right)$. Since we only do stuff on the generators (not doing specific things depending on $\mathbb{K}$) it is clear that writing $m = \frac{n!}{r!(n-r)!}$ we get $E^r_{\mathbb{C}} \cdot c_n = c_m \cdot E^r_{\mathbb{R}}$ (this of course acts as a prelude to proving property iv)).

We're going to put these $E^r_{\mathbb{K}}$'s to act just like taking all basis elements to the power k. We do it in the following way: since the function $\sum_{i=1}^n (x_i)^k$ is symmetric, we can write it in terms of the symmetric polynomials σ_1 up to σ_n (σ_0 is excluded since it only induces a constant term), in particular we get:

$$\sum_{i=1}^{n} (x_i)^k = Q^k_n(\sigma_1, ..., \sigma_n)$$

for some polynomial Q^k_n. Now we define for $k \geq 0$:

$$\psi^k_{\mathbb{K},n} = Q^k_n(E^1_{\mathbb{K}}, ..., E^n_{\mathbb{K}})$$

evaluating Q^k_n in the ring $K'_{\mathbb{K}}(\mathrm{GL}(n, \mathbb{K}))$.

For $k = -1$ we set $\psi^k_{\mathbb{K},n}$ by precisely what we want to have, namely $\psi^{-1}_{\mathbb{K},n}(M) = (M^T)^{-1}$, and to finish of for $k > 1$ we define $\psi^{-k}_{\mathbb{K},n} = \psi^k_{\mathbb{K},n} \cdot \psi^{-1}_{\mathbb{K},n}$ (it is clear that is this defintion if well-defined, since $(M^T)^{-1}$ has the same dimensions as M). Having made the definition, we go over the desired properties:

Claim 1. (property i)): $\psi^k_{\mathbb{K},n}$ has degree n.

Proof. For $k \geq 0$ we plug in the identity matrix and look at what dimensions the outcome is. This coincides with plugging in $x_i = 1$ in the defining polynomial, and looking at what comes out, we get that $\deg \psi^k_{\mathbb{K},n} = \sum_{i=1}^n 1^k = n$. □

Claim 2. (property vii)): $\psi^1_{\mathbb{K},n}$ is the identity, while $\psi^0_{\mathbb{K},n}$ is the trivial represetation of degree n and $\psi^{-1}_{\mathbb{K},n}$ is given by $\psi^{-1}_{\mathbb{K},n}(M) = (M^T)^{-1}$.

Proof. The statement about $k = -1$ is true per definition. For $k = 1$ we note that $Q^1_n(\sigma_1, ..., \sigma_n) = \sigma_1$ and hence $\psi^1_n = E^1_n$. Since $E^1(\mathbb{R}^n) = \mathbb{R}^n$ we see that E^1_n is the identity and hence ψ^1_n is aswell. For $k = 0$ we note that $\sum_{i=1}^n (x_i)^k = n$ and hence Q^0_n is just the constant term n, and hence $\psi^0_{\mathbb{K},n} = n = \oplus_n 1$, i.e. sending everything to the n-fold direct sum of 1, hence to the identity $n \times n$-matrix. □

Claim 3. (property iii)): $\psi^k_{\mathbb{K},1}$ is the k'th exterior power of the identity representation.

Proof. For $k \geq 0$ we get that when $n = 1$ we get $Q^k_1(\sigma^1) = \sigma^k_1$ and hence $\psi^k_{\mathbb{K},1} = (E^1_{\mathbb{K}})^k$ and since $E^1_{\mathbb{K}} = \mathbb{1}_{\mathrm{GL}(1,\mathbb{K})}$ the claim follows. For $k = -1$ we see that since every 1×1-matrix is symmetric (it really is a number) we get that $\psi^{-1}_{\mathbb{K},1}(M) = M^{-1}$ and hence the multiplicative inverse of the identity representation. The claim for $k < -1$ follows by defintion $\psi^k_{\mathbb{K},1}$ for such k's and the results for $k = -1$ and $k \geq 0$. □

Claim 4. (property iv)): $\Psi^k_{\mathbb{C}} \cdot c = c \cdot \Psi^k_{\mathbb{R}}$.

Proof. Since complexification commutes with exterior products and sums, this follows for $k \geq 0$ from the above note on $E^r_{\mathbb{K}}$ and c. For $k = -1$ it is clear and hence $k < -1$ follows from the defintion of Ψ^k for $k < -1$. □

Next we note that the character $\chi(\alpha)$ of a representation $\alpha : G \to \mathrm{GL}(n, \mathbb{K})$ is given by $\chi(\alpha)g = \mathrm{Tr}(\alpha g)$. We have $\chi(\alpha \oplus \beta) = \chi(\alpha) + \chi(\beta)$ and $\chi(\alpha \otimes \beta) = \chi(\alpha)\chi(\beta)$. The following claim we leave without proof (it involves a lot of intricacies with matrices which I do need think is interesting to write here, though it does act as a motiviation for the definition of $\psi^k_{\mathbb{K},n}$)

Claim 5. [1, Prop 4.3, p. 613]: For any matrix $M \in \mathrm{GL}(n, \mathbb{K})$ we have: $\chi(\psi^k_{\mathbb{K},n})M = \mathrm{Tr}(M^k)$.

We use this claim for the following:

Claim 6. For every representation $\alpha : G \to \mathrm{GL}(n, \mathbb{K})$ and each $g \in G$ we have $\chi(\psi^k_{\mathbb{K},n} \cdot \alpha)g = \chi(\alpha)g^k$.

Proof. We have
$$\chi(\psi^k_{\mathbb{K},n} \cdot \alpha)g = \chi(\psi^k_{\mathbb{K},n})(\alpha g),$$
since $\alpha g \in \mathrm{GL}(\mathbb{K}, n)$ we use the previous claim and get:
$$\chi(\psi^k_{\mathbb{K},n})(\alpha g) = \mathrm{Tr}((\alpha g)^k) = \mathrm{Tr}(\alpha(g^k)) = \chi(\alpha)(g^k)$$
□

We note that this property extends to virtual representations by definition of virtual representations, and hence prorperty vi) follows.
Using the characterization of the behaviour of the character on this represenations, we'll prove the perhaps most important property of this sequence:

Claim 7. (property ii)): The sequence $\Psi^k_{\mathbb{K}}$ is additive and multiplicative.

Proof. Recall that $\mathrm{pr}_1 : \mathrm{GL}(m, \mathbb{K}) \times \mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(m, \mathbb{K})$ and $\mathrm{pr}_2 : \mathrm{GL}(m, \mathbb{K}) \times \mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})$ are the representations that just project to the first respectively second factor.
We want to prove:

a) $\psi^k_{\mathbb{K},m+n} \cdot (\mathrm{pr}_1 \oplus \mathrm{pr}_2) = (\psi^k_{\mathbb{K},m} \cdot \mathrm{pr}_1) + (\psi^k_{\mathbb{K},n} \cdot \mathrm{pr}_2)$

b) $\psi^k_{\mathbb{K},mn} \cdot (\mathrm{pr}_1 \otimes \mathrm{pr}_2) = (\psi^k_{\mathbb{K},m} \cdot \mathrm{pr}_1) \otimes (\psi^k_{\mathbb{K},n} \cdot \mathrm{pr}_2)$

We first check that the characters coincide. In particular using the previous claims we get:
$$\begin{aligned}
\chi[\psi^k_{\mathbb{K},m+n} \cdot (\mathrm{pr}_1 \oplus \mathrm{pr}_2)]g &= \chi(\mathrm{pr}_1 \oplus \mathrm{pr}_2)g^k \\
&= \chi(\mathrm{pr}_1)g^k + \chi(\mathrm{pr}_2)g^k \\
&= \chi(\psi^k_{\mathbb{K},m} \cdot \mathrm{pr}_1)g + \chi(\psi^k_{\mathbb{K},n} \cdot \mathrm{pr}_2)g \\
&= \chi[(\psi^k_{\mathbb{K},m} \cdot \mathrm{pr}_1) + (\psi^k_{\mathbb{K},n} \cdot \mathrm{pr}_2)]g,
\end{aligned}$$

when we look at case a), if we look at case b) we get:

$$\begin{aligned}\chi[\psi^k_{\mathbb{K},mn}\cdot(\mathrm{pr}_1\otimes\mathrm{pr}_2)]g = &= \chi(\mathrm{pr}_1\otimes\mathrm{pr}_2)g^k\\ &= \chi(\mathrm{pr}_1)g^k\cdot\chi(\mathrm{pr}_2)g^k\\ &= \chi(\psi^k_{\mathbb{K},m}\cdot\mathrm{pr}_1)g\cdot\chi(\psi^k_{\mathbb{K},n}\cdot\mathrm{pr}_2)g\\ &= \chi((\psi^k_{\mathbb{K},m}\cdot\mathrm{pr}_1)\otimes(\psi^k_{\mathbb{K},n}\cdot\mathrm{pr}_2))g\end{aligned}$$

The fact that the representations are equal when the characters are the same follows from looking at it restricted to compact subgroups. The exact argument can be found in [1, Prop 4.5, p.614]. □

Now we just have to check property v) and that is the last part of the proof:

Claim 8. (property v)): $\Psi^k_{\mathbb{K}}\cdot\psi^l_{\mathbb{K},n}=\psi^{kl}_{\mathbb{K},n}$.

Proof. We again just check the characters and use the same argument as above to see that then the equality of the representations follow. We have:

$$\begin{aligned}\chi(\Psi^k_{\mathbb{K},n}\cdot\psi^l_{\mathbb{K},n})M &= \chi(\psi^l_{\mathbb{K},n})M^k && \text{(Claim 6)}\\ &= \mathrm{Tr}(M^{kl}) && \text{(Claim 5)}\\ &= \chi(\psi^{kl}_{\mathbb{K},n})M && \text{(Claim 5)}\end{aligned}$$

□

Having checked that this definition satisfies all the properties we have finished the proof of Theorem 7.14. □

Having constructed these representation, we can discuss what they do in K-Theory:

Proposition 7.15. The sequences $\Psi^k_{\mathbb{K}}$ induce operations $\Psi^k_{\mathbb{K}}: K_{\mathbb{K}}(X)\to K_{\mathbb{K}}(X)$ on the real or complex K-theory of some space X such that:

i) $\Psi^k_{\mathbb{K}}$ commutes with continuous maps, that is, the following diagram commutes:

$$\begin{tikzcd} K_{\mathbb{K}}(X) \arrow[r, "\Psi^k_{\mathbb{K}}"] & K_{\mathbb{K}}(X) \\ K_{\mathbb{K}}(Y) \arrow[u, "f^*"] \arrow[r, "\Psi^k_{\mathbb{K}}"] & K_{\mathbb{K}}(Y) \arrow[u, "f^*"'] \end{tikzcd}$$

ii) $\Psi^k_{\mathbb{K}}$ is a homomorphism of rings with units

iii) If ξ is a line bundle over X then $\Psi^k_{\mathbb{K}}\xi=\xi^k$

iv) The operations $\Psi^k_{\mathbb{R}}$ and $\Psi^k_{\mathbb{C}}$ commute with complexification, that is, the following diagram commutes:

$$\begin{tikzcd} KO(X) \arrow[r, "\Psi^k_{\mathbb{R}}"] \arrow[d, "c"] & KO(X) \arrow[d, "c"] \\ K(X) \arrow[r, "\Psi^k_{\mathbb{C}}"] & K(X) \end{tikzcd}$$

v) $\Psi_{\mathbb{K}}^k \circ \Psi_{\mathbb{K}}^l = \Psi_{\mathbb{K}}^{kl}$

vi) For $\kappa \in K(X)$ we have $\mathrm{ch}^q(\Psi_{\mathbb{C}}^k \kappa) = k^q \mathrm{ch}^q(\kappa)$ where $\mathrm{ch}^q(\kappa) \in H^{2q}(X;\mathbb{R})$ is the $2q$-dimensional component of the Chern character of κ.

vii) $\Psi_{\mathbb{K}}^1$ and $\Psi_{\mathbb{R}}^{-1}$ are the identity functions, $\Psi_{\mathbb{C}}^{-1} = t$ where t is the operation of Example 7.13, while $\Psi_{\mathbb{K}}^0$ is the function that maps a bundle over X to the trivial bundle of the same dimension.

Proof. We see that i) and ii) follow directly from i) and ii) of Proposition 7.14. Part iii) follows from iii) of 7.14 since a line bundle has cocycle-maps in $\mathrm{GL}(1,\mathbb{K})$ and tensor products on both side coincide. The same holds for iv) and v), in that it follows from iv) respectively v) of 7.14 using the arithmatic lemmas. The proof of the parts about $\Psi_{\mathbb{K}}^1$ and $\Psi_{\mathbb{K}}^0$ of vii) follow directly from vii) of 7.14 while the part about $\Psi_{\mathbb{K}}^{-1}$ follows from discussion about the map $M \mapsto (M^T)^{-1}$.
Here we note that since $O(n)$ is a deform rectract of $GL(n,\mathbb{R})$ since it's a maximal compact subgroup, any real bundle is equivalent to a bundle with cocyle-maps in $O(n) \subset \mathrm{GL}(n,\mathbb{R})$ where $(M^T)^{-1} = M$. In the complex case we have the same but now with $U(n)$ so if we have a complex vector bundle we get an equivalent one with cocyle-maps mapping into $U(n) \subset \mathrm{GL}(n,\mathbb{C})$ where $(M^T)^{-1} = \overline{M}$, so we see that the claim follows.

As for vi), we will only prove it in the case of a sum of line bundels, since with the Splitting Principle we can always reduce to that case by iv). So if $\xi = \eta_1 \oplus \cdots \oplus \eta_n$ we have:

$$\mathrm{ch}(\xi) = e^{c_1(\eta_1)} + \cdots e^{c_1(\eta_n)};\ \ \mathrm{ch}^q(\xi) = \frac{1}{q!}\left[c_1(\eta_1)^q + \cdots c_1(\eta_n)^q\right]$$

Now using the fact that both ch and $\Psi_{\mathbb{C}}^k$ are ring homomorphisms and part iii) we get:

$$\begin{aligned} \mathrm{ch}(\Psi_{\mathbb{C}}^k \xi) &= \mathrm{ch}(\Psi_{\mathbb{C}}^k(\eta_1 \oplus \cdots \oplus \eta_n)) \\ &= \mathrm{ch}((\Psi_{\mathbb{C}}^k \eta_1) \oplus \cdots \oplus (\Psi_{\mathbb{C}}^k \eta_n)) \\ &= \mathrm{ch}(\eta_1^k \oplus \cdots \oplus \eta_n^k) \\ &= \mathrm{ch}(\eta_1)^k + \cdots + \mathrm{ch}(\eta_n)^k \\ &= e^{kc_1(\eta_1)} + \cdots e^{kc_1(\eta_n)} \end{aligned}$$

And when looking at the q'th component we see:

$$\mathrm{ch}^q(\Psi_{\mathbb{C}}^k \xi) = \frac{1}{q!}\left[(kc_1(\eta_1))^q + ...(kc_1(\eta_n))^q\right] = \frac{k^q}{q!}\left[c_1(\eta_1)^q + ... + c_1(\eta_n)^q\right] = k^q \mathrm{ch}^q(\xi)$$

□

We will finish this part of the chapter with an example to see how these operations come to life on a particular space. First though we note that we can also define operations on the reduced K-theory. Since that is defined such that $K_{\mathbb{K}}(X) = \widetilde{K}_{\mathbb{K}}(X) \oplus K_{\mathbb{K}}(\text{point}) = \widetilde{K}_{\mathbb{K}}(X) \oplus \mathbb{Z}$ and since on a point all vector bundles are trivial, we see that the operations must act on the reduced part, so we can all the same talk about the operations $\Psi_{\mathbb{K}}^k$ on $\widetilde{K}_{\mathbb{K}}(X)$. On the even dimensional spheres (which are the most interesting cases of spehres since there the K-theory is isomorphic to $\mathbb{Z}$) the operations then become as follows:

Proposition 7.16. The operations $\Psi_{\mathbb{C}}^k : \widetilde{K}_{\mathbb{C}}(S^{2q}) \to \widetilde{K}_{\mathbb{C}}(S^{2q})$ and (for q even) $\Psi_{\mathbb{R}}^k : \widetilde{K}_{\mathbb{R}}(S^{2q}) \to \widetilde{K}_{\mathbb{R}}(S^{2q})$ are given by $\Psi_{\mathbb{K}}^k(\kappa) = k^q \kappa$.

Proof. The proof of the complex case follows directly from:

Claim 1. the map $\mathrm{ch}^q : \widetilde{K}_{\mathbb{C}}(S^{2q}) \to H^{2q}(S^{2q}; \mathbb{Q})$ is injective.

Using this claim we get from Proposition 7.15.vi) that:

$$\mathrm{ch}^q(k^q\kappa) = k^q(\mathrm{ch}^q\kappa) = \mathrm{ch}^q(\Psi^k_{\mathbb{C}}\kappa)$$

Since ch^q injective we get that $\Psi^k_{\mathbb{C}}\kappa = k^q\kappa$.

If q is even the real case follows from the following claim:

Claim 2. the map $c : \mathbb{Z} = \widetilde{KO}(S^{4t}) \to \widetilde{K}(S^{4t}) = \mathbb{Z}$ is injective.

We will use this on the following consequence of 7.15.iv):

$$c(\Psi^k_{\mathbb{R}}(\kappa)) = \Psi^k_{\mathbb{C}}(c(\kappa)) = k^q c(\kappa) = c(k^q\kappa)$$

Injectivity of c now implies that $\Psi^k_{\mathbb{R}}(\kappa) = k^q\kappa$.

Proof of Claim 1. This is a direct consequence of Proposition 6.21.

Proof of Claim 2. We know that c sends the generator 1 of $\widetilde{KO}(S^{4t})$, namely the trivial real bundle, to a certainly non-trivial complex bundle. So $c(1) \neq 0$ and hence c is injective. It follows from Bott's work that $c(1) = 1$ if t is even and $c(1) = 2$ if t is odd. □

This example, apart from being an example of how the operations work out, is also indicative for the parts to come, where we will mostly derive properties in complex K-theory and use maps like c and r to derive results in real K-theory. This order of work is a result of the easiness of complex K-theory as compared to the messy properties of real K-theory (in particular on the spheres).

8 An important technical result

Having set up the machinery we can now prove the theorem that we reduced to in chapter 4 to prove that there are no $\rho(n)$ vector fields on S^{n-1}. The statement reads as follows:

Theorem 8.1. For any $m \geq 1$ the space $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ is not co-reducible.

Recall that we saw earlier that $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ not being co-reducible is equivalent to proving that we cannot find a map $g : \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1} \to S^m$ such that g precomposed with the inclusion $S^m = \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{m-1} \to \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ has degree 1.

8.1 Calculations of K-Theory of projective spaces

In this part we'll calculate the K-Theory of projective spaces, in particular the complex K-Theory of $\mathbb{C}\mathrm{P}^m/\mathbb{C}\mathrm{P}^n$ and the real and complex K-Theory of $\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n$. Establishing rigid generator-relation-forms of these groups we'll then show that a map g as described above cannot exist.

To ease the choice of generators with going from complex to real we first prove the following lemma:

Lemma 8.2. Let ξ_k be the tautological real line bundle over $\mathbb{R}\mathrm{P}^k$ and let η_k be the tautological complex line bundle over $\mathbb{C}\mathrm{P}^k$ and $\pi : \mathbb{R}\mathrm{P}^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ be the obvious projection. Then $c\xi_{2n-1} = \pi^*\eta_{n-1}$ (as elements of $K(\mathbb{R}\mathrm{P}^{2n-1})$).

Proof. The case $n = 1$ is trivial and hence we see that $H^2(\mathbb{R}\mathrm{P}^{2n-1}; \mathbb{Z}) = \mathbb{Z}_2$ and we have $c_1\pi^*\eta_{n-1} = \pi^*c_1\eta_{n-1} \neq 0$. We know that complex line bundle are classified by their first Chern class and since $\mathbb{Z}_2$ has only one non-zero element we hence have to show that $c\xi_{2n-1}$ is not trivial. For this we look at $rc\xi_{2n-1}$, from Example 7.13 it follows that $rc\xi_{2n-1} = \xi_{2n-1} \oplus \xi_{2n-1}$. We know that $w(\xi_{2n-1}) = 1 + x$ with x the generator of $H^1(\mathbb{R}\mathrm{P}^{2n-1}; \mathbb{Z}_2)$ and hence we have $w(rc\xi_{2n-1}) = (1+x)^2 = 1 + x^2 \in H^*(\mathbb{R}\mathrm{P}^{2n-1}; \mathbb{Z}_2)$. If $c\xi_{2n-1}$ were to be trivial, then $rc\xi_{2n-1}$ would be aswell and then $w(rc\xi_{2n-1})$ would be 1, which it obviously isn't so we see that $c\xi_{2n-1} = \pi^*\eta_{n-1}$ $\square$

Using these line bundles we'll define the following elements of the K-Theories to use in the upcoming calculations. For clarity we'll drop the index (though we should certainly not forget about it!) and define:

$$\lambda = \lambda_n = \xi_n - 1 \in \widetilde{KO}(\mathbb{R}\mathrm{P}^n)$$
$$\mu = \mu_n = \xi_\eta - 1 \in \widetilde{K}(\mathbb{C}\mathrm{P}^n)$$
$$\nu = \nu_n = c\lambda_n \in \widetilde{K}(\mathbb{R}\mathrm{P}^n)$$

Having defined these special elements of the K-Theories we now start the fun part with calculating the complex K-Theory of $\mathbb{C}\mathrm{P}^m/\mathbb{C}\mathrm{P}^n$:

Theorem 8.3. $K(\mathbb{C}\mathrm{P}^m)$ is the polynomial ring $\mathbb{Z}[\mu]/(\mu^{m+1})$, i.e. the polynomial ring generated by μ with coefficients in $\mathbb{Z}$ with the relation $\mu^{m+1} = 0$. The projection $\mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^m/\mathbb{C}\mathrm{P}^n$ maps $\widetilde{K}(\mathbb{C}\mathrm{P}^m/\mathbb{C}\mathrm{P}^n)$ isomorphically onto the subgroup of $K(\mathbb{C}\mathrm{P}^m)$ generated by μ^{n+1}, ..., μ^n. In either case the ring homomorphisms $\Psi^k_{\mathbb{C}}$ are given by:

$$\Psi^k_{\mathbb{C}} \cdot \mu^s = ((1+\mu)^k - 1)^s$$

If k is negative we see $(1+\mu)^k$ as the formal power series $1 + k\mu + k(k-1)/2\mu^2 + ...$ which is a finite sum since $\mu^{m+1} = 0$.

Proof. Since by Proposition 6.21 ch is injective we will use the structure of $H^*(\mathbb{C}\mathrm{P}^m; \mathbb{Q}) \simeq \mathbb{Q}[x]/(x^{n+1})$ to prove things for $\mathrm{ch}(K(\mathbb{C}\mathrm{P}^m))$ and then conclude the same things for $K(\mathbb{C}\mathrm{P}^m)$.
We denote by x be the generator of $H^*(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$ (and hence also a generator of $H^*(\mathbb{C}\mathrm{P}^m; \mathbb{Q})$). Since the line bundle η_m has Chern class $1 + x$, we have $\mathrm{ch}(\eta_n) = e^x$ and hence we have $\mathbb{Z}[e^x]/(x^{m+1}) \subset \mathrm{ch}(K(\mathbb{C}\mathrm{P}^m))$.
Now from the exact sequence of the pair in K-Theory we have the exact sequence:

$$0 \to \mathrm{ch}(K(\mathbb{C}\mathrm{P}_m, \mathbb{C}\mathrm{P}_{m-1})) \to \mathrm{ch}(K(\mathbb{C}\mathrm{P}_m)) \to \mathrm{ch}(K(\mathbb{C}\mathrm{P}_{m-1}) \to 0.$$

While $\mathbb{C}\mathrm{P}_m/\mathbb{C}\mathrm{P}_{m-1} = S^{2m}$ we get that:

$$\mathrm{ch}(K(\mathbb{C}\mathrm{P}_m, \mathbb{C}\mathrm{P}_{m-1}) = \mathbb{Z}[x^m]/(x^{m+1}) = \mathbb{Z}[(e^x - 1)^m]/(x^{m+1}) \subset \mathbb{Z}[e^x]/(x^{m+1})$$

Now the suppose by induction hypothesis that $\mathrm{ch}(\mathbb{C}\mathrm{P}^{m-1}) = \mathbb{Z}[e^x]/(x^m)$ (the case $m = 1$ is trivial), we get that $\mathbb{Z}[e^x]/(x^{m+1})$ is a subgroup of $\mathrm{ch}(K(\mathbb{C}\mathrm{P}^m))$ which contains $\mathrm{ch}(K(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^{m-1})$ and maps onto $\mathrm{ch}(K(\mathbb{C}\mathrm{P}^{m-1})$, so we know that $\mathbb{Z}[e^x]/(x^{m+1})$ is a subgroup of $\mathrm{ch}(K(\mathbb{C}\mathrm{P}^m))$ for which the short exact sequence works, hence it must be the whole group and we get:

$$\mathrm{ch}(K(\mathbb{C}\mathrm{P}^m)) = \mathbb{Z}[e^x]/(x^{m+1}) = \mathbb{Z}[e^x - 1]/(x^{m+1}) = \mathbb{Z}[\mathrm{ch}(\mu)]/((\mathrm{ch}(\mu))^{m+1})$$

so plugging back with ch^{-1} we see that $K(\mathbb{C}\mathrm{P}^m) = \mathbb{Z}[\mu]/(\mu^{m+1})$.

Now since the exact sequence of the pair $\widetilde{K}(\mathbb{C}\mathrm{P}^m/\mathbb{C}\mathrm{P}^n) \to K(\mathbb{C}\mathrm{P}^m) \to K(\mathbb{C}\mathrm{P}^n)$ becomes: $\widetilde{K}(\mathbb{C}\mathrm{P}^m/\mathbb{C}\mathrm{P}^n) \to \mathbb{Z}[\mu]/(\mu^{m+1}) \to \mathbb{Z}[\mu]/(\mu^{n+1})$, the result for $\widetilde{K}(\mathbb{C}\mathrm{P}^m/\mathbb{C}\mathrm{P}^n)$ is clear.

As for $\Psi^k_{\mathbb{C}}$ we know that $\Psi^k_{\mathbb{C}}(\eta) = \eta^k$ since η is a line bundle, now using the ring homomorphism properties of $\Psi^k_{\mathbb{C}}$ we get:

$$\Psi^k_{\mathbb{C}}(\mu^s) = \left(\Psi^k_{\mathbb{C}}(\mu)\right)^s = \left(\Psi^k_{\mathbb{C}}(\mu+1) - \Psi^k_{\mathbb{C}}1\right)^s = ((1+\mu)^k - 1)^s \qquad \square$$

We will now use the complex K-Theory of the complex projective spaces to calculate the complex K-Theory of the real projective spaces. To do this we will extensively use the projection $\pi : \mathbb{R}\mathrm{P}^{2n+1} \to \mathbb{C}\mathrm{P}^n$.
This map factors to maps $\omega' : \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2s+1} \to \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^s$ and $\omega : \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2s} \to \mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^s$. Using these maps we will define generators of the complex K-Theory of real projective spaces. First we note that we define $\mu^{(s+1)}$ for the element of $\widetilde{K}(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^s)$ that gets mapped to μ^{s+1} in $K(\mathbb{C}\mathrm{P}^n)$ (see the preceding theorem). Now we set $\nu'^{(s+1)} = \omega'^*\mu^{(s+1)} \in \widetilde{K}(\mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2s+1})$ and $\nu^{(s+1)} = \omega^*\mu^{(s+1)} \in \widetilde{K}(\mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2s})$. Now is clear from the defintions that the induced maps from $\mathbb{R}\mathrm{P}^{2n+1} \to \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2s} \to \mathbb{R}\mathrm{P}^{2n+1}/\mathbb{R}\mathrm{P}^{2s+1}$ maps $\nu'^{(s+1)}$ to $\nu^{(s+1)}$ to ν^{s+1}, which should act as a justification for the notation. Using these elements as generators, we get the following result on the complex K-Theory of the real projective space:

Theorem 8.4. Let $n = 2t$ for some integer t and let $f = \lfloor \frac{1}{2}(m-n) \rfloor$, then $\widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n) = \mathbb{Z}_{2^f}$. In the case that $n = 0$ we can describe $K(\mathbb{R}\mathrm{P}^m)$ by the generator ν and the relations $\nu^2 = -2\nu$ and $\nu^{f+1} = 0$. Otherwise $\widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n)$ is generated by $\nu^{(t+1)}$ and the projection $\mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n$ maps $\widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n)$ isomorphically onto the subgroup of $K(\mathbb{R}\mathrm{P}^m)$ generated by ν^{t+1}.
If $n = 2t-1$ we have $\widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n) = \mathbb{Z} \oplus \widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{n+1})$ where the factor $\mathbb{Z}$ is the subgroup generated by $\nu'^{(t)}$ while the second factor comes from the projection $\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{n+1}$.
Furthermore the operations $\Psi^k_{\mathbb{C}}$ are given by:

$$\begin{array}{llll} \text{i)} & \Psi^k_{\mathbb{C}}\nu^{(t+1)} & = & \begin{cases} 0 & \text{if } k \text{ even} \\ \nu^{(t+1)} & \text{if } k \text{ odd} \end{cases} \\ \text{ii)} & \Psi^k_{\mathbb{C}}\nu'^{(t)} & = & k^t\nu'^{(t)} + \begin{cases} \frac{1}{2}k^t\nu^{(t+1)} & \text{if } k \text{ even} \\ \frac{1}{2}(k^t - 1)\nu^{(t+1)} & \text{if } k \text{ odd} \end{cases} \end{array}$$

Proof. The main calculation of the ring structure comes from spectral sequences, which is beyond the scope of this thesis. For this part one can look at [1, Thm 7.3, p.622]. We can however discuss the relations of the ring $K(\mathbb{R}\mathrm{P}^m)$. For this we first detour into real vector bundles. Since real line bundles are identified by their first Stiefel-Whitney class, we have that for the tautological line bundle ξ over $\mathbb{R}\mathrm{P}^m$ that $\xi^2 = 1$. This follows since by definition $w_1(\xi) \neq 0$ and $H^1(\mathbb{R}\mathrm{P}^m) = \mathbb{Z}_2$, so $w_1(\xi^2) = 2w_1(\xi) = 0 = w_1(1)$, so we get $\xi^2 = 1$ and hence $(\lambda+1)^2 = 1$ or $\lambda^2 = -2\lambda$. Now

applying c we get $c(\lambda^2) = c(-2\lambda)$, that is $(c\lambda)^2 = -2c(\lambda)$ and hence $\nu^2 = -2\nu$. On the other hand we know that ν^{f+1} is the image of $\nu'^{(f+1)} \in \widetilde{K}(\mathbb{R}\mathrm{P}^N/\mathbb{R}\mathrm{P}^{2f+1})$ from the canonical quotient. Now since $2f+1 \geq n$ we see that has to get mapped to zero since we pound the vector bundle down to something trivial (indeed $k = f+1$ is the smallest number such that $2k-1 \geq m$, so the order of ν is exactly $f+1$).
We also note that the identity $\widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n) = \mathbb{Z} \oplus \widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{n+1})$ follows from the exact sequence of the pair $(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{2t-1}, \mathbb{R}\mathrm{P}^{2t}/\mathbb{R}\mathrm{P}^{2t-1})$, and hence from the exact sequence (which in this case ends with zeroes by results from the spectral sequence):

$$0 \to \widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{2t}) \to \widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{2t-1}) \to \widetilde{K}(\mathbb{R}\mathrm{P}^{2t}/\mathbb{R}\mathrm{P}^{2t-1}) = \widetilde{K}(S^{2t}) = \mathbb{Z} \to 0$$

Next for the operations $\Psi^k_{\mathbb{C}}$ we note that for a line bundle η we have $\Psi^k_{\mathbb{C}}\eta = \eta^k$, on the other hand we already noted that for the tautological line bundle ξ we have $(c\xi)^2 = 1$, so we get:

$$\Psi^k_{\mathbb{C}}(c\xi) = \begin{cases} 1 & \text{if } k \text{ even} \\ c\xi & \text{if } k \text{ odd} \end{cases}$$

Plugging in the defintion of ν this means that:

$$\Psi^k_{\mathbb{C}}(1+\nu) = \begin{cases} 1 & \text{if } k \text{ even} \\ 1+\nu & \text{if } k \text{ odd} \end{cases},$$

therefore using the fact that $\Psi^k_{\mathbb{C}}$ is a ring homomorphism:

$$\Psi^k_{\mathbb{C}}(\nu) = \begin{cases} 0 & \text{if } k \text{ even} \\ \nu & \text{if } k \text{ odd} \end{cases},$$

so in particular we have:

$$\Psi^k_{\mathbb{C}}(\nu^s) = \begin{cases} 0 & \text{if } k \text{ even} \\ \nu^s & \text{if } k \text{ odd} \end{cases}$$

Since $\widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{2t})$ maps isomorphically into $K(\mathbb{R}\mathrm{P}^m)$ the result for $\Psi^k_{\mathbb{C}}\nu^{(t+1)}$ follows from the above.

Now for $\nu'^{(t)}$. We know that since it has to land in the appropriate group we have $\Psi^k_{\mathbb{C}}\nu'^{(t)} = a\nu'^{(t)} + b\nu^{(t+1)}$ for some a and b. First we push it through the induced map from the injection $p : \mathbb{R}\mathrm{P}^{2t}/\mathbb{R}\mathrm{P}^{2t-1} \to \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{2t-1}$, which only leaves the $a\nu'^{(t)}$, in particular we have:

$$\Psi^k_{\mathbb{C}}(p^*\nu'^{(t)}) = p^*(\Psi^k_{\mathbb{C}}(\nu'^{(t)})) = ap^*(\nu'^{(t)})$$

But now $\mathbb{R}\mathrm{P}^{2t}/\mathbb{R}\mathrm{P}^{2t-1} = S^{2t}$ and we know by Proposition 7.16 that on S^{2t} the operations are given by multiplication with k^t, hence we see that $a = k^t$.
Next we project onto $\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{2t-2}$. By definition $\nu'^{(t)}$ gets mapped into $\nu^{(t)}$ while using the structure of $\widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{2t-2})$ we have that $\nu^{(t+1)} = -2\nu^{(t)}$. Now using the this and the preceding results on $\Psi^k_{\mathbb{C}}(\nu^{(t)})$ we get:

$$\Psi^k_{\mathbb{C}}(\nu^{(t)}) = (k^t - 2b)\nu^{(t)} = \begin{cases} 0 & \text{if } k \text{ even} \\ \nu^{(t)} & \text{if } k \text{ odd} \end{cases}$$

Since $\widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{2t-2}) = \mathbb{Z}_{2^{f+1}}$, we solve for b and get:

$$b \equiv \begin{cases} \frac{1}{2}k^t & \text{if } k \text{ even} \\ \frac{1}{2}(k^t - 1) & \text{if } k \text{ odd} \end{cases} \mod 2^f$$

Since $\widetilde{K}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{2t}) = \mathbb{Z}_{2^f}$ and hence the modulo 2^f does not matter the result follows. $\square$

Now at last we calculate the real K-Theory of the real projective space. This will get quite messy and the results follows from more shady arguments, but since it gets more involved it is particulary useful in proving Theorem 8.1.
First we define $\phi(m,n)$ to be integers s such that $n < s \leq m$ and $s \equiv 0, 1, 2$ or $4 \bmod 8$. Note that we have the identities $\phi(m,n) = \phi(m,0) - \phi(n,0)$ and $\phi(8t,0) = 4t$. Then we have the following results:

Theorem 8.5. Assume $n \not\equiv -1 \bmod 4$. Then we $\widetilde{KO}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n) = \mathbb{Z}_{2^f}$ where $f = \phi(m,n)$. If $n = 0$ then $KO(\mathbb{R}\mathrm{P}^m)$ can be described by the generator λ and the relations: $\lambda^2 = -2\lambda$ and $\lambda^{f+1} = 0$. Otherwise the projection $\mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n$ maps $\widetilde{KO}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^n)$ isomorphically onto the subgroup of $KO(\mathbb{R}\mathrm{P}^m)$ generated by λ^{g+1} where $g = \phi(n,0)$.
In the case that $n \equiv -1 \bmod 4$ we have:

$$\widetilde{KO}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{4t-1}) = \mathbb{Z} + \widetilde{KO}(\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{4t})$$

Here the second factor is induced by the projection $\mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{4t-1} \to \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{4t}$, and the first factor is the subgroup generated by some element $\lambda'^{(g)}$ (again $g = \phi(n,0) = \phi(4t,0)$). The operations $\Psi^k_{\mathbb{R}}$ are as follows:

$$\begin{array}{llcll}
\text{i)} & \Psi^k_{\mathbb{R}}\lambda^{(g+1)} & = & & \begin{cases} 0 & \text{if } k \text{ even} \\ \lambda^{(g+1)} & \text{if } k \text{ odd} \end{cases} \\
\text{ii)} & \Psi^k_{\mathbb{R}}\lambda'^{(g)} & = & k^{2t}\lambda'^{(g)} \; + & \begin{cases} \frac{1}{2}k^{2t}\lambda^{(g+1)} & \text{if } k \text{ even} \\ \frac{1}{2}(k^{2t}-1)\lambda^{(g+1)} & \text{if } k \text{ odd} \end{cases}
\end{array}$$

The element $\lambda'^{(g)}$ has properties which are very much alike the properties of $\nu'^{(t)}$ in Theorem 8.4. The exact construction of this element is tideous and can be found in [1, Lemmata 7.5-7.7, pp.625-628].

Proof. As for the ring structure, this again follows from arguments from spectral sequences, see [1, Thm 7.4,p.625].
Since Proposition 7.16 also works in the real case for multiples of 4 (which we have here since we will project to $\mathbb{R}\mathrm{P}^{4t}/\mathbb{R}\mathrm{P}^{4t-1} = S^{4t}$, the proof of the formulas for the operations are similar to the proof of Theorem 8.4. $\square$

8.2 Proof of Theorem 8.1

Now having a representation for $\widetilde{KO}(\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1})$ we will use this to disprove that there is a map $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1} \xrightarrow{f} S^m$ such that the composition $S^m = \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{m-1} \xrightarrow{i} \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}) \xrightarrow{f} S^m$ has degree 1. The argument that we will use only works for m a multiple of 8 but denoting $m = (2a+1)2^b$ the result for $b \leq 3$ can be proven with so called Steenrod operations which are beyond the scope of this thesis (the interested reader can look at [13]).
Now we look at $\phi(m+\rho(m), m)$, hence the number of integers s such that $m < s \leq m + \rho(m)$ and $s \equiv 0, 1, 2$ or $4 \bmod 8$. Identifying $m = (2a+1)2^{c+4d}$ with $0 \leq c \leq 3$ such that $\rho(m) = 2^c + 8d$, we split the range up is the section $m+1 \leq s \leq m+8$, $m+9 \leq s \leq m+16$, ..., $m + 8(d-1) + 1 \leq s \leq m + 8d$ and $m + 8d + 1 \leq s \leq m + 8d + 8$. Now apart from the last section all sections have 4 'good numbers' in it since m is a multiple of 8, namely $m + 8t + 1$, $m + 8t + 2$, $m + 8t + 4$ and $m + 8t + 8$ (for $t = 0, ..., d-1$), while in the last section we have $m + 8d + 2^t$ for $0 \leq t \leq c$ as the 'good numbers'. So we see that in total we have $4d + c + 1$ 'good numbers' between m and

$m+\rho(m)$, so we get $\phi(m+\rho(m),m) = 4d+c+1 = b+1$.

Next suppose we have a composite:

$$S^m = \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{m-1} \xrightarrow{i} \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}) \xrightarrow{f} S^m$$

This induces a sequence in K-Theory:

$$\widetilde{KO}(S^m) \xrightarrow{f^*} \widetilde{KO}(\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}) \xrightarrow{i^*} \widetilde{KO}(S^m)$$

Since $m \equiv 0 \bmod 8$ we have $\widetilde{KO}(S^m) = \mathbb{Z}$ and by Theorem 8.5 we have $\widetilde{KO}(\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}) = \mathbb{Z}\oplus\mathbb{Z}_{2^{b+1}}$. Now we choose a generator γ for $\widetilde{KO}(S^m)$, while we note that the factors of $\widetilde{KO}(\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1})$ are generated by $\lambda'^{(g)}$ and $\lambda^{(g+1)}$ where in this case $g = \frac{1}{2}m$. In particular we get:

$$\begin{tikzcd}
\widetilde{KO}(S^m) \arrow[r, "f^*"] \arrow[d, "\cong"] & \widetilde{KO}(\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}) \arrow[r, "i^*"] \arrow[d, "\cong"] & \widetilde{KO}(S^m) \arrow[d, "\cong"] \\
\mathbb{Z} \arrow[r, "f^*"] \arrow[d, "\cong"] & \mathbb{Z}\oplus\mathbb{Z}_{2^{b+1}} \arrow[r, "i^*"] \arrow[d, "\cong"] & \mathbb{Z} \arrow[d, "\cong"] \\
\langle\gamma\rangle \arrow[r, "f^*"] & \langle \lambda'^{(g)}, \lambda^{(g+1)} | 2^{b+1}\lambda^{(g+1)} = 0\rangle_{\mathrm{ab}} \arrow[r, "i^*"] & \langle\gamma\rangle
\end{tikzcd}$$

Now supposing that fi has degree 1, we'd have $i^*f^* = \mathbb{1}^* = \mathbb{1}$, so we'd have $i^*f^*(\gamma) = \gamma$.
We determine the structure of i^* and f^*. They are characterised by $i^*(\lambda'^{(g)})$, $i^*(\lambda^{(g+1)})$ and $f^*(\gamma)$. Since $\lambda^{(g+1)}$ has degree 2^{b+1} and since $\mathbb{Z}$ has apart from 0 no elements with finite order, we have $i^*(\lambda^{(g+1)}) = 0$. On the other hand we can write $f^*(\gamma) = M_1\lambda'^{(g)} + N\lambda^{(g+1)}$ and $i^*(\lambda'^{(g)}) = M_2\gamma$ and we want to determine the integers M_1, M_2 and N. Using the fact that $i^*f^*(\gamma) = \gamma$ we get:

$$\gamma = i^*f^*(\gamma) = i^*(M_1\lambda'^{(g)} + N\lambda^{(g+1)}) = M_1M_2\gamma$$

So we see that $M_1M_2 = 1$ and hence $M_1 = M_2 = \pm 1$. Now on the other hand we have $f^*(\Psi^k_{\mathbb{R}}\gamma) = \Psi^k_{\mathbb{R}}(f^*(\gamma))$. Using Proposition 7.16 (and the fact that $m \equiv 0 \bmod 8$) this becomes:

$$f^*(\Psi^k_{\mathbb{R}}\gamma) = f^*(k^{\frac{m}{2}}\gamma),$$

while using the structure of f^* we have:

$$\Psi^k_{\mathbb{R}}(f^*(\gamma)) = \Psi^k_{\mathbb{R}}(\pm\lambda'^{(g)} + \lambda^{(g+1)}),$$

and

$$f^*(k^{\frac{m}{2}}\gamma) = \pm k^{\frac{m}{2}}\lambda'^{(g)} + k^{\frac{m}{2}}N\lambda^{(g+1)}$$

Using Theorem 8.5 we can get rid of the $\Psi^k_{\mathbb{R}}$ and get:

$$\Psi^k_{\mathbb{R}}(\pm\lambda'^{(g)} + \lambda^{(g+1)}) = \pm k^{\frac{m}{2}}\lambda'^{(g)} \pm \frac{1}{2}\left(k^{\frac{m}{2}} - \epsilon\right)\lambda^{(g+1)} + \epsilon N\lambda^{(g+1)},$$

where ϵ is 0 if k is even and 1 if k is odd. Now plugging all this together we get:

$$\pm k^{\frac{m}{2}}\lambda'^{(g)} + k^{\frac{m}{2}}N\lambda^{(g+1)} = \pm k^{\frac{m}{2}}\lambda'^{(g)} \pm \frac{1}{2}\left(k^{\frac{m}{2}} - \epsilon\right)\lambda^{(g+1)} + \epsilon N\lambda^{(g+1)},$$

and scrambling all the terms we get:

$$\left(N \mp \frac{1}{2}\right)\left(k^{\frac{m}{2}} - \epsilon\right)\lambda^{(g+1)} = 0,$$

that is:

$$\left(N \mp \frac{1}{2}\right)\left(k^{\frac{m}{2}} - \epsilon\right) \equiv 0 \bmod 2^{b+1}$$

If a suitable function f exists this equation should hold for all k. To show that it doesn't suppose we have a k such that:

$$k^{\frac{m}{2}} - \epsilon \equiv 2^{b+1} \bmod 2^{b+2}$$

in that case we'd have:

$$\begin{aligned}
\left(N \mp \frac{1}{2}\right)\left(k^{\frac{m}{2}} - \epsilon\right) &= \left(N \mp \frac{1}{2}\right)(2^{b+1} + t2^{b+2}) \\
&= 2^{b+1}\left(N \mp \frac{1}{2}\right)(1 + 2t) \\
&= 2^b (2n \mp 1)(1 + 2t) \\
&= 2^b (2n + 4nt \mp 1 \mp 2t) \\
&= 2^{b+1}(n + 2nt \mp t) \mp 2^b \\
&\equiv 2^b \bmod 2^{b+1} \\
&\not\equiv 0 \bmod 2^{b+1}
\end{aligned}$$

So the last part will be finding such a k. The answer comes from number theory:

Lemma 8.6. If $n = (2a+1)2^f$ with $f \geq 1$ then $3^n - 1 \equiv 2^{f+2} \bmod 2^{f+3}$.

Note that this thus is the case $k = 3$. Also note that since the exponent of k is $m/2$ we have $m = 2n$ and hence $m = (2a+1)2^b = (2a+1)2^{f+1}$ and hence $b = f+1$. We may use this lemma since $m \equiv 0 \bmod 8$ and hence $b \geq 3$.

Proof. We will prove by induction. But first note that since $3^2 \equiv 1 \bmod 8$ we know that $3^{2n} \equiv 1 \bmod 8$ and hence $3^{2n} + 1 \equiv 2 \bmod 8$. Now we will use induction on the statement:

$$3^{(2^f)} - 1 \equiv 2^{f+2} \bmod 2^{f+4}$$

The case $f = 1$ is true since $3^{(2^1)} = 8$ and hence certainly 8 mod 32. Now suppose that it is true for some f then:

$$3^{(2^{f+1})} - 1 = (3^{2^f} - 1)(3^{2^f} + 1)$$

Now using the induction hypothesis $3^{2^f} - 1 = 2^{f+2} + x2^{f+4}$ and by the note above we have $3^{2^f} + 1 = 2 + y2^3$ so:

$$\begin{aligned}
3^{(2^{f+1})} - 1 &= (2^{f+2} + x2^{f+4})(2 + y2^3) \\
&= 2^{f+3} + x2^{f+5} + y2^{f+5} + xy2^{f+7} \\
&\equiv 2^{f+3} \bmod 2^{f+5}
\end{aligned}$$

Now we know that $3^{(2^{f+1})} - 1 \equiv 2^{f+3} \bmod 2^{f+5}$ and hence certainly it holds that $3^{(2^{f+1})} \equiv 1 \bmod 2^{f+3}$.

Next using this result we see that $3^{(2a)2^f} \equiv 1 \bmod 2^{f+3}$.

Now multiplying with an extra $3^{(2^f)}$ we get:

$$3^{(2a+1)2^f} \equiv 3^{(2^f)} \bmod 2^{f+3}$$

So in particular we get that:

$$3^{(2a+1)2^f} - 1 \equiv 3^{(2^f)} - 1 \bmod 2^{f+3}$$

By the first argument in this proof we now get that $3^{(2^f)} - 1 \equiv 2^{f+2} \bmod 2^{f+3}$ (mod 2^{f+3} or mod 2^{f+4} does not matter here) and hence:

$$3^n - 1 = 3^{(2a+1)2^f} - 1 \equiv 2^{f+2} \bmod 2^{f+3}$$

□

So we see that the case $k = 3$ indeed yields the desired contradiction, so we see that there is no map $f : \mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1} \to S^m$ such that fi has degree 1 and this concludes the proof of Theorem 8.1. □

9 Conclusion and outlook

Now recalling from chapter 4 that if there would we some m such that $\mathbb{R}\mathrm{P}^{m+\rho(m)}/\mathbb{R}\mathrm{P}^{m-1}$ would be co-reducible if there were some n such that there would be $\rho(n)$ pointswise independent vector fields on S^{n-1}, we see that this can never happen and hence we find that there are exactly $\rho(n) - 1$ pointwise independent vector fields on S^{n-1} and we have case closed.

One can wonder if we can use this proof and it's ideas to determine the maximum number of vector fields on any manifold M. This turns out to be not completely the case since we extensively used that the tangent space of the sphere can be very well described within Euclidean space. In particular the connection between vector fields and Theorem 8.1 is something that is really specific to the case with spheres.

We can however always make some statements on the number of vector fields. In particular using the Poincare-Hopf Theorem we can always determine whether there is one nowhere vanishing vector field at all, cutting off the search very early if $\chi(M) \neq 0$. Also we recall the statement on the Stiefel-Whitney class of a vector bundle and the number of independent vector fields (proposition 6.11). To use this one can determine $w(TM)$ and then look how many of the top elements are zero. In particular if $w_i(TM) \neq 0$ $(i > 0)$ then we know that there are at most $\dim(M) - i$ vector fields on M. So while the ideas of this thesis do not fully come in use for general M, some statements made in this text can be used to at least determine an upper and lower bound.

A Appendix on general topology

A.1 Construction on spaces

This section will give a quick overview of non-trivial constructions on spaces which will be used in the text. This list turns out to be one, namely:

Definition A.1. Given two pointed spaces (X, x_0) and (Y, y_0) we define the *smash product* $X \wedge Y$ by:

$$X \wedge Y = (X \times Y)/(X \times \{x_0\} \cup (\{x_0\} \times Y))$$

A.2 Vector bundles and fibre bundles

We assume that the reader has some knowledge of the basics of differential topology (charts, atlases, smooth maps, etc.), but we will recall some facts about vector and fibre bundles because they are heavily used throughout the text.

Definition A.2. A *fibre bundle* over a (connected) topological space X with fibre F is a topological space E with a surjective continuous map $\pi : E \to X$ such that:

i) For every $x \in X$ we have that $E_x = \pi^{-1}(x) \simeq F$, that is, on every point of X we have attached a copy of F.

ii) There is an open cover $\{U_a\}_{a \in A}$ of X such that for every $a \in A$ there is a homeomorphism Φ_a between $\pi^{-1}(U_a)$ and $U_a \times F$, such that the first factor of $\Phi_a(y)$ is equal to $\pi(y)$.

In the case that F is a linear space, we get the notion of a vector bundle, for which the second property gets a bit more refined:

Definition A.3. A *vector bundle* over a (connected) topological space X with fiber some linear space V is a topological space E with a surjective continuous map $\pi : E \to X$ such that:

i) For every $x \in X$ we have that $E_x = \pi^{-1}(x) \simeq V$

ii) There is an open cover $\{U_a\}_{a \in A}$ such that for every $a \in A$ there is a homeomorphism Φ_a between $\pi^{-1}(U_a)$ and $U_a \times V$ such that for every $x \in U_a$, Φ_a restricts to a linear isomorphism between E_x and $\{x\} \times V$.

Remarks:

i) When $V = \mathbb{R}^k$, E is called a real vector bundle, and when $V = \mathbb{C}^k$, E is called a complex vector bundle.

ii) When $V = \mathbb{R}$ or $V = \mathbb{C}$, E is called a real respectively complex line bundle.

iii) One could also do this in the smooth sense, in which case we want X, F and E to be smooth manifolds and all maps to be smooth and all homeomorphisms to be diffeomorphisms.

Now one could ask when two vector bundles are related or the same. For this we define the following:

Definition A.4. Let $f : X \to Y$ be a continuous map and E and F be vector bundles over X and Y respectively. A *vector bundle homomorphism* between E and F covering f is a map $\overline{f} : E \to F$ such that

i) The following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\overline{f}"] \arrow[d, "\pi_E"] & F \arrow[d, "\pi_F"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

i.e. that $\overline{f}$ maps a fiber E_x into the fiber $F_{f(x)}$.

ii) The map $\overline{f}|_{E_x} : E_x \to F_{f(x)}$ is linear.

When $\overline{f}$ is a homeomorphism then it is called a vector bundle isomorphism.

If we have a vector bundle E (with fiber V), we get a cover $\{U_a\}$ and maps $\Phi_a : \pi^{-1}(U_a) \to U_a \times V$. Taking two of those maps Φ_a and Φ_b we get maps:

$$\Phi_{ab} = \Phi_a \circ \Phi_b^{-1} : U_{ab} \times V \to U_{ab} \times V$$

(where $U_{ab} = U_a \cap U_b$). Since the Φ's map take a fibre E_x into $\{x\} \times V$ and are fibrewise linear we get that:

$$\Phi_{ab}(p, v) = (p, \phi_{ab}(p)v)$$

for some $\phi_{ab}(p) \in GL(V)$. Now since Φ_{ab} is continuous it follows that $\phi_{ab} : U_{ab} \to GL(V)$ is also continuous. The collection $\{\phi_{ab}\}$ is called the (Čech) cocyle of E. This cocyle satisfies the property that $\phi_{ab}\phi_{bc}\phi_{ca}(p) = \mathbb{1}$ for every p in $U_a \cap U_b \cap U_c$. Also it follows that if two vector bundles have isomorphic cocyles (i.e. all maps differ by a fixed invertible matrix), that they're isomorphic themselves. So we see that cocycles are a way to describe vector bundles.

Now we want to freely play with these cocycles, so we also that for any open cover $\{U_a\}_{a\in A}$ with maps $\phi_{ab} : U_{ab} \to GL(V)$ satisfying $\phi_{ab}\phi_{bc}\phi_{ca}(p) = \mathbb{1}$ for any $p \in U_a \cap U_b \cap U_c$ we have a vector bundle with cocycle $\{\phi_{ab}\}$. It follows that we can actually do this:

Theorem A.5. Given a open cover $\{U_a\}_{a\in A}$ with maps $\phi_{ab} : U_{ab} \to GL(V)$ satisfying $\phi_{ab}\phi_{bc}\phi_{ca}(p) = \mathbb{1}$ for any $p \in U_a \cap U_b \cap U_c$, there is a vector bundle E over X with fibre V which has cocycle-maps $\{\phi_{ab}\}$.

Proof. We set $E = (\bigsqcup_{a\in A} U_a \times V)/\sim$ where $\sim$ is an equivalence relation set by $U_a \times V \ni (p, v) \sim (p, v') \in U_b \times V$ when $v = \phi_{ab}(v')$. That this is an equivalence relation follows from the assumption on ϕ_{ab}. That E is a vector bundle over X with cocycles ϕ_{ab} follows more or less directly. $\square$

In particular if we have a group homomorphism $\rho : GL(V) \to GL(W)$ and a vector bundle E over X with fibres V we get a vector bundle F over X with fibres W by looking at the cocycles ϕ_{ab} of E and setting the cocycles of F on $\rho \circ \phi_{ab}$. It follows that $\rho(\phi_{ab})\rho(\phi_{bc})\rho(\phi_{ca}) = \rho(\phi_{ab}\phi_{bc}\phi_{ca}) = \rho(\mathbb{1}) = \mathbb{1}$, so we can use the previous theorem to make F.

B Appendix on algebraic topology

In this apppendix we will recall homotopy groups, singular homology and singular cohomology and state the definition of generalized (co)homology theories.

B.1 Homotopy groups

Homotopy groups measure how extremely a certain space X is formed. While the most used example, the fundamental group π_1 consisting of homotopy classes of paths, is particulary easy to compute, the higher order homotopy groups are harder to determine, though they still encode valuable information about a space.
The discussion centers about maps $f : (I^n, \delta I^n) \to (X, x_0)$, i.e. $f(y) = x_0$ for any $y \in \delta I^n$. Two such maps, f_0 and f_1, are homotopic there is a map $F : I^n \times I \to X$ such that $F(\cdot, 0) = f_0$ and $F(\cdot, 1) = f_1$ and $f(\delta I^n, t) = x_0$ for any $t \in I$. Maps being homotopic is an equivalence relation, and we denote $[f]$ for the equivalence class of a map f. Set-theoretically we now have:

Definition B.1. The *n'th homotopy group* of X at x_0 is defined by

$$\pi_n(X, x_0) = \{[f], f : (I^n, \delta I^n) \to (X, x_0)\}$$

The group operation comes from concaternation of maps from I^n. So let $[f]$ and $[g]$ be two elements of $\pi_n(X, x_0)$ we define:

$$(f \star g)(s_1, ..., s_n) = \begin{cases} f(2s_1, s_2, ..., s_n) & \text{if } 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, ..., s_n) & \text{if } \frac{1}{2} \leq s_1 \leq 1 \end{cases}$$

It follows that the homotopy class of this map does not depend on the choice of f and g, so it induces a group operation on $\pi_n(X, x_0)$.
We note that when $n > 1$ we have 'enough space' to exchange f and g, so for $n > 1$ $\pi_n(X, x_0)$ is abelian. Also when X is path-connected, the isomorphism class of $\pi_n(X, x_0)$ does not depend on x_0 and we just call it $\pi_n(X)$
The homotopy groups satisfy: $\pi_i(S^n) = 0$ for $i < n$ and $\pi_n(S^n) = \mathbb{Z}$, but $\pi_i(S^n)$ for $i > n$ turn out to be a messy state of affairs, so we define another, less intuitive, but more computable way to describe the shape of a space.

B.2 Singular homology

Singular homology is a way to describe how many holes a space X has. Just like homotopy groups it is an algebraic invariant (i.e. homotopy equivalent spaces have isomorphic homology groups) which measures how the space is formed, but it is much more computable. While homotopy groups have the downside that the higher order ones are very hard to compute, homology groups will turn out to have the property that for a n-dimensional CW-complex X we have $H_i(X) = 0$ for $i > n$. First we let $\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1 \text{ and } t_i \geq 0\}$, the smallest convex subset of $\mathbb{R}^{n+1}$ containing all basis vectors e_i. In very much the same way we can also define it to be $\Delta^n = [v_0, ..., v_n]$ where $[v_0, ..., v_n]$ is the smallest convex subset containing all v_i (for $\{v_i\}_i$ linearly independent). It we forget one v_i, say $[v_0, ..., \hat{v}_i, ..., v_n]$, it is understood that we get a Δ^{n-1}-simplex. Now a singular n-simplex is a map $\sigma : \Delta^n \to X$ and we let $C_n(X; G)$ be the group generated by all maps σ with coefficients in an (abelian) group G, that is the group of formal sums $\sum_i n_i \sigma_i$ with $n_i \in G$ and $\sigma_i : \Delta^n \to X$. There is a boundary map $\partial_n : C_n(X; G) \to C_{n-1}(X; G)$ defined by:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, ..., \hat{v}_i, ..., v_n]}$$

Here it is understood that $\sigma|_{[v_0,\dots,\hat{v_i},\dots,v_n]}$ can be seen as a map $\Delta^{n-1} \to X$. Also we define C_{-1} to be the trivial group and hence $\partial_0 = 0$.

Now $\partial^2 = 0$ and we define the n'th homology group by

$$H_n(X;G) = \ker(\partial_n)/\mathrm{im}(\partial_{n-1}).$$

We can also put an extra G between $C_0(X;G)$ and 0, and replace ∂_0 with first a map $\epsilon : C_0(X;G) \to G$ that sends $\sum_i n_i\sigma_i$ to $\sum_i n_i$ and then the zero map $G \to 0$. Since $\epsilon\partial_1 = 0$ we define the reduced homology by:

$$\widetilde{H}_0(X;G) = \ker(\epsilon)/\mathrm{im}(\partial_0)$$

while $\widetilde{H}_n(X;G) = H_n(X;G)$ for $n > 0$. It follows that $\widetilde{H}_0(X;G) \otimes G = H_0(X;G)$.

We can also define relative homology. Here we 'forget' about a subset A and define $C_n(X,A;G) = C_n(X;G)/C_n(X;A)$. Since ∂ factors through it to maps $\partial_n : C_n(X,A;G) \to C_{n-1}(X,A;G)$ we define the n'th relative homology group by $H_n(X,A;G) = \ker\partial/\mathrm{im}\partial$.

Now it follows that singular homology has the following properties:

Proposition B.2 (additivity)**.** If X is made up of connected components X_a then $H_n(X;G) = \oplus_a H_n(X_a;G)$.

Proposition B.3 (functoriality)**.** For any map $f : X \to Y$ there is a map $f_* : H_n(X;G) \to H_n(Y;G)$ satisfying $(fg)_* = f_*g_*$ and $\mathbb{1}_* = \mathbb{1}$. Furthermore if $f \simeq g$ then $f_* = g_*$. In particular it is a (covariant) functor from spaces to groups.

Here we first put a map $f_\sharp$ on $C_n(X;G)$ which sends a simplex σ to the simplex $f\sigma \in C_n(Y;G)$. This map satisfies $\partial f_\sharp = f_\sharp\partial$ and hence factors through the quotients into maps $f_* : H_n(X;G) \to H_n(Y;G)$.

Proposition B.4 (long exact sequence of the pair)**.** A pair (X,A) with $A \subset X$ induces the exact sequence of the pair

$$\cdots \to H_n(A;G) \to H_n(X;G) \to H_n(X,A;G) \to H_{n-1}(A;G) \to \cdots.$$

Proposition B.5 (excision)**.** Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, the inclusion $(X - Z, A - Z) \to (X,A)$ induces isomorphisms

$$H_n(X - Z, A - Z;G) \to H_n(X,A;G)$$

for all n. In particular if we have an open cover $\{A,B\}$ of X the inclusion $(B, A \cap B) \to (X,A)$ induces isomorphisms

$$H_n(B, A \cap B;G) \to H_n(X,A;G)$$

for all n.

We also note that $\widetilde{H}_i(S^n;G) = G$ for $i = n$ and 0 otherwise, and this in principal induces all the values for any CW-complex.

Having defined singular homology we also define the degree of a map $f : S^n \to S^n$. Since $H_n(S^n;\mathbb{Z}) = \mathbb{Z}$ the induced map f_* is multiplication by some $d \in \mathbb{Z}$, that is $f_*(n) = dn$, and we define $\deg(f) = d = f_*(1)$. It follows that $\deg(f) = \deg(g)$ if and only if f and g are homotopic.

B.3 Singular cohomology

Cohomology is the dual of homology. This means that there is no real new information, but some information gets shuffeld in such a way that we can more easily read it, just like when we ditched homotopy for homology.
We define $C^n(X;G)$, the group of singular n-cochains with coefficients in G to be $\mathrm{Hom}(C_n(X;\mathbb{Z}),G)$, that is group homomorphisms from $C_n(X;\mathbb{Z})$ to G. I.e. a n-cochain ψ takes a n-chain σ to $\psi(\sigma) \in G$. This closely relates the construction of the dual of a vector space. The boundary map δ is defined as the dual of ∂, that is $\delta\psi(\sigma) = \sigma(\partial\sigma)$. Since δ^2 is the dual of $\partial^2 = 0$ we see that $\delta^2 = 0$ and we define the n'th cohomology group by $H^n(X;G) = \ker(\delta)/\mathrm{im}(\delta)$. Relative and reduced cohomology are defined in the same manner. More or less all properties of homology come over to cohomology, but now that everything is contravariant and hence all the arrows flip:

Proposition B.6 (additivity)**.** If X is made up of connected components X_a then $H^n(X;G) = \oplus_a H^n(X_a;G)$.

Proposition B.7 (functoriality)**.** For any map $f : X \to Y$ there is a map $f^* : H^n(Y;G) \to H^n(X;G)$ satisfying $(fg)_* = g_* f_*$ and $\mathbb{1}_* = \mathbb{1}$. Furthermore if $f \simeq g$ then $f^* = g^*$. In particular it is a contravariant functor from spaces to groups.

Proposition B.8 (long exact sequence)**.** A pair (X,A) with $A \subset X$ induces the exact sequence of the pair

$$\cdots \to H^{n-1}(A;G) \to H^n(X,A;G) \to H^n(X;G) \to H^n(A;G) \to \cdots .$$

Proposition B.9 (excision)**.** Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, the inclusion $(X - Z, A - Z) \to (X,A)$ induces isomorphisms

$$H^n(X - Z, A - Z; G) \simeq H^n(X,A;G)$$

for all n. In particular if we have an open cover $\{A, B\}$ of X the inclusion $(B, A \cap B) \to (X,A)$ induces isomorphisms

$$H^n(B, A \cap B; G) \simeq H^n(X,A;G)$$

for all n.

Also we again have that $\widetilde{H}^i(S^n;G) = G$ for $n = i$ and 0 otherwise.

B.4 (Co)homology theories

Having defined singular (co)homology, we note that most of the time we calculate (co)homology of a space X by using the various propositions and the values on spheres. This axiomatic approach leads to the notion of generalized (co)homology theories, where we leave the propositions in tact, but vary the values on spheres:

Definition B.10. A *generalized homology theory* is a sequence of covariant functors h_n from the category of pairs (X,A) to the category of abelian groups, together with a natural transformation $\partial : h_n(X,A) \to h_{n-1}(A)$ (using the notaton A for $(A,\emptyset)$ such that:

i) A map $f : (X, A) \to (Y, B)$ induces a map $f_* : h_n(X, A) \to h_n(Y, B)$ such that $(fg)_* = f_* g_*$ and $\mathbb{1}_* = \mathbb{1}$.

ii) If (X, A) is a pair and U is a subset of X such that $\overline{U} \subset \mathring{A}$, then the inclusion $(X-U, A-U) \to (X, A)$ induces isomorphisms $h_n(X - U, A - U) \to h_n(X, A)$.

iii) If $X = \bigsqcup_a X_a$, then $h_n(X) \simeq \oplus_a h_n(X_a)$

iv) Each pair (X, A) induces a long exact sequence via the inclusion $i : A \to X$ and $j : X \to (X, A)$: $\cdots \to h_n(A) \xrightarrow{i_*} h_n(X) \xrightarrow{j_*} h_n(X, A) \xrightarrow{\partial} h_{n-1}(A) \to \cdots$

Since the last axiom induces strategies to calculate the groups h_n for a CW-complex if one knows $h_i(S^n)$, a homology theory is normally characterised by the values on spheres.
Also we note that sometimes one demands that $h_n(\{x_0\}) = 0$ for all $n \neq 0$, to 'normalize' the results, but some interesting homology theories do to satisfy this axiom, so we obmit it here.

Flipping the direction of the arrows we also get the definition of a generalized (sometimes called extraordinary) cohomology theory:

Definition B.11. A *generalized cohomology theory* is a serie of contravariant functors h^n from the category of pairs (X, A) to the category of abelian groups, together with a natural transformation $\delta : h^n(A) \to h^{n+1}(X, A)$ (using the notation $h^n(A)$ for $h^n(A, \emptyset)$) such that:

i) A map $f : (X, A) \to (Y, B)$ induces a map $f^* : h_n(Y, B) \to h_n(X, A)$ such that $(fg)^* = g^* f^*$ and $\mathbb{1}^* = \mathbb{1}$.

ii) If (X, A) is a pair and U is a subset of X such that $\overline{U} \subset \mathring{A}$, then the inclusion $(X-U, A-U) \to (X, A)$ induces isomorphisms $h^n(X - U, A - U) \to h^n(X, A)$.

iii) If $X = \bigsqcup_a X_a$, then $h^n(X) \simeq \oplus_a h^n(X_a)$

iv) Each pair (X, A) induces a long exact sequence via the inclusion $i : A \to X$ and $j : X \to (X, A)$: $\cdots \to h_{n-1}(A) \xrightarrow{\delta} h_n(X, A) \xrightarrow{j^*} h_n(X) \xrightarrow{i^*} h_n(A) \to \cdots$

C Appendix on group theory

This section is dedicated to a specific construction of a group from a sequence of groups. For this let $\{G_i\}_{i\in\mathbb{N}}$ be a sequence of groups with homomorphisms $f_i : G_i \to G_{i+1}$. From these we can make homomorphisms $f_{ij} : G_i \to G_j$ for $i < j$ defined as $f_{ij} = f_{j-1} \circ \cdots \circ f_i$. We have $f_{ii} = \mathbb{1}_{G_i}$ and $f_{ik} = f_{jk} \circ f_{ij}$. Now we have:

Definition C.1. The *direct limit* of G_i, $\varinjlim G_i$ is defined by: $\varinjlim G_i = \oplus_i G_i / \sim$ where for $g_i \in G_i$ and $g_j \in G_j$ we have $g_i \sim g_j$ if and only if there is a $k \geq \max(i, j)$ such that $f_{ik}(g_i) = f_{jk}(g_j)$.

Heuristically this means that we have a lot of groups, where each maps into the next and that identify two elements of two groups if there evantually map into the same element.

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