Topologie 2009/2010
Lecturer: Marius Crainic

CHAPTER 1

Introduction: some standard spaces

1. Keywords for this course

In this course we study topological spaces. One may remember that in group theory one studies groups- and a group is a set G together with some extra-structure (the group operation) which allows us to multiply the elements of G. Similarly, but roughly speaking, a topological space is a set X together with some "extra-structure" which allows us to make sense of "two points getting close to each other" or, even better, it allows us to make sense of statements like: a sequence $(x_n)_{n\geq 1}$ of elements of X converges to $x \in X$. Of course, if X is endowed with a "metric" (i.e. a way to measure, or to give sense to, "the distance between two points of X"), then such statements have a clear intuitive meaning and can easily be made precise. However, the correct extra-structure that is needed is a bit more subtle- it is the notion of *topology on X* which will be explained below.

The interesting functions in topology are the continuous functions. One may remember that, in group theory, the interesting functions between two groups G_1 and G_2 are not all arbitrary functions $f : G_1 \longrightarrow G_2$, but just those which "respect the group structure" (group homomorphisms). Similarly, in topology, the interesting maps between two topological spaces X and Y are those functions $f : X \longrightarrow Y$ which are continuous. Continuous means that "it respects the topological structures"- and this will be made precise later. But roughly speaking, f being continuous means that it maps convergent sequences to convergent sequences: if $(x_n)_{n\geq 1}$ is a sequence in X converging to $x \in X$, then the sequence $(f(x_n))_{n\geq 1}$ of elements of Y converges to $f(x) \in Y$.

The correct notion of isomorphism in topology is that of homeomorphism. In particular, we do not really distinguish between spaces which are *homeomorphic*. Thinking again back at group theory, there we do not really distinguish between groups which are isomorphic- and there the notion of isomorphism was: a bijection which preserves the group structure. Similarly, in topology, a homeomorphism between two topological spaces X and Y is a bijection $f : X \longrightarrow Y$ so that f and f^{-1} are both continuous (note the apparent difference with group theory: there, a group isomorphism was a bijection $f : G_1 \longrightarrow G_2$ such that f is a group homomorphism. However, one can prove that from this it does follow that f^{-1} is itself a group homomorphism).

One of the main problems in topology is to have tools that help us decide when two topological spaces are homeomorphic. In this course we will see two methods of tackling this problem.

The first one consists of understanding special properties of topological spaces, called topological properties (such as connectedness, compactness, being normal, etc). For instance, a space which is compact (or connected, or etc) can never be homeomorphic to one which is not. The first part of the course is devoted to the study of such topological properties.

The second method consists of finding ways associating topological invariants to topological spaces, so that, if two spaces have distinct topological invariant, they cannot be homeomorphic.

The topological invariants could be numbers (such as "the number of distinct connected components", or "the number of wholes"), but they can also be more complicated algebraic objects such as groups. For instance, in the last part of the course, we will introduce one of the most important topological invariants- *the fundamental group*.

What we will also pay attention to in this course are *topological constructions*- i.e. methods that allow us to construct new topological spaces out of ones that we already know (such as taking the product of two topological spaces, the cone of a space, quotients).

We start this course with various examples of "topological spaces" without introducing the formal definition (but trying to point out the need of such a notion). Hence, I would like to emphasize that some of the statements made in this chapter are rather loose (un-precise)- and I try to make that clear by using quotes. Throughout this chapter, the spaces are rather explicit and intuitive, and when saying "space" (as opposed to set) we have in mind the underlying set (of elements, also called points) as well as the fact that we can talk (at least intuitively) about its points "getting closer to each other " (or, even better, about convergence of sequences of points in the set). With this in mind, the notion of continuous function between spaces will have a precise meaning- it is, by definition, any function which sends convergent sequences to convergent sequences. Similarly, by homeomorphism between two spaces we mean any bijective function with the property that both f and f^{-1} are continuous (see also the outline above).

The most intuitive examples of spaces are the real line $\mathbb{R}$, the plane $\mathbb{R}^2$, the space $\mathbb{R}^3$ or, more generally, the Euclidean space

$$\mathbb{R}^k = \{(x^1, \ldots, x^k) : x^1, \ldots, x^n \in \mathbb{R}\}$$

defined for any integer $k \geq 1$.

To make things more precise, one uses the Euclidean metric and the notion of convergence and continuity with respect to this metric. Let us first recall the notion of metric space.

DEFINITION 1.1. *Let X be a set. A metric on X is a function*

$$d : X \times X \longrightarrow \mathbb{R}$$

which associates to a pair (x, y) of points x and y of X, a real number $d(x, y)$, called the distance between x and y, such that the following conditions hold:

(M1) $d(x, y) \geq 0$ *for all* $x, y \in X$.
(M2) $d(x, y) = 0$ *if and only if* $x = y$.
(M3) $d(x, y) = d(y, x)$.
(M4) *(triangle inequality)* $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in X$.

A metric space is a pair (X, d) consisting of a set X together with a metric d.

Metric spaces are particular cases of "topological spaces"- since they allow us to talk about convergence and continuity. Recall that, given a metric space (X, d), and a sequence $(x_n)_{n\geq 1}$ of points of X, we say that $(x_n)_{n\geq 1}$ converges to $x \in X$ (in (X, d), or with respect to d) if $\lim_{n\to\infty} d(x_n, x) = 0$. The notion of continuous function between two metric spaces is defined by the general principle: $f : X \longrightarrow Y$ is continuous if $(x_n)_{n\geq 1}$ converges to x implies that $(f(x_n))_{n\geq 1}$ converges to $f(x)$.

Back to the Euclidean spaces, one has the standard (or Euclidean) metric given by:

$$d(x, y) = \sqrt{(x^1 - y^1)^2 + \ldots + (x^k - y^k)^2}$$

(for $x = (x^1, \ldots, x^k)$, $y = (y^1, \ldots, y^k) \in \mathbb{R}^k$). Another interesting metric on $\mathbb{R}^k$ is the square metric ρ, defined by:

$$\rho(x, y) = max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

The next exercise exercise shows that, although the notion of metric allows us to talk about convergence, metrics do not encode convergence faithfully (two very different looking metrics can induce the same convergent sequences, hence the same "space"). The key of understanding the "topological content" of metrics (i.e. the one that allows us to talk about convergent sequences) is the notion of open subsets- to which we will come back later.

EXERCISE 1.1. *Show that a sequence of points of $\mathbb{R}^n$ is convergent with respect to the Euclidean metric if and only if it is convergent with respect to the square metric.*

Inside these Euclidean spaces sit other interesting topological spaces such as intervals, circles, spheres, etc. In general, any subset

$$X \subset \mathbb{R}^k$$

can naturally be viewed as a "space"; note also that the Euclidean metric induces a metric on X.

EXERCISE 1.2. *Show that, for any two numbers $a < b$*

(1) *the interval $[a, b]$ is homeomorphic to $[0, 1]$.*
(2) *the interval $[a, b)$ is homeomorphic to $[0, 1)$ and also to $[0, \infty)$.*
(3) *the interval (a, b) is homeomorphic to $(0, 1)$ and also to $(0, \infty)$ and to $\mathbb{R}$.*

2. The unit circle

In $\mathbb{R}^2$ one has the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

the open disk

$$\overset{\circ}{D}{}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

the closed disk

$$D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Next, we mentione the standard parametrization of the unit circle: any point on the circle can be written as

$$e^{it} := (cos(t), sin(t))$$

for some $t \in \mathbb{R}$. This gives rise to a function

$$e : \mathbb{R} \longrightarrow S^1, \quad e(t) = e^{it}$$

which is continuous (explain why!). A nice picture of this function is obtained by first spiraling $\mathbb{R}$ above the circle and then projecting it down, as in Figure 1.

EXERCISE 1.3. *Make Figure 1 more precise. More precisely, find an explicit subspace $S \subset \mathbb{R}^3$ which looks like the spiral, and a homeomorphism h between $\mathbb{R}$ and S, so that the map e above is obtained by first applying h and then applying the projection p ($p(x, y, z) = (x, y)$).*
(Hint: $\{(x, y, z) \in \mathbb{R}^3 : x = cos(z), y = sin(z)\}$).

Note also that, if one restricts to $t \in [0, 2\pi)$, we obtain a continuous bijection

$$f : [0, 2\pi) \longrightarrow S^1.$$

However, $[0, 2\pi)$ and S^1 behave quite differently as topological spaces, or, more precisely, they are not homeomorphic. Note that this does not only mean that f is not a homeomorphism; it means that neither f nor any other bijection between $[0, 2\pi)$ and S^1 is a homeomorphism. It

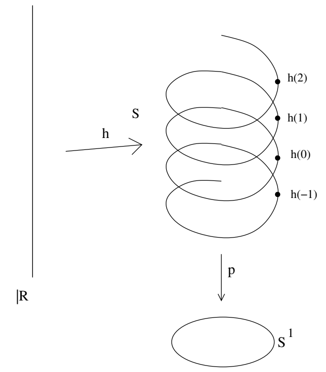


FIGURE 1.

will be only later, after some study of topological properties (e.g. compactness), that we will be able to prove this statement. At this point however, one can solve the following

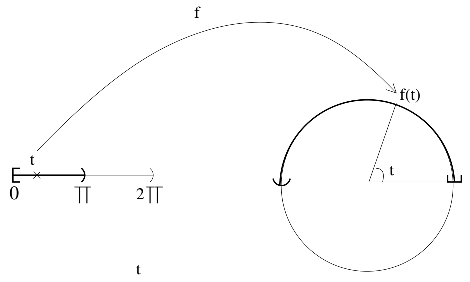


FIGURE 2.

EXERCISE 1.4. *Show that the map $f : [0, 2\pi) \longrightarrow S^1$ is not a homeomorphism.*

Next, there is yet another way one can look at the unit circle: as obtained from the unit interval $[0, 1]$ by "banding it" and "gluing" its end points, as pictured in Figure 3. This "gluing process" will be made more precise later and will give yet another general method for constructing interesting topological spaces.

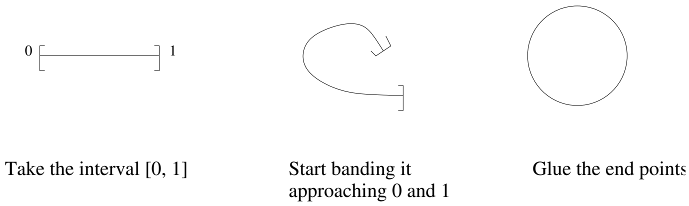


FIGURE 3.

In general, by a (topological) circle we mean any space which is homeomorphic to S^1. In general, they may be placed in the space in a rather non-trivial way. Some examples of circles are:

- circles S^1_r with a radius $r > 0$ different from 1, or other circles placed somewhere else in the plane.
- pictures obtained by twisting a circle in the space, such as in Figure 4.

- even pictures which, in the space, are obtained by braking apart a circle, knotting it, and then gluing it back (see Figure 5).

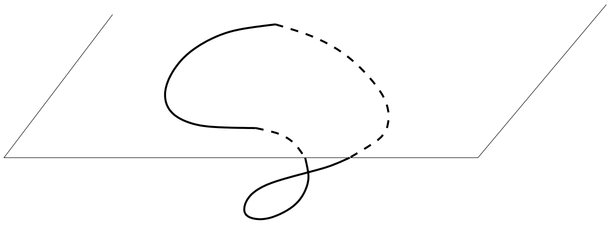


FIGURE 4.

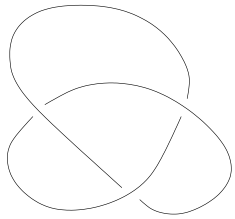


FIGURE 5.

EXERCISE 1.5. *Explain on pictures that all the spaces enumerated above are homeomorphic to* S^1*. If you find it strange, try to explain to yourself what makes it look strange (is it really the circles, or is it more about the ambient spaces in which you realize the circles?).*

Similarly, by a (topological) disk we we mean any space which is homeomorphic to D^2. For instance, the unit square

$$[0,1] \times [0,1] = \{(x,y) \in \mathbb{R}^2 : x, y \in [0,1]\}$$

is an important example of topological disk. More precisely, one has the following:

EXERCISE 1.6. *Show that the unit disk* D^2 *is homeomorphic to the unit square, by a homeomorphism which restricts to a homeomorphism between the unit circle* S^1 *and the boundary of the unit square.*

EXERCISE 1.7. *Explain why all the three subset of the plane drawn in Figure 7 are homeomorphic.*

EXERCISE 1.8. *Which of the subset of the plane drawn in Figure 8 do you think are homeomorphic? (be aware that, at this point, we do not have enough tools to prove which two are not!).*

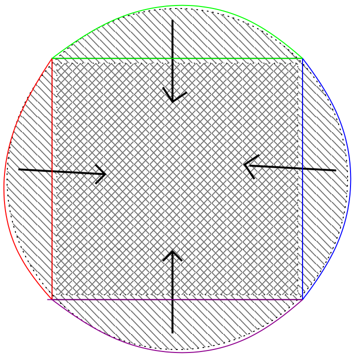
3. The sphere and its higher dimensional versions

For each n, we have the <u>n-sphere</u>

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : (x_0)^2 + \ldots + (x_n)^2 = 1\} \subset \mathbb{R}^{n+1},$$

the <u>open $(n+1)$-disk</u>

$$\overset{\circ}{D}{}^{n+1} = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : (x_0)^2 + \ldots + (x_n)^2 < 1\} \subset \mathbb{R}^{n+1},$$



The unit disk and the square are homeomorphic

FIGURE 6.

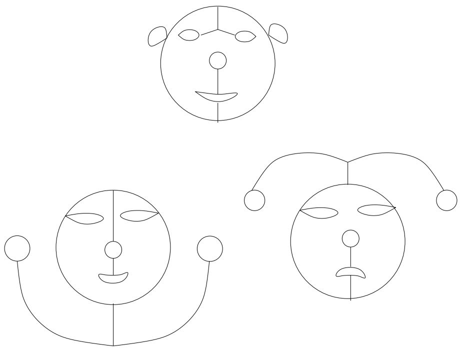


FIGURE 7.

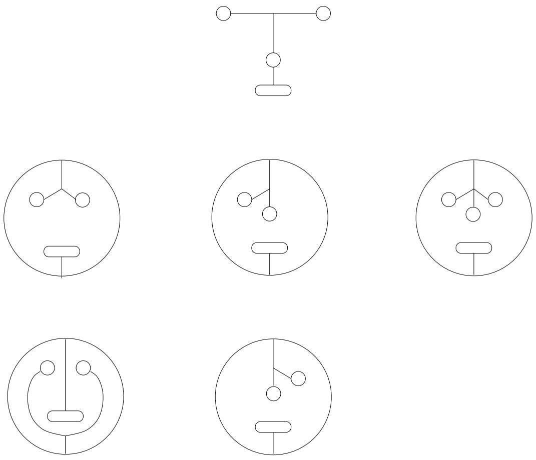


FIGURE 8.

and similarly the closed $(n+1)$-disk D^{n+1}.

The points

$$p_N = (0, \dots, 0, 1), p_S = (0, \dots, 0, -1) \in S^n$$

are usually called the north and the south pole, respectively, and

$$S^n_+ = \{(x_0, \ldots, x_n) : x_n \geq 0, S^n_- = \{(x_0 \ldots, x_n) : x_n \leq 0\}$$

are called the north and the south hemisphere, respectively. See figure 9.

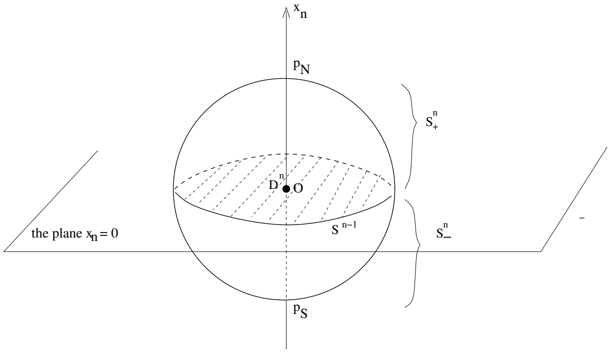


FIGURE 9.

EXERCISE 1.9. *Explain on a picture that* $S^n_+ \cap S^n_-$ *is homeomorphic to* S^{n-1}*, and* S^n_+ *and* S^n_- *are both homeomorphic to* D^n*.*

As with the circle, we will call a sphere (or an n-sphere) any space which is homeomorphic to S^2 (or S^n). Also, we call a disk (or n-disk, or open n-disk) any space which is homeomorphic to D^2 (or D^n, or $\mathring{D}^n$). For instance, the previous exercise shows that the two hemispheres S^n_+ and S^n_- are n-disks, and S^n is the union of two n-disks whose intersection is an $n-1$-sphere. This also indicates that S^n can actually be obtained by gluing two copies of D^n along their boundaries. And, still as for circles, there are many subspaces of Euclidean spaces which are spheres (or disks, or etc), but look quite different from the actual unit sphere (or unit disk, or etc). An important example has already been seen in Exercise 1.6 .

Another interpretation of S^n is as adding to $\mathbb{R}^n$ "a point at infinity". (to be made precise later on). This can be explained using the stereographic projection

$$f : S^n - \{p_N\} \longrightarrow \mathbb{R}^n$$

which associates to a point $p \in S^n$ the intersection of the line $p_N p$ with the horizontal hyperplane (see Figure 10).

EXERCISE 1.10. *Explain on the picture that the stereographic projection is a homeomorphism between* $S^n - \{p_N\}$ *and* $\mathbb{R}^n$*, and that it cannot be extended to a continuous function defined on the entire* S^n*. Then try to give a meaning to: "*S^n *can be obtained from* $\mathbb{R}^n$ *by adding a point at infinity to". Also, find the explicit formula for* f*.*

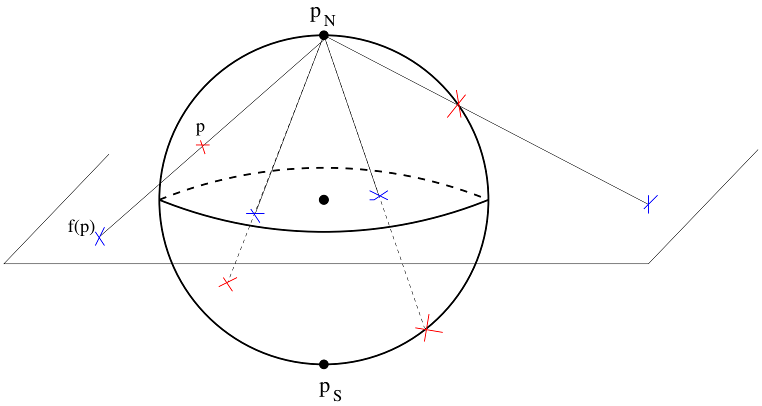
Here is anther construction of the n-sphere. Take a copy of D^n, grab its boundary $S^{n-1} \subset D^n$ and glue it together (so that it becomes a point). You then get S^n (see Figure 11).

EXERCISE 1.11. *Find explicitly the function*

$$f : D^2 \longrightarrow S^2$$

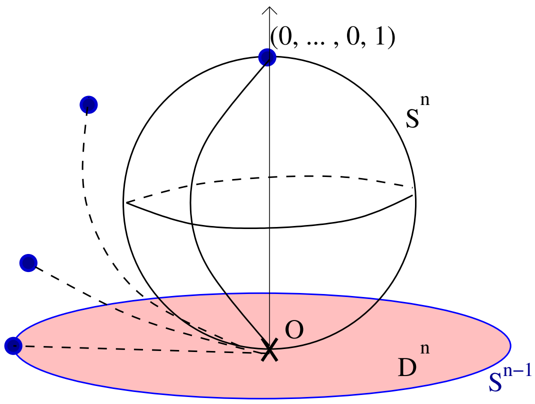
from Figure 11, check that $f^{-1}(p_N)$ *is precisely* $S^1 \subset D^2$*, then generalize to arbitrary dimensions.*

Another interesting way of obtaining the sphere S^2 is by taking the unit disk D^2, dividing its boundary circle S^1 into two equal sides and gluing the two half circles as indicated in the Figure 12.



The stereographic projection (sending the red points to the blue ones)

FIGURE 10.



glue the boundary of D n to a point to obtain S n

FIGURE 11.

A related construction of the sphere, which is quite important, is the following: take the disk D^2 and divide now it boundary circle into four equal sides, or take the unit square and its sides, and label them as in Figure 13. Glue now the two arcs denoted by a and the two arcs denoted by b.

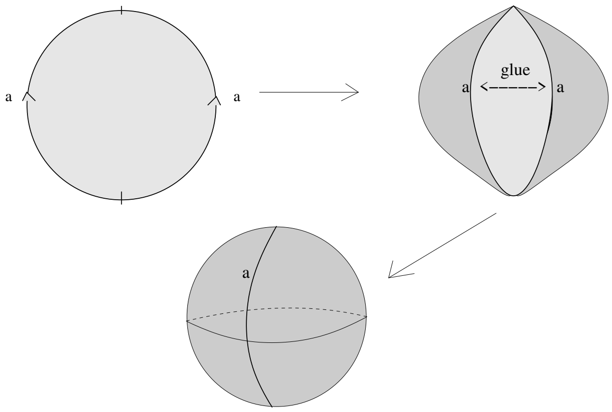
4. The Moebius band

"The Moebius band" is a standard name for subspaces of $\mathbb{R}^3$ which are obtained from the unit square $[0,1]\times[0,1]$ by "gluing" two opposite sides after twisting the square one time, as shown in Figure 14. As in the discussion about the unit circle (obtained from the unit interval by "gluing" its end points), this "gluing process" should be understood intuitively, and the precise meaning in topology will be explained later. The following exercise provides a possible parametrization of the Moebius band (inside $\mathbb{R}^3$).

EXERCISE 1.12. *Consider*

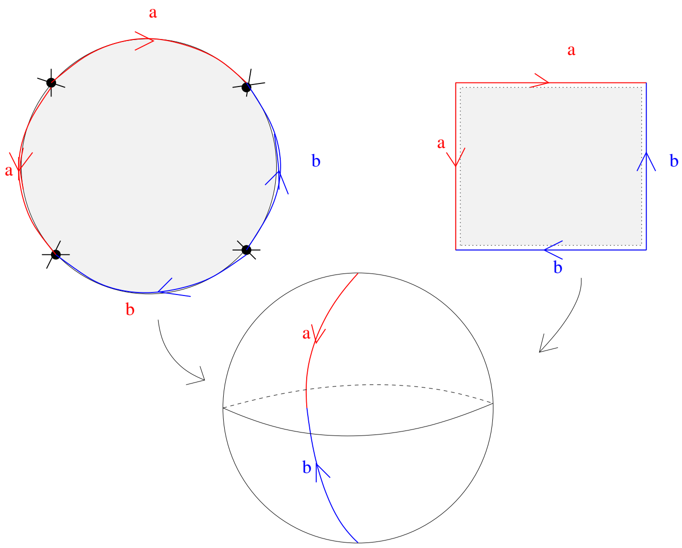
$$f : [0,1] \times [0,1] \longrightarrow \mathbb{R}^3,$$

$$f(t,s) = ((2+(2s-1)sin(\pi t))cos(2\pi t), (2+(2s-1)sin(\pi t))sin(2\pi t), (2s-1)cos(\pi t)).$$



The sphere obtained from the disk by glueing the two half circles denoted by "a" on the picture

FIGURE 12.



The sphere obtained from a disk or a square glueing as indicated in the pictu

FIGURE 13.

You may want to check that $f(t, s) = f(t', s')$ holds only in the following cases:

(1) $(t, s) = (t', s')$.
(2) $t = 0$, $t' = 1$ *and* $s' = 1 - s$.
(3) $t = 1$, $t' = 0$ *and* $s' = 1 - s$.

(but this also follows from the discussion below). Based on this, explain why the image of f can be considered as the result of gluing the opposite sides of a square with the reverse orientation.

To understand where these formulas come from, and to describe explicit models of the Moebius band in $\mathbb{R}^3$, we can imagine the Moebius band as obtained by starting with a segment in $\mathbb{R}^3$ and rotating it around its middle point, while its middle point is being rotated on a circle. See Figure 15.

The rotations take place at the same time (uniformly), and while the segment rotates by $180°$, the middle point makes a full rotation ($360°$). To write down explicit formulas, assume that

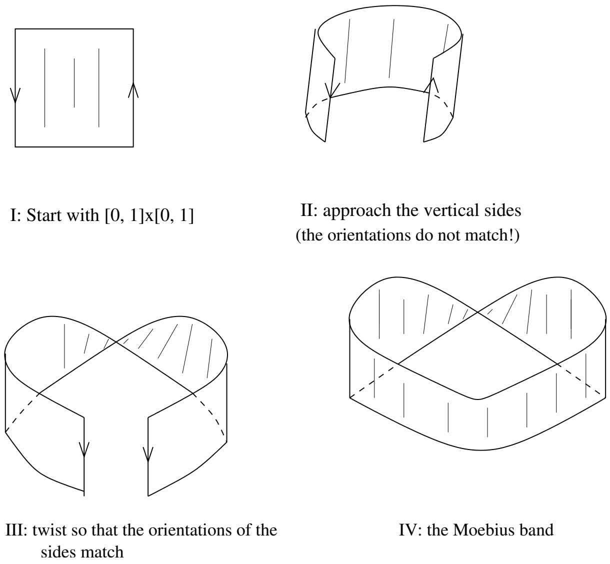


FIGURE 14.

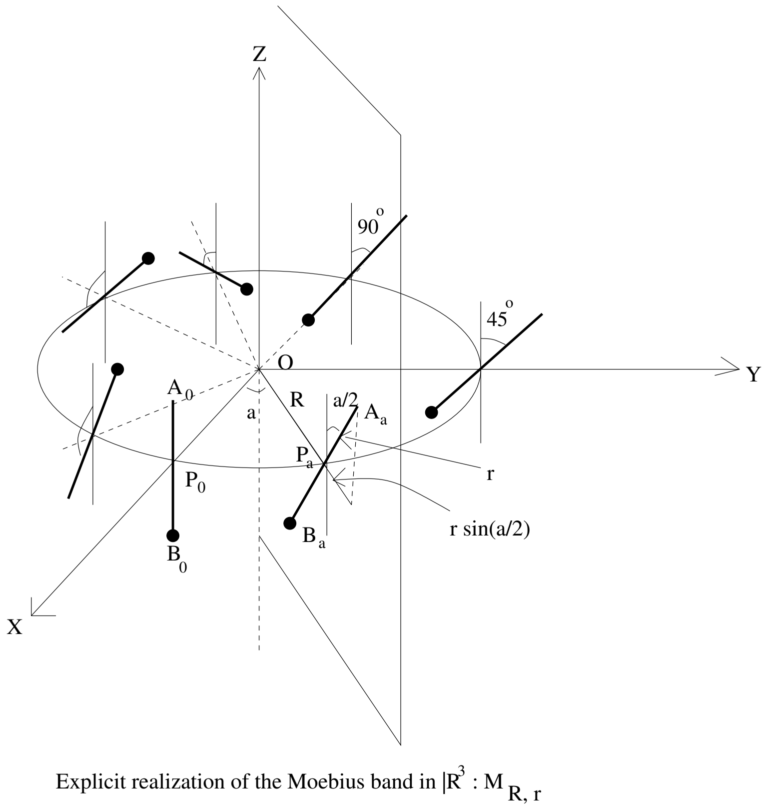


FIGURE 15.

- the circle is situated in the XOY plane, is centered at the origin, and has radius R.
- the length of the segment is $2r$ and the starting position A_0B_0 of the segment is perpendicular on XOY with middle point $P_0 = (R, 0, 0)$.

- at any moment, the segment stays in the plane through the origin and its middle point, which is perpendicular on the XOY plane.

We denote by $M_{R,r}$ the resulting subspace of $\mathbb{R}^3$ (note that we need to impose the condition $R > r$). To parametrize $M_{R,r}$, we parametrize the movement by the angle a which determines the middle point on the circle:

$$P_a = (Rcos(a), Rsin(a), 0).$$

At this point, the precise position of the segment, denoted A_aB_a, is determined by the angle that it makes with the perpendicular on the plane XOY through P_a; call it b. This angle depends on a. Due to the assumptions (namely that while a goes from 0 to 2π, b only goes from 0 to π, and that the rotations are uniform), we have $b = a/2$ (see 15). We deduce

$$A_a = \{(R + rsin(a/2))cos(a), (R + rsin(a/2))sin(a), rcos(a/2)),$$

and a similar formula for B_a (obtained by replacing r by $-r$). Then, the Moebius band $M_{R,r}$ is:

$$M_{R,r} = \{(R+tsin(a/2))cos(a), (R+tsin(a/2))sin(a), tcos(a/2)) : a \in [0, 2\pi], t \in [-r, r]\} \tag{4.1}$$

Note that, although this depends on R and r, different choices of R and r produce homeomorphic spaces. To fix one example, one usually takes $R = 2$ and $r = 1$.

EXERCISE 1.13. *Do the following:*

(1) *Make a model of the Moebius band and cut it through the middle circle. You get a new connected object. Do you think it is a new Moebius band? Then cut it again through the middle circle and see what you get.*
(2) *Prove (without using a paper model) that if you cut the Moebius band through the middle circle, you obtain a (space homeomorphic to a) cylinder. What happens if you cut it again?*

5. The torus

"The torus" is a standard name for subspaces of $\mathbb{R}^3$ which look like a doughnut- drawn in Figure 16.

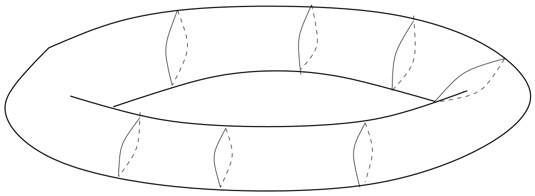


FIGURE 16.

EXERCISE 1.14. *Show that the surface of a cup with a handle is homeomorphic to the torus (what about a cup with no handle?).*

The simplest construction of the torus is by a gluing process: one starts with the unit square and then one glues each pair of opposite sides, as shown in Figure 17.

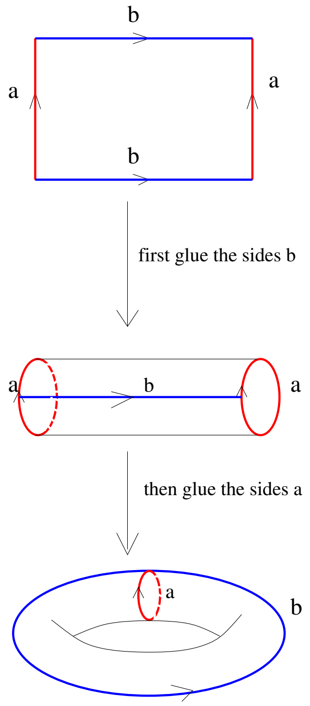


FIGURE 17.

As in the case of circles, spheres, disks, etc, by a torus we mean any space which is homeomorphic to the doughnut. Let's find explicit models (in $\mathbb{R}^3$) for the torus. To achieve that, we will build it by placing our hand in the origin in the space, and use it to rotate a rope which at the other end has attached a non-flexible circle. The surface that the rotating circle describes is clearly a torus (see Figure 18).

To describe the resulting space explicitly, we assume that the rope rotates inside the XOY plane (i.e. the circle rotates around the OZ axis). Also, we assume that the initial position of

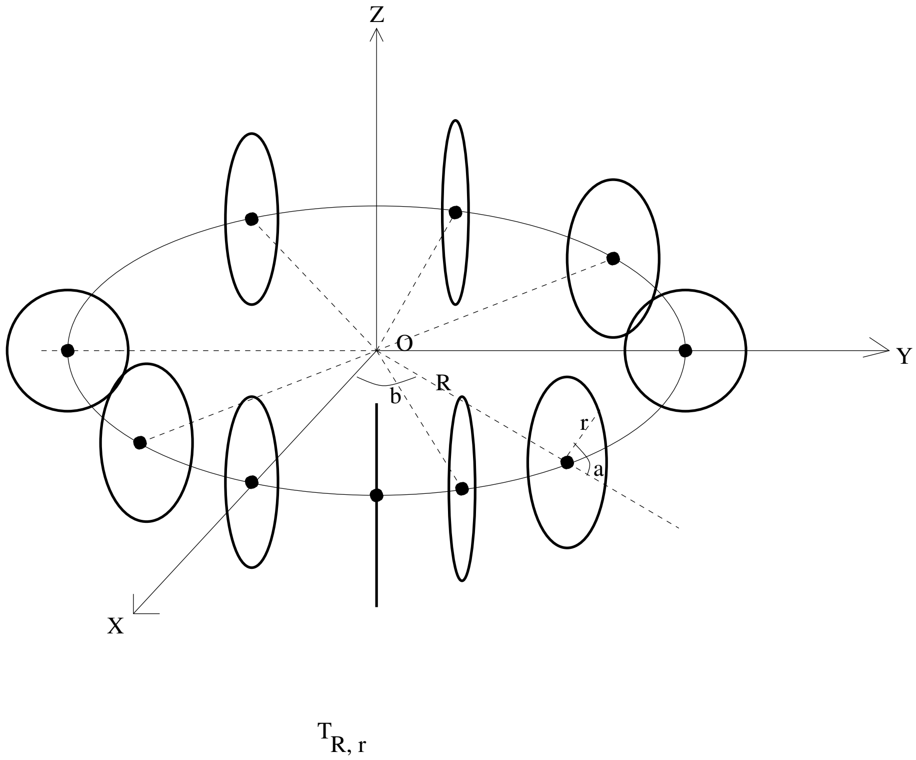


FIGURE 18.

the circle is in the XOZ plane, with center of coordinates $(R, 0, 0)$, and let r be the radius of the circle ($R > r$ because the length of the rope is $R - r$). We denote by $T^2_{R,r}$ the resulting subspace of $\mathbb{R}^3$. A point on $T^2_{R,r}$ is uniquely determined by the angles a and b indicated on the picture (Figure 18), and we find the parametric description:

$$T^2_{R,r} = \{(R + rcos(a))cos(b), (R + rcos(a))sin(b), rsin(a)) : a, b \in [0, 2\pi]\} \subset \mathbb{R}^3. \tag{5.1}$$

EXERCISE 1.15. *Show that*

$$T^2_{R,r} = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\}. \tag{5.2}$$

Although $T^2_{R,r}$ depends on R and r, different choices of R and r produce homeomorphic spaces.

There is yet another interesting interpretation of the torus, as the Cartesian product of two circles:

$$S^1 \times S^1 = \{(z, z') : z, z' \in S^1\} \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4.$$

Note that, priory, this product is in the 4-dimensional space. The torus can be viewed as a homeomorphic copy inside $\mathbb{R}^3$.

EXERCISE 1.16. *Show that*

$$f : S^1 \times S^1 \longrightarrow T^2_{R,r},\ \ f(e^{ia}, e^{ib}) = (R + rcos(a))cos(b), (R + rcos(a))sin(b), rsin(a))$$

is a bijection. Explain the map also on the picture, and convince yourself that it is a homeomorphism.

Proving directly that f is a homeomorphism is not really pleasant, , but the simplest way of proving that it is actually a homeomorphism will require the notion of compactness (and we will come back to this at the appropriate time).

Related to the torus is the double torus, pictured in Figure 19. Similarly, for each $g \geq 1$ integer, one can talk about the torus with g-holes.

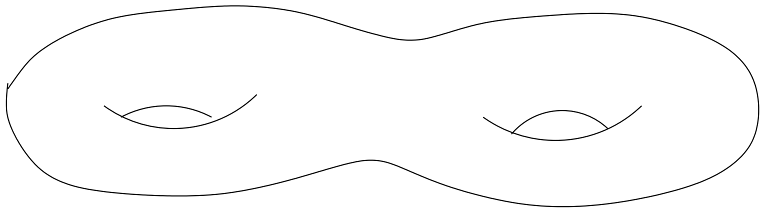


FIGURE 19.

EXERCISE 1.17. *Show that the surface of a cup with two handles is homeomorphic to the double torus.*

Note that the double torus can be realized from two disjoint copies of the torus, by removing a small ball from each one of them, and then gluing them along the resulting circles- see Figure 20.

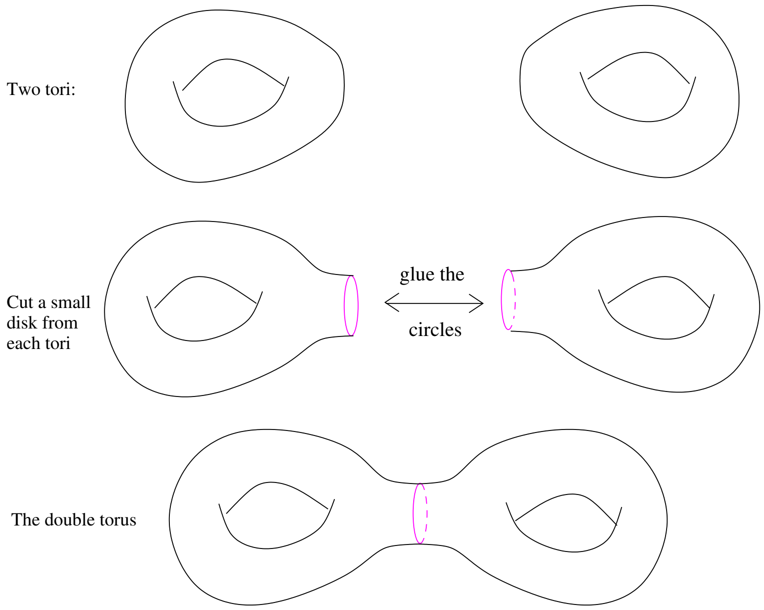


FIGURE 20.

EXERCISE 1.18. *How should one glue the sides of a Pentagon so that the result is a cut torus? (Hint: see Figure 21 and try to understand it. Try to make a paper model).*

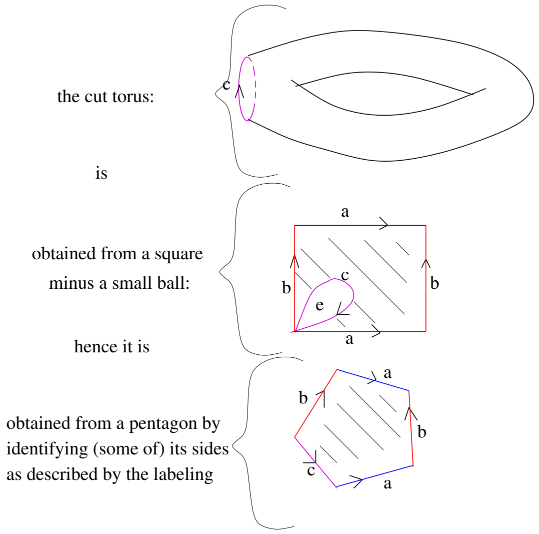


FIGURE 21.

EXERCISE 1.19. *Show that the double torus can be obtained from an octagon by gluing some of its sides.*
(Hint: see Figure 22 and try to understand it. Try to make a paper model).

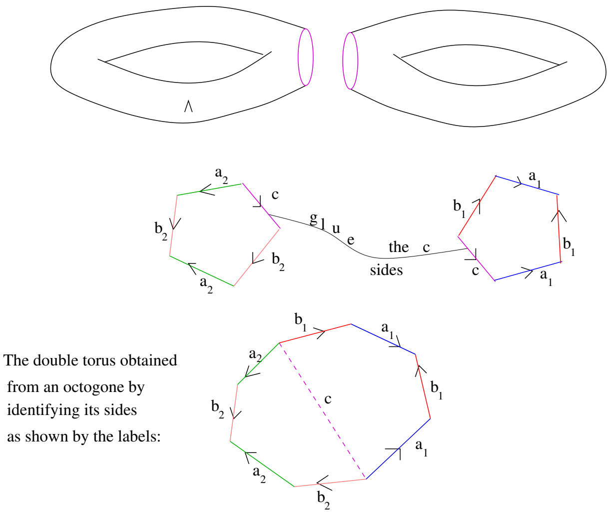


FIGURE 22.

6. The Klein bottle

We have seen that, starting from an unit square and gluing some of its sides, we can produce the sphere (Figure 13), the Moebius band (Figure 14) or the torus (Figure 17). What if we try to glue the sides differently? The next in this list of example would be the space obtained by gluing the opposite sides of the square but reversing the orientation for one of them, as indicated in Figure 23. The resulting space is called the Klein bottle, denoted here by K. Trying to repeat what we have done in the previous examples, we have trouble when "twisting the cylinder". Is that really a problem? It is now worth having a look back at what we have already seen:

- starting with an interval and gluing its end points, although the interval sits on the real line, we did not require the gluing to be performed without leaving the real line (it would not have been possible). Instead, we used one extra-dimension to have more freedom and we obtained the circle.
- similarly, when we constructed the Moebius band or the torus, although we started in the plane with a square, we did not require the gluing to take place inside the plane (it wouldn't even have been possible). Instead, we used an extra-dimension to have more freedom for "twisting", and the result was sitting in the space $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$.

Something very similar happens in the case of the Klein bottle: what seems to be a problem only indicates the fact that the gluing cannot be performed in $\mathbb{R}^3$; instead, it indicates that K cannot be pictured in $\mathbb{R}^3$, or, more precisely, that K cannot be embedded in $\mathbb{R}^3$. Instead, K can be embedded in $\mathbb{R}^4$. The following exercise is an indication of that.

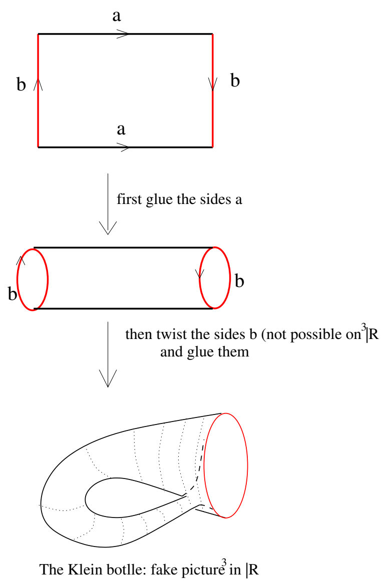


FIGURE 23.

EXERCISE 1.20. *Consider the map*

$$\tilde{f} : [0,1] \times [0,1] \longrightarrow \mathbb{R}^4,$$

$$\tilde{f}(t,s) = ((2 - cos(2\pi s)cos(2\pi t), (2 - cos(2\pi s)sin(2\pi t), sin(2\pi s)cos(\pi t), sin(2\pi s)sin(\pi t)).$$

Explain why the image of this map in $\mathbb{R}^4$ can be interpreted as the space obtained from the square by gluing its opposite sides as indicated in the picture (hence can serve as a model for the Klein bottle).

EXERCISE 1.21. *Explain how can one obtain a Klein bottle by starting with two Moebius bands and gluing them along their boundaries.*
(Hint: look at Figure 24).

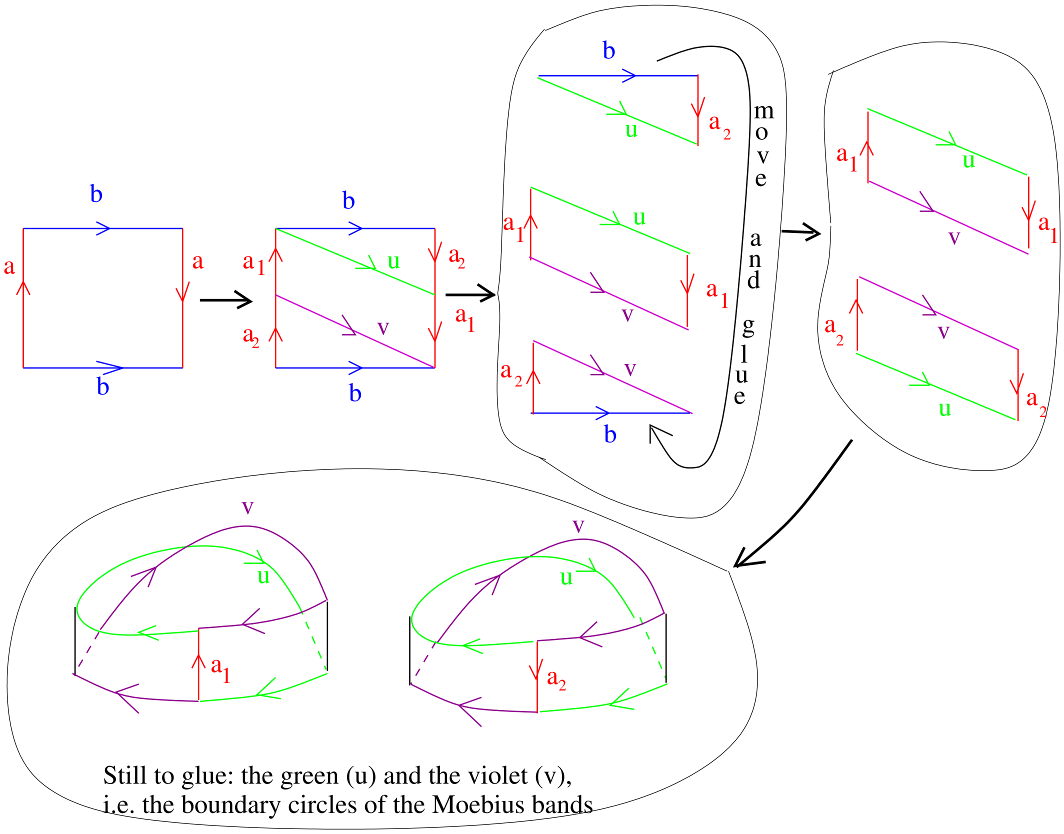


FIGURE 24.

7. The projective plane $\mathbb{P}^2$

In the same spirit as that of the Klein bottle, let's now try to glue the sides of the square as indicated on the left hand side of Figure 25.

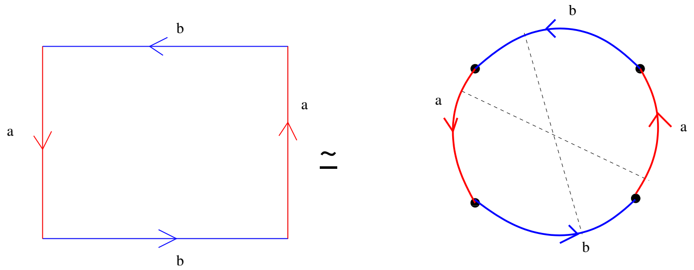


FIGURE 25.

Still as in the case of the Klein bottle, it is difficult to picture the result because it cannot be embedded in $\mathbb{R}^3$ (but it can be embedded in $\mathbb{R}^4$!). However, the result is very interesting: it can be interpreted as the set of all lines in $\mathbb{R}^3$ passing through the origin- denoted $\mathbb{P}^2$. Let's us adopt here the standard definition of the projective space:

$$\mathbb{P}^2 := \{l : l \subset \mathbb{R}^3 \text{ is a line through the origin}\},$$

(i.e. $l \subset \mathbb{R}^3$ is a one-dimensional vector subspace). Note that this is a "space" in the sense that there is a clear intuitive meaning for "two lines getting close to each other". We will explain how $\mathbb{P}^2$ can be interpreted as the result of the gluing that appears in Figure 25.

Step 1: First of all, there is a simple map:

$$f : S^2 \longrightarrow \mathbb{P}^2$$

which associates to a point on the sphere, the line through it and the origin. Since every line intersects the sphere exactly in two (antipodal) points, this map is surjective and has the special property:

$$f(z) = f(z') \Longleftrightarrow z = z' \text{ or } z = -z' \ (z \text{ and } z' \text{ are antipodal}).$$

In other words, $\mathbb{P}^2$ can be seen as the result of gluing the antipodal points of the sphere.

Step 2: In this gluing process, the lower hemisphere is glued over the upper one. We see that, the result of this gluing can also be seen as follows: start with the upper hemisphere S^2_+ and then glue the antipodal points which are on its boundary circle.

Step 3: Next, the upper hemisphere is homeomorphic to the horizontal unit disk (by the projection on the horizontal plane). Hence we could just start with the unit disk D^2 and glue the opposite points on its boundary circle.

Step 4: Finally, recall that the unit ball is homeomorphic to the square (by a homeomorphism that sends the unit circle to the contour of the square). We conclude that our space can be obtained by the gluing indicated in the initial picture (Figure 25).

Note that, since $\mathbb{P}^2$ can be interpreted as the result of gluing the antipodal points of S^2, the following exercise indicates why $\mathbb{P}^2$ can be seen inside $\mathbb{R}^4$:

EXERCISE 1.22. *Show that*

$$\tilde{f} : S^2 \longrightarrow \mathbb{R}^4 \ , \tilde{f}(x, y, z) = (x^2 - y^2, xy, xz, yz)$$

has the property that, for $p, p' \in S^2$, $f(p) = f(p')$ holds if and only if p and p' are either equal or antipodal.

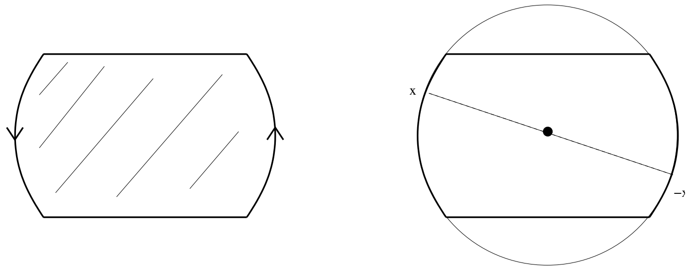
Note also that (a model of) the Moebius band can be seen as sitting inside (a model of) the projective plane $M \hookrightarrow \mathbb{P}^2$. To see this, recall that $\mathbb{P}^2$ can be seen as obtained from D^2 by gluing the antipodal points on its boundary. Consider inside D^2 the "band"

$$B = \{(x, y) \in D^2 : -\frac{1}{2} \leq y \leq \frac{1}{2}\}. \subset D^2.$$

The gluing process that produces $\mathbb{P}^2$ affects B in the following way: it glues the "opposite curved sides" of B as in the picture (Figure 26), and gives us the Moebius band.

Paying attention to what happens to $D^2 - B$ in the gluing process, you can now try the following.

EXERCISE 1.23. *Indicate how $\mathbb{P}^2$ can be obtained by starting from a Moebius band and a disk, and glue them along the boundary circle.*



The Moebius band inside the projective plane

FIGURE 26.

8. Gluing (or quotients)

We have already seen some examples of spaces obtained by gluing some of their points. When the gluing becomes less intuitive or more complicated, we start asking ourselves:

(1) But what gluing really means?
(2) But what is really the result of such a gluing?

Here we address these questions. In examples such the circle, torus or Moebius band, the answer was clear intuitively:

(1) Gluing had the intuitive meaning- done effectively by using paper models.
(2) the result was a new object, or rather a shape (it depends on how much we twist and pull the piece of paper).

Emphasize again that there was no preferred torus or Moebius band, but rather models for it (each two models being homeomorphic). Of course, one can fix a preference- the one that uses a homogeneous piece of paper, an uniformly distributed force of our hand, and a completely symmetric handling of the paper- but that becomes problematic as soon as our objects fail to be too symmetric. Moving to the Klein bottle, things started to become less intuitive, since the result of the gluing cannot be pictured in $\mathbb{R}^3$ (and things become probably even worse in the case of the projective plane). But, as we explained above, if we use an extra-dimension, the Klein bottle exist in $\mathbb{R}^4$- and Exercise 1.20 produces a subset of $\mathbb{R}^4$ which is an explicit *model* for it.

And things become much worse if we now start performing more complicated gluing of more complicated objects (one can even get "spaces" which cannot be "embedded" in any of the spaces $\mathbb{R}^n$!). It is then useful to a have a more conceptual (but abstract) understanding of what "gluing" and "the result of a gluing" means.

Start with a set X and assume that we want to glue some of its points. Which points we want to glue form the initial "gluing data", which can be regarded as a subset

$$R \subset X \times X$$

consisting of all pairs (x, y) with the property that we want x and y to be glued. This subset must have some special properties:

(1) If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.
(2) If $(x, y) \in R$ then also $(y, x) \in R$.
(3) $(x, x) \in R$ for all $x \in X$.

This is what is usually called *an equivalence relation* on the set X. Hence, as a gluing data we start with any equivalence relation on X.

What should the result of the gluing be? First of all, it is going to be a new set Y. Secondly, any point of X should give a point in Y, i.e. there should be a function $\pi : X \longrightarrow Y$ which should be surjective (the gluing should not introduce new points). Finally, π should really reflect the gluing, in the sense that $\pi(x) = \pi(y)$ should only happen when $(x, y) \in R$. Here is the formal definition.

DEFINITION 1.2. *Given an equivalence relation R on a set X, a quotient of X modulo R is a pair (Y, π) consisting of a set Y and a surjection $\pi : X \longrightarrow Y$ (called the quotient map) with the property that*

$$\pi(x) = \pi(y) \Longleftrightarrow (x, y) \in R.$$

Hence, a quotient of X modulo R can be viewed as a model for the set obtained from X by gluing its points according to R. Such models (quotients) are unique in the following sense.

EXERCISE 1.24. *Show that if (Y_1, π_1) and (Y_2, π_2) are two quotients of X modulo R, then there exists and is unique a bijection $f : Y_1 \longrightarrow Y_2$ such that $f \circ \pi_1 = \pi_2$.*

In full generality (for arbitrary X and R), there is an abstract construction of a such model (quotient). Namely, for each $x \in X$ we define the R-equivalence class of x as

$$R(x) := \{y \in X : (x, y) \in R\}$$

(a subset of X) and define

$$X/R = \{R(x) : x \in X\}$$

(a new set whose elements are subsets of X). There is a simple function

$$\pi_R : X \longrightarrow X/R, \quad \pi_R(x) = R(x),$$

called the canonical projection.

EXERCISE 1.25. *Show that:*

(1) *For any equivalence relation R on a set X, $(X/R, \pi_R)$ is a quotient of X modulo R.*
(2) *For any surjective map, $f : X \longrightarrow Y$, there is a unique relation R on X such that (Y, f) is a quotient of X modulo R.*

This discussion has been set-theoretical, so let's now go back to the case that X is a subset of some $\mathbb{R}^n$, and R is some equivalence relation on X. It is clear that, in such a "topological setting", we do not look for arbitrary models (quotients), but only for those which are in agreement with our intuition. In other words we are looking for "topological quotients" ("topological models"). What that really means will be made precise later on (since it requires the precise notion of topology). As a first attempt one could look for quotients (Y, π) of X modulo R with the property that

(1) Y is itself is a subspace of some $\mathbb{R}^k$.
(2) π is continuous.

These requirements pose two problems:

- Insisting that Y is a subspace of some $\mathbb{R}^k$ is too strong- see e.g. the exercise below. (Instead, Y will be just "topological space").
- The list of requirements is not complete. This can already be seen when $R = \{(x, x) : x \in X\}$ (i.e. when no gluing is required). Clearly, in this case, a "topological model" is X itself, and any other model should be homeomorphic to X. However, the requirements above only say that $\pi : X \longrightarrow Y$ is a continuous bijection which, as we have already seen, does not imply that π is a homeomorphism.

As we already said, the precise list of requirements will be made precise later on. (There is a good news however: if X is "compact", then any quotient (Y, π) of X modulo R which satisfies (1) and (2) above, is a good topological model!).

EXERCISE 1.26. *Let $X = S^1$, and we want to glue any two points $e^{ia}, e^{ib} \in S^1$ with the property that $b = a + 2\pi\sqrt{2}$. Show that there is no model (Y, π) with Y-a subset of some space $\mathbb{R}^n$ and $\pi : X \longrightarrow Y$ continuous.*

EXERCISE 1.27. *Let*

$$\mathbb{P}^n := \{l : l \subset \mathbb{R}^{n+1} \text{ is a line through the origin}\},$$

(i.e. $l \subset \mathbb{R}^{n+1}$ is a one-dimensional vector subspace). Explain how $\mathbb{P}^n$ (with the appropriate quotient maps) can be seen as:

(1) *a quotient of $\mathbb{R}^{n+1}$ modulo an equivalence relation that you have to specify (see second part of Exercise 1.25).*
(2) *obtained from S^n by gluing every pairs of its antipodal points.*
(3) *obtained from D^n by gluing every pair of antipodal points situated on the boundary sphere S^{n-1}.*

9. Back to metric spaces

As we have already indicated in Exercise 1.1, metric spaces do not encode convergence faithfully. The key of understanding the "topological content" of metrics (i.e. the one that allows us to talk about convergent sequences) is the notion of opens with respect to a metric. This is the first step toward the abstract notion of topological space.

DEFINITION 1.3. *Let (X, d) be a metric space. For $x \in X$, $\epsilon > 0$ one defines the open ball with center x and radius ϵ (with respect to d):*

$$B_d(x, \epsilon) = \{y \in X : d(y, x) < \epsilon\}.$$

A a set $U \subset X$ is called open with respect to d if

$$\forall \; x \in U \;\; \exists \;\; \epsilon > 0 \;\; \text{such that} \;\; B(x, \epsilon) \subset U. \tag{9.1}$$

We denote by $\mathcal{T}_d$ the collection of all subsets of X which are open with respect to d and we call it the topology induced by d.

With this we have:

EXERCISE 1.28. *Let d and d' be two metrics on the set X. Show that convergence in X with respect to d coincides with convergence in X with respect to d' if and only if $\mathcal{T}_d = \mathcal{T}_{d'}$.*

It is not a surprise now that the notion of convergence and continuity can be rephrased using only opens.

EXERCISE 1.29. *Let (X, d) be a metric space, $(x_n)_{n \geq 1}$ a sequence in X, $x \in X$. Then $(x_n)_{n \geq 1}$ converges to x (in (X, d)) if and only if: for any open $U \in \mathcal{T}_d$ containing x, there exists an integer n_U such that*

$$x_n \in U \quad \forall \quad n \geq n_U.$$

EXERCISE 1.30. *Let (X, d) and (Y, d') be two metric spaces, and $f : X \longrightarrow Y$ a function. Then f is continuous if and only if*

$$f^{-1}(U) \in \mathcal{T}_d \quad \forall \quad U \in \mathcal{T}_{d'}.$$