CHAPTER 2

Topological spaces

1. Topological spaces

We start with the abstract definition of topological spaces.

DEFINITION 2.1. *A topology on a set X is a collection $\mathcal{T}$ of subsets of X, satisfying the following axioms:*

(T1) *$\emptyset$ and X belong to $\mathcal{T}$.*
(T2) *The intersection of any two sets from $\mathcal{T}$ is again in $\mathcal{T}$.*
(T3) *The union of any collection of sets of $\mathcal{T}$ is again in $\mathcal{T}$.*

A topological space is a pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X; when there is no danger of confusion (e.g. when the topology $\mathcal{T}$ is clear from the context), we will simply say that X is a topological space without making reference to $\mathcal{T}$.

Given the topological space $(X, \mathcal{T})$, a subset $U \subset X$ is called open in X with respect to $\mathcal{T}$, or open in the topological space $(X, \mathcal{T})$ (or simply: open in X- when no confusion arises) if $U \in \mathcal{T}$. A subset $A \subset X$ is called closed in X with respect to $\mathcal{T}$ (or simply: closed in X- when no confusion arises) if its complement $X - A$ is open.

Given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on X, we say that $\mathcal{T}'$ is finer then $\mathcal{T}$, or that $\mathcal{T}$ is smaller then $\mathcal{T}'$, if $\mathcal{T} \subset \mathcal{T}'$.

EXAMPLE 2.2. *(Extreme topologies)* In general, on any set X we can define the following topologies:

(1) The trivial topology on X, denoted $\mathcal{T}_{\text{triv}}$, is the topology whose open sets are only $\emptyset$ and X.
(2) The discrete topology on X, denoted $\mathcal{T}_{\text{dis}}$, is the topology whose open sets are all the subsets of X.
(3) The co-finite topology on X, denoted $\mathcal{T}_{\text{cf}}$, is the topology whose open sets are the empty set and complements of finite subsets of X.
(4) The co-countable topology on X, denoted $\mathcal{T}_{\text{cc}}$, is the topology whose open sets are the empty set and complements of subsets of X which are at most countable.

EXERCISE 2.1. Show that, in a topological space $(X, \mathcal{T})$, any finite intersection of open sets is open: for each $k \geq 1$ integer, $U_1, \ldots, U_k \in \mathcal{T}$, one must have $U_1 \cap \ldots \cap U_k \in \mathcal{T}$. Is it true that arbitrary intersections of opens sets is open?

EXERCISE 2.2. How many distinct topologies can there be defined on a set with two elements? But with three? But with four?

EXERCISE 2.3. Consider the set $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ (the set of strictly positive integers to which we add the infinity) and, for each $n \in \mathbb{N}$ put

$$U_n := \{k \in \overline{\mathbb{N}} : k \geq n\} = \{n, n+1, \ldots\} \cup \{\infty\}.$$

Show that

$$\mathcal{T}_{\text{seq}} := \{\emptyset, U_1, U_2, U_3, \ldots\}$$

is a topology on $\overline{\mathbb{N}}$.

PROPOSITION 2.3. *For any metric space (X, d), the family $\mathcal{T}_d$ of opens in X with respect to d is a topology on X. Moreover, this is the smallest topology on X with the property that it contains all the balls*

$$B_d(x; r) = \{y \in X : d(x, y) < r\}$$

for all $x \in X$, $r > 0$.

PROOF. Axiom (T1) is immediate. To prove (T2), let $U, V \in \mathcal{T}_{\mathrm{d}}$ and we want to prove that $U \cap V \in \mathcal{T}_{\mathrm{d}}$. We have to show that, for any point $x \in U \cap V$, there exists $r > 0$ such that $B_d(x, r) \subset U \cap V$. So, let $x \in U \cap V$. That means that $x \in U$ and $x \in V$. Since $U, V \in \mathcal{T}_{\mathrm{d}}$, there exists $r_1 > 0$ and $r_2 > 0$ such that

$$B_d(x, r_1) \subset U, B_d(x, r_2) \subset V.$$

Taking $r = \min\{r_1, r_2\}$, $B_d(x, r)$ is inside both U and V, i.e. $B_d(x, r) \subset U \cap V$, hence r has the desired properties.

To prove axiom (T3), let $\{U_i : i \in I\}$ be a family of elements $U_i \in \mathcal{T}_{\mathrm{d}}$ (indexed by a set I) and we want to prove that $U := \cup_{i \in I} U_i \in \mathcal{T}_{\mathrm{d}}$. We have to show that, for any point $x \in U$, there exists $r > 0$ such that $B_d(x, r) \subset U$. So, let $x \in U$. That means that there exists $i \in I$ such that $x \in U_i$. Since $U_i \in \mathcal{T}_{\mathrm{d}}$, we find $r > 0$ such that $B_d(x, r) \subset U_i$. Since $U_i \subset U$, it follows that r has the desired property $B_d(x, r) \subset U$.

For the last part, assume that $\mathcal{T}$ is a topology on X which contains all the balls and we prove that $\mathcal{T}_d \subset \mathcal{T}$. Let $U \in \mathcal{T}_d$, and we want to prove that $U \in T$. From the definition of $\mathcal{T}_d$, for each $x \in U$ we find $r_x > 0$ such that

$$\{x\} \subset B(x; r_x) \subset U.$$

Taking the union over all $x \in U$ we deduce that

$$U \subset \cup_x B(x; r_x) \subset U.$$

Hence the two inclusions become equalities. Since all the balls are in $\mathcal{T}$ is a topology, axiom (T3) implies that U must belong to $\mathcal{T}$. $\square$

DEFINITION 2.4. *Given a topological space $(X, \mathcal{T})$ we say that it is metrizable if there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$.*

REMARK 2.5. One of the important problems in topology, which stimulated many important developments, is the following question

which topological spaces are metrizable?

More exactly, one would like to understand which are the special topological properties that a topology must have so that it is induced by a metric. Such properties will be discussed throughout the first half of the course (Hausdorffness, first countability, etc).

The first example of "space" which we have seen in the previous chapter was the Euclidean space. There we have used the intuitive notion of convergence, which was based on the Euclidean metric on $\mathbb{R}^n$. Hence, according to the previous proposition, we did look at a topological space-where the topology is the one induced by the metric. To avoid confusion in the terminology, we adopt the following convention.

DEFINITION 2.6. *The topology induced by the Euclidean metric on $\mathbb{R}^n$ is called the Euclidean topology on $\mathbb{R}^n$, denoted $\mathcal{T}_{eucl}$. From now on, unless we specify it otherwise, $\mathbb{R}^n$ will be viewed as a topological spaces endowed with the Euclidean topology $\mathcal{T}_{eucl}$.*

Note that, in particular, the previous proposition shows that the Euclidean topology on $\mathbb{R}$ is the smallest topology on $\mathbb{R}$ which contains all the open intervals (a, b). There are many other topologies on $\mathbb{R}$.

EXERCISE 2.4. Show the following:

(1) Given a set X and a family $\mathcal{S}$ of subsets of X, there exists a topology $\mathcal{T}(\mathcal{S})$ on X which contains $\mathcal{S}$ and is the smallest with this property.
(2) On $\mathbb{R}$ define topology $\mathcal{T}_l$ as the smallest topology which contains all the intervals of type $[a, b)$ with $a, b \in \mathbb{R}$. Then $\mathcal{T}_l$ is finer then $\mathcal{T}_{\text{eucl}}$.
(3) Show that a subset $D \subset \mathbb{R}$ belongs to $\mathcal{T}_l$ if and only if the following condition holds: for any $x \in D$ there exists an interval $[a, b)$ such that

$$x \in [a, b) \subset \mathbb{D}.$$

EXERCISE 2.5. On $\mathbb{R}$ define topology $\mathcal{T}_r$ as the smallest topology which contains all the intervals of type $(a, b]$ with $a, b \in \mathbb{R}$. Show that:

(1) $\mathcal{T}_r$ is finer then $\mathcal{T}_{\text{eucl}}$, but $\mathcal{T}_l$ and $\mathcal{T}_r$ are not comparable.
(2) $\mathcal{T}_{\text{dis}}$ is the only topology on $\mathbb{R}$ which contains both $\mathcal{T}_l$ and $\mathcal{T}_r$.

EXERCISE 2.6. On $\mathbb{R}^2$ we define the topology $\mathcal{T}_l \times \mathcal{T}_l$ as the smallest topology which contains all subsets of type

$$[a, b) \times [c, d)$$

with $a, b, c, d \in \mathbb{R}$. Define similarly $\mathcal{T}_r \times \mathcal{T}_r$, $\mathcal{T}_l \times \mathcal{T}_r$ and $\mathcal{T}_r \times \mathcal{T}_l$. Show that any two of these four topologies are not comparable.

2. Continuous functions; homeomorphisms

Inspired by metric spaces, we can now talk about continuous functions between arbitrary topological spaces,

DEFINITION 2.7. *Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, and a function $f : X \longrightarrow Y$, we say that f is continuous (with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$) of:*

$$f^{-1}(U) \in \mathcal{T}_X \quad \forall \quad U \in \mathcal{T}_Y.$$

EXAMPLE 2.8. Some extreme examples first:

(1) If Y is given the trivial topology then, for any other topological space $(X, \mathcal{T}_X)$, any function $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_{\text{triv}})$ is automatically continuous.
(2) If X is given the discrete topology then, for any other topological space $(Y, \mathcal{T}_Y)$, any function $f : (X, \mathcal{T}_{\text{dis}}) \longrightarrow (Y, \mathcal{T}_Y)$ is automatically continuous.
(3) The composition of two continuous functions is continuous: If $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ and $g : (Y, \mathcal{T}_Y) \longrightarrow (Z, \mathcal{T}_Z)$ are continuous, then so is $g \circ f : (X, \mathcal{T}_X) \longrightarrow (Z, \mathcal{T}_Z)$. Indeed, for any $U \in \mathcal{T}_Z$, $V := g^{-1}(U) \in \mathcal{T}_Z$, hence

$$(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U)) = g^{-1}(V)$$

must be in $\mathcal{T}_X$.
(4) For any topological space $(X, \mathcal{T})$, the identity map $\mathrm{Id}_X : (X, \mathcal{T}) \longrightarrow (X, \mathcal{T})$ is continuous. More generally, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, then the identity map $\mathrm{Id}_X : (X, \mathcal{T}_1) \longrightarrow (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_2$ is smaller then $\mathcal{T}_1$.

EXAMPLE 2.9. Given $f : X \longrightarrow Y$ a map between two metric spaces (X, d) and (Y, d'), Exercise 1.30 says that $f : X \longrightarrow Y$ is continuous as a map between metric spaces (in the sense discussed in the previous chapter) if and only if $f : (X, \mathcal{T}_d) \longrightarrow (Y, \mathcal{T}_{d'})$ is continuous as a map between topological spaces. In particular. all functions between Euclidean spaces which are built out of the standard functions (polynomial, or using exp, log, sin, cos, etc) are automatically continuous).

EXERCISE 2.7. Let $\mathbb{R}$ be endoweed with the topology $\mathcal{T}_l$ from Exercise 2.4. Which one of the following functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous:

(i) $f(x) = x + 1$
(ii) $f(x) = -x$.
(iii) $f(x) = x^2$.

EXERCISE 2.8. Consider

$$D := \{(x, y) : e^x > sin(y)cos(x)\},$$

$$A := \{(x, y) : x^7 - sin(y^7) \geq \frac{1}{x^2 + y^2 + 1}\}.$$

Show that D is open in $\mathbb{R}^2$, while A is closed in $\mathbb{R}^2$.

EXERCISE 2.9. Show that a map $f : X \longrightarrow Y$ between two topological spaces (X with some topology $\mathcal{T}_X$, and Y with some topology $\mathcal{T}_Y$) is continuous if and only if $f^{-1}(A)$ is closed in X for any closed subset A of Y.

EXERCISE 2.10. *(gluing continuous functions)* Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces and let $\{A_i : i \in I\}$ be a collection of subsets of X which cover X (i.e. $X = \cup_i A_i$). Assume that

$$f_i : A_i \longrightarrow Y$$

are continuous functions (where each A_i is endowed with the topology induced from X), and assume that

$$f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$$

for all i and j. Consider the function $f : X \longrightarrow \mathbb{R}$ defined by $f|_{A_i} = f_i$. Then f is continuous in each of the following cases:

(1) If all A_i are open.
(2) If all A_i are closed and I is finite.

DEFINITION 2.10. *Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a homeomorphism between them is a bijective function $f : X \longrightarrow Y$ with the property that both f and f^{-1} are continuous.*

We say that two topological spaces are homeomorphic if there exists a homeomorphism beteween them.

REMARK 2.11. We would like to emphasize that the notion of "homeomorphism" is the correct notion "isomorphism in the topological world". A homeomorphism $f : X \longrightarrow Y$ allows us to be able to move from X to Y and backwards any topological argument (i.e. any argument which is based on the notion of opens) and without loosing any topological information. For this reason, in topology, homeomorphic spaces are not viewed as being different from each other.

REMARK 2.12. In the definition of the notion of homeomorphism (and as we have seen already in the previous chapter), it is not enough to require that $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous and bijective (it may happen that f^{-1} is not continuous). For f^{-1} to be continuous, one would need that for each $U \subset X$ open, $f(U) \subset Y$ is open. Functions with this property (that send opens to opens) are called open maps.

For instance, the function

$$f : [0, 2\pi) \longrightarrow S^1, f(t) = (cos(t), sin(t)),$$

that we also discussed in the previous chapter, is continuous and bijective, but it is not a homeomorphism. More precisely, it is not open: $[0, \pi)$ is open in X, while $f([0, 2\pi))$ is a half circle closed at one end and open at the other- hence not open (see Figure 2 in the previous Chapter).

EXERCISE 2.11. From the topologies that you found in Exercise 2.2, how many non homeomorphic ones are there?

REMARK 2.13. Actually, many notions/constructions in topology are motivated by the question:

how do we decide if two spaces are homeomorphic or not?

One possible way to attack the problem is to understand the possible topological properties of a space, or to construct topological invariants. Roughly speaking, a topological property (or topological invarian) is a property (or number, or structure) that can be characterized using open sets only. The precise defintion actually makes use of the notion of homeomorphism (see below).

Given a topological property P, if X satisfies P and Y does not (or if the topological invariant associated to X is different from the one associated to Y), we can conclude that X and Y are not homeomorphic. All the topological properties we discuss in this course (the countability axioms, Hausdorffness, connectedness, compactness, etc) could be motivated by this problem. For instance, try to prove now that $(0, 2)$ and $(0, 1) \cup (1, 2)$ are not homeomorphic (if you manage, you have probably discovered the notion of connectedness). Then try to prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Next, try to prove that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic (this is much more difficult). Or try something a bit different: $(0, 1)$ and $[0, 1]$ are not homeomorphic. Later on we will see how to prove such results.

DEFINITION 2.14. *We call topological property any property $\mathcal{P}$ of topological spaces (that a space may or may not have) such that, if X and Y are homeomorphic, then X has the property $\mathcal{P}$ if and only if Y has it.*

We call topological numerical invariant any correspondence which associates to a topological space X a number $n(X)$ such that $n(X) = n(Y)$ whenever X and Y are homeomorphic.

For instance, the property of being metrizable (see Definition 2.4) is a topological property:

EXERCISE 2.12. Let X and Y be two homeoomorphic topological spaces. Show that X is metrizable if and only if Y is.

As we have already mentioned, one of the interesting problems in topology is to understand which are the special topological properties that a topology must satisy so that it is induced by a metric. The most basic property is that of Hausdorffness.

DEFINITION 2.15. *We say that a topological space $(X, \mathcal{T})$ is Hausdorff if for any $x, y \in X$ with $x \neq y$, there are opens $V \in \mathcal{T}$ containing x and $W \in \mathcal{T}$ containing y, such that $V \cap W = \emptyset$.*

EXERCISE 2.13. Show that Hausdorffness is a topological property.

As promised, one has:

PROPOSITION 2.16. *Any metric space is Hausdorff.*

PROOF. Given (X, d), $x, y \in X$ distinct, we must have $r := d(x, y) > 0$, We then choose $V = B(x; \frac{r}{2})$, $W = B(y; \frac{r}{2})$. We claim these are disjoint. If not, we find z in their intersection, i.e. $z \in Z$ such that $d(x, z)$ and $d(y, z)$ are both less then $\frac{r}{2}$. From the triangle inequality for d we deduce that

$$r = d(x, y) < d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r,$$

which is a contradiction. □

EXAMPLE 2.17. Looking at the extreme topologies: $\mathcal{T}_{\text{triv}}$ is not Hausdorff (unless X is empty or consists of one point only), while $\mathcal{T}_{\text{dis}}$ is Hausdorff.

In the light of the Hausdorffness property, the cofinal topology $\mathcal{T}_{cf}$ becomes more interesting:

EXERCISE 2.14. Given a set X, show that any Hausdorff topology on X contains $\mathcal{T}_{cf}$. When is $\mathcal{T}_{cf}$ Hausdorff? When does it exist a smallest Hausdorff topology on X (i.e. a Hausdorff topology which is contained in all other Hausdorff topologies on X)?

Coming back to the notion of homeomorphism, we point out the following exercise whose first part is extremely useful for some of the later exercises.

Exercise 2.15 (the "removing a point trick"). *If X and Y are homeomorphic prove that, for any $x \in X$ there exists $y \in Y$ such that $X - \{x\}$ is homeomorphic to $Y - \{y\}$.*
Also explain that, "there exists $y \in Y$" cannot be replaced with "for any $y \in Y$".

3. Neighborhoods and convergent sequences

Inspired by metric spaces, we can also talk about convergence in an arbitrary topological space.

Definition 2.18. *Given a topological space $(X, \mathcal{T})$, and a point $x \in X$, an open neighborhood of x is any open $U \in \mathcal{T}$ which contains x. We denote by $\mathcal{T}(x)$ the set of all open neighborhoods of x:*

$$\mathcal{T}(x) := \{u \in \mathcal{T} : X \in u\}.$$

A neighborhood of x (in the topological space $(X, \mathcal{T})$ is any subset $V \subset X$ with the property that there exists an open U such that

$$x \in U \subset V.$$

We denote by $\mathcal{V}(x)$ the set of all neighborhoods of x.
Given a sequence $(x_n)_{n\geq 1}$ of elements of X, and $x \in X$, we say that $(x_n)_{n\geq 1}$ converges to x, and we write $\lim_{n\to\infty} x_n = x$ (in the topological space $(X, \mathcal{T})$)) if,

$$\forall\ V \in \mathcal{V}(x),\ \exists\ n_V \in \mathbb{N} : x_n \in V\ \forall\ n \geq n_V. \tag{3.1}$$

Remark 2.19. What are neighborhoods good for? We have already used them in order to talk about convergence. More generally, they are the " topological pieces" which are relevant when looking at properties which are "local", in the sense that they depend only on what happens "near points". For instance, the notion of continuity is a local property: a function $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous if and only if it is continuous at each point, in the sense that for any $x \in X$ and any $V \in \mathcal{V}(f(x))$, $f^{-1}(V)$ is in $\mathcal{V}(x)$. (Prove this!)

Example 2.20. Given a metric space (X, d), Exercise 1.29 says that a sequence $(x_n)_{n\geq 1}$ converges to x in the sense of the previous chapter if and only if it converges in the topological space $(X, \mathcal{T}_d)$.

Example 2.21. If X is given the trivial topology then any sequence $(x_n)_{n\geq 1}$ of points in X converges to any $x \in X$. If X is given the discrete topology then a sequence $(x_n)_{n\geq 1}$ converges to a $x \in X$ if and only if $(x_n)_{n\geq 1}$ is stationary equal to x, i.e. there exists n_0 such that $x_n = x$ for all $n \geq n_0$.

Exercise 2.16. Let $X = (0, 1)$ endowed with the Euclidean topology. Is the sequence

$$x_n = \frac{1}{n}$$

convergent in the topological space X?

Exercise 2.17. Let $\mathbb{R}$ be endoweed with the topology $\mathcal{T}_l$ from Exercise 2.4. Study the convergence of the sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ where

$$x_n = \frac{1}{n},\ y_n = -\frac{1}{n}.$$

Exercise 2.18. What about convergence in $(X, \mathcal{T}_{\mathrm{cf}})$. But about $(X, \mathcal{T}_{\mathrm{cc}})$?

EXERCISE 2.19. In this exercise we use the notations from Exercise 2.3. Let $(X, \mathcal{T})$ be a topological space. Given a sequence $(x_n)_{n\geq 1}$ in X, and $x \in X$, we re-interpret this data as a function

$$f : \overline{\mathbb{N}} \longrightarrow X$$

which sends $n \in \mathbb{N}$ to x_n and ∞ to x. Show that $(x_n)_{n\geq 1}$ converges to x if and only if f is continuous.

Next, we discuss the relationship between sequences and the notions of continuity and Hausdorffness.

THEOREM 2.22. *Let $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ be a map between two topological spaces. If f is continuous then it sends convergent sequences to convergent sequences (i.e. if for any sequence $(x_n)_{n\geq 1}$ convergent to x in $(X, \mathcal{T}_X)$, the sequence $(f(x_n))_{n\geq 1}$ converges to $f(x)$ in $(Y, \mathcal{T}_Y)$).*

PROOF. Assume that f is continuous and $(x_n)_{n\geq 1}$ converges to x. To show that $(f(x_n))_{n\geq 1}$ converges to $f(x)$, we start with $V \in \mathcal{V}(f(x))$ arbitrary and we have to find n_V such that $f(x_n) \in V$ for all $n \geq n_V$. Since f is continuous, we must have $f^{-1}(V) \in \mathcal{V}(x)$ (cf. remark 2.19) and since $(x_n)_{n\geq 1}$ converges to x we find n_V such that $x_n \in f^{-1}(V)$ for all $n \geq n_V$. Clearly, this n_V has the desired properties. □

THEOREM 2.23. *Let (X, T) be a topological space. If X is Hausdorff, then every sequence $(x_n)_{n\geq 1}$ has at most one limit in X.*

PROOF. Assume that X is Hausdorff. Assume that there exists a sequence $(x_n)_{n\geq 1}$ in X converging both to $x \in X$ and $y \in X$, with $x \neq y$; the aim is to reach a contradition. Choose $V \in \mathcal{V}(x)$ and $W \in \mathcal{V}(y)$ such that $V \cap W = \emptyset$. Then we find n_V and n_W such that $x_n \in V$ for all $n \geq n_V$, and similarly for W. Choosing $n > \max\{n_V, n_W\}$, this will contradict the fact that V and W are disjoint. □

DEFINITION 2.24. *Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A basis of neighborhoods of x (in the topological space $(X, \mathcal{T})$) is a collection $\mathcal{B}_x$ of neighborhoods of x with the property that*

$$\forall\ V \in \mathcal{V}(x)\ \exists\ B \in \mathcal{B}_x :\ B \subset V.$$

EXAMPLE 2.25. If (X, d) is a metric space, and $x \in X$, then the family of all balls centered at x,

$$\mathcal{B}_d(x) := \{B(x; r) : r > 0\}, \tag{3.2}$$

is a basis of neighborhoods of x.

REMARK 2.26. What are bases of neighborhoods good for? They are collections of neighborhoods which are "rich enough" to characterize local properties. I.e. instead of proving/checking conditions for all neighborhoods of x, it is enough to do it only for the elements of a basis (of neighborhoods). Here is an illustration of this principle in checking convergence: if $\mathcal{B}_x$ is a basis of neighborhoods of x in the topological space $(X, \mathcal{T})$, the condition (3.1) for the convergence of x_n to x is equivalent to

$$\forall\ V \in \mathcal{B}(x),\ \exists\ n_V \in \mathbb{N} : x_n \in V\ \forall\ n \geq n_V.$$

(why!). Similarly, the continuity at each point x (cf. Remark 2.19) can be checked using only neighborhoods in a given basis at $f(x)$.

Note that, in the case of metric spaces, this principle says that it is enough to use the balls. Using the basis (3.2) above, we see that this condition of convergence of a sequence (x_n) in a metric space (X, d) is equivalent to:

$$\forall\ \epsilon > 0,\ \exists\ n_\epsilon \in \mathbb{N} : d(x_n, x) < \epsilon\ \forall\ n \geq n_\epsilon.$$

4. Inside a topological space: closure, interior and boundary

DEFINITION 2.27. *Let $(X, \mathcal{T})$ be a topological space. Given $A \subset X$, define:*

- *the interior of A:*

$$\overset{\circ}{A} = \bigcup_{U-\text{open contained in } A} U.$$

(The union is over all the subsets of U of A which are open in $(X, \mathcal{T})$). It is sometimes denoted by Int(A). Note that $\overset{\circ}{A}$ is open, is contained in A, and its is the largest set with these properties.

- *the closure of A:*

$$\overline{A} = \bigcap_{F-\text{closed containing } A} F.$$

(The intersection is over all the subsets A of X which contain A and are closed with in $(X, \mathcal{T})$). It is sometimes denoted by Cl(A). Note that $\overline{A}$ is closed, contains A, and its is the smallest set with these properties.

- *the boundary of A:*

$$\partial(A) = \overline{A} - \overset{\circ}{A}.$$

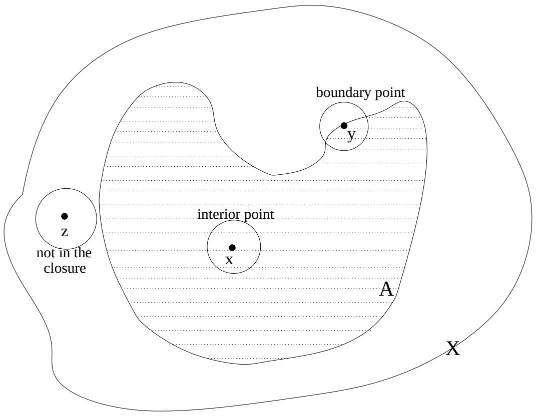

FIGURE 1.

LEMMA 2.28. *Let $(X, \mathcal{T})$ be a topological space, $x \in X$, and assume that $\mathcal{B}_x$ is a basis of neighborhoods around x (e.g. $\mathcal{B}_x = \mathcal{T}(x)$). Then:*

(i) *$x \in \overset{\circ}{A}$ if and only if there exists $U \in \mathcal{B}_x$ such that $U \subset A$.*
(ii) *$x \in \overline{A}$ if and only if, for all $U \in \mathcal{B}_x$, $U \cap A \neq \emptyset$.*
(iii) *If $(X, \mathcal{T})$ is metrizable, then $x \in \overline{A}$ if and only if there exists a sequence $(a_n)_{n \geq 1}$ of elements of A such that $a_n \to a$.*

See Figure 1.

PROOF. (of the lemma) You should first convince yourself that (i) is easy; we prove here (ii) and (iii). To prove the equivalence in (ii), is sufficient to prove the equivalence of the negations, i.e.

$$[x \notin \overline{A}] \Longleftrightarrow [\exists\, U \in \mathcal{B}_x : U \cap A = \emptyset].$$

From the definition of $\overline{A}$, the left hand side is equivalent to:

$$\exists\ F - \text{closed} : A \subset F, x \notin F.$$

Since closed sets are of those of type $F = X - U$ with U-open, this is equivalent to

$$\exists\ U - \text{open} : A \cap U = \emptyset, x \in U,$$

i.e.: there exists $U \in \mathcal{T}(x)$ such that $U \cap A = \emptyset$. On the other hand, any $U \in \mathcal{T}(x)$ contains at least one $B \in \mathcal{B}_x$, and the condition $U \cap A = \emptyset$ will not be destroyed if we replace U by B. This concludes the proof of (ii). For (iii), first assume that $x = lima_n$ for some sequence of elements of A. Then, for any $U \in \mathcal{T}(x)$, we find n_U such that $a_n \in U$ for all $n \geq n_U$, which shows that $U \cap A \neq \emptyset$, By (ii), $x \in \overline{A}$. For the converse, one uses that fact that $B(x, \frac{1}{n}) \cap A \neq \emptyset$ hence, for each n, we find $a_n \in A$ with $d(a_n, x) < \frac{1}{n}$. Clearly $a_n \to a$. $\square$

Example 2.29. Take the "open disk" in the plane

$$A = \overset{\circ}{D}{}^2\ (= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}).$$

Then the interior of A is A itself (it is open!), the closure is the "closed disk"

$$A = D^2 (= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}),$$

while the boundary is the unit circle

$$\partial(A) = S^1 (= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}).$$

See Figure 2. The same is true for balls in any metric space (check this!).

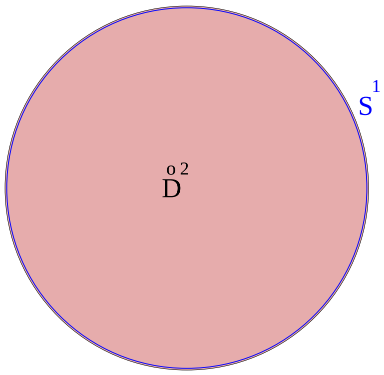

Figure 2.

Example 2.30. Take $A = [0,1) \cup \{2\} \cup [3,4)$ in $X = \mathbb{R}$. Using the lemma (and the basis given by open intervals) we find

$$\overset{\circ}{A} = (0,1) \cup (3,4),\ \ \overline{A} = [0,1] \cup \{2\} \cup [3,4],\ \ \partial(A) = \{0,1,2,3,4\}.$$

However, considering A inside $X' = [0,4)$ (with the topology induced from $\mathbb{R}$),

$$\overset{\circ}{A} = [0,1) \cup (3,4),\ \ \overline{A} = [0,1] \cup \{2\} \cup [3,4), \partial(A) = \{1,2,3\}.$$

Exercise 2.20. Show that the following three spaces are homeomorphic (giving the explicit homeomorphisms):

$$X = \{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$$
$$Y = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2} < x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$$

$$Z = S^1 \times (0,1] = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 < z \leq 1\} \subset \mathbb{R}^3.$$

Then compute the interiors and the boundaries of X, Y (in $\mathbb{R}^2$) and Z (in $\mathbb{R}^3$). How comes that, although X, Y and Z are homeomorphic, their interiors and boundaries are quite different?

EXERCISE 2.21. Show that

(1) $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e. $\overline{\mathbb{Q}} = \mathbb{R}$.
(2) If two continuous functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy $f(q) = g(q)$ for all $q \in \mathbb{Q}$, then $f = g$.

Then generalize the second part.

EXERCISE 2.22. Exhibit an embedding $f : M \longrightarrow M$ of the Moebius band into itself which is not surjective. What is the boundary of $f(M)$ in M?

EXERCISE 2.23. Describe (on the picture) a subset C of the torus T such that

(1) There is a continuous bijection $f : \mathbb{R} \longrightarrow C$
(2) C is dense in T (i.e. $\overline{C} = T$).

EXERCISE 2.24. Find the interior and the closure of $\mathbb{Q}$ in $\mathbb{R}$ in each of the cases: when $\mathbb{R}$ is endowed with

- the Euclidean topology.
- the discrete topology.
- the cofinite topology.
- the co-countable topology.

EXERCISE 2.25. Let $\mathcal{T}_l$ be the topology from Exercise 2.4. Find the closure and the interior in $(\mathbb{R}, \mathcal{T}_l)$ of each of the intervals

$$[0,1), (0,1], (0,1), [0,1].$$

EXERCISE 2.26 (part of Exam A, 2008). Let $\mathcal{T}_l$ be the topology from Exercise 2.4.

(i) In the topological space $(\mathbb{R}, \mathcal{T})$, find the closure, the interior and the boundary of

$$A = (0,1) \cup [2,3]. \qquad \textit{(1.5p)}$$

(ii) Show that $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T}_{\text{eucl}})$ are not homeomorphic.*(0.5p)*

EXERCISE 2.27 (from Exam A, 2009). Compute the interior, the closure and the boundary of

$$A = (0,1] \times [0,1)$$

in the topological space $X = \mathbb{R}^2$ endowed with the topology $\mathcal{T}_l \times \mathcal{T}_l$ (see Exercise 2.6).

EXERCISE 2.28. Let A, B and B_i denote subsets of a topological space X, where i runs in a set of indexes I. Prove that

(a) If $A \subset B$ then $\overline{A} \subset \overline{B}$.
(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(c) $\cup_i \overline{A_i} \subset \overline{\cup_i A_i}$; give an example where the equality fails. When does the equality hold?

EXERCISE 2.29 (Exam A, 2006). Show that, in any topological space X, for any subspace $A \subset X$, one has

$$\partial(A) = \partial(X - A).$$

For the case of metric spaces, let us point out the following corollary. To state it, recall that given a metric space (X, d), $A \subset X$, and $x \in X$, one defines the distance between x and A as

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

COROLLARY 2.31. *If A is a subset of a metric space (X, d), $x \in X$, then the following are equivalent:*

(1) $x \in \overline{A}$.
(2) *there exists a sequence (a_n) of elements of A such that $a_n \to x$.*
(3) $d(x, A) = 0$.

PROOF. The equivalence of (1) and (2) follows directly from (iii) of the lemma. Next, the condition (3) means that, for all $\epsilon > 0$, there exists $a \in A$ such that $d(x, a) < \epsilon$. In other words, $A \cap B(x, \epsilon) \neq \emptyset$ for all $\epsilon > 0$. Using (iii) of the lemma (with $\mathcal{B}_x$ being the collection of all balls centered at x), we find that (3) is equivalent to (1). □

PROPOSITION 2.32. *(closed sets as level sets) If (X, d) is a metric space and $A \subset X$, then the function*

$$d_A : X \longrightarrow \mathbb{R}, \quad d_A(x) = d(x, A)$$

is continuous.

In particular, any closed set of a metric space X can be realized as the level set $f^{-1}(0)$ of some continuous function $f : X \longrightarrow \mathbb{R}$.

PROOF. In general,

$$d(x, A) \leq d(x, y) + d(y, A)$$

for all x and y. Indeed, from the definition of $d(y, A)$, we find a sequence $(a_n)_{n \geq 1}$ of points of A such that $d(y, a_n) \to d(y, A)$. Then

$$d(x, A) \leq d(x, a_n) \leq d(x, y) + d(y, a_n)$$

for all n, hence making $n \to \infty$, we find the desired inequality. Next, remark that this inequality implies that d_A satisfies

$$|d_A(x) - d_A(y)| \leq d(x, y)$$

for all x and y. It is now clear that, if $d(x_n, x)$ converges to zero, then $d_A(x_n)$ converges to $d_A(x)$. □

EXERCISE 2.30 (based on an exercise from Exam A, 2006). Let (X, d) be a metric space and A, B closed subsets of X such that

$$d(A, B) = 0.$$

(Recall that $d(A, B) =: \inf\{d(a, b) : a \in A, b \in B\}$).

Is it true that A and B must have a common point (i.e. $A \cap B \neq \emptyset$)? What if we assume that both A and B are closed?

5. The first countability axiom

In the first chapter (where our discussion was less precise- based on the intuitive understanding of "space"), when talking about the continuity of a function what we had in mind was the property that "it sends convergent sequences to convergent sequences". With the precise notion of topological space, convergent sequence and continuity at hand now, we were able to prove that, indeed, continuous functions have that property. For the converse however, one needs an extra-hypothesis- the so called first countability axiom (which, since it is satisfied by all metric spaces, it is also satisfied by all the examples we have seen in the first chapter). The role of this axiom can be stated more generally: "it is under this axiom that the notion of sequence can be used in its full power".

DEFINITION 2.33. *We say that $(X, \mathcal{T})$ satisfies the first countability axiom, or that it is first-countable, if for each point $x \in X$ there exists a countable basis of neighborhoods of x.*

EXERCISE 2.31. Show that the first-countability is a topological property,

EXAMPLE 2.34. Any topology induced by a metric is first countable. More precisely, if (X, d) is a metric space, and $x \in X$, then the family

$$\mathcal{B}'_d(x) := \{B(x; \frac{1}{n}) : n \in \mathbb{N}\}$$

is a countable basis of neighborhoods of x.

EXERCISE 2.32. Let $(X, \mathcal{T})$ be a topological space and $x \in X$. Show that if x admits a countable basis of neighborhoods, then one can also find a decreasing one, i.e. one of type

$$\mathcal{B}_x = \{B_1, B_2, B_3, \ldots\}, \text{with } \ldots \subset B_3 \subset B_2 \subset B_1.$$

(Hint: $B_n = V_1 \cap V_2 \cap \ldots \cap V_n$).

THEOREM 2.35. *Let $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ be a map between two topological spaces, with X-first countable.*
Then f is continuous if and only if it sends convergent sequences to convergent sequences.

PROOF. The direct implication was already proven without any assumption on X. Let's now assume that X is first countable and prove the converse implication. We do it by contradiction: assume there f is sequentially continuous but not continuous. Since it is not continuous, we find $x \in X$ and $V \in \mathcal{V}(f(x))$ such that $f^{-1}(V)$ is not in $\mathcal{V}(x)$. Let $\{B_i : i \in \mathbb{N}\}$ be a countable and increasing basis of neighborhoods of x (see the previous exercise). Since $f^{-1}(V)$ is not a neighborhood of x, it does not contain any of the B_n's, hence for each n we find $x_n \in B_n - f^{-1}(V)$. Since $x_n \in B_n$, it follows that $(x_n)_{n \geq 1}$ (see Remark 2.26), hence by the hypothesis on f, $(f(x_n))_{n \geq 1}$ must converge to $f(x)$. Use now the V (neighborhood of $f(x)$) and find n_V such that $f(x_n) \in V$ for all $n \geq n_V$. This contradicts the construction of the sequence $(x_n)_{n \geq 1}$ (each x_n was not in $f^{-1}(V)$). □

THEOREM 2.36. *Let (X, T) be a first countable topological space.*
Then X is Hausdorff if and only if every sequence $(x_n)_{n \geq 1}$ has at most one limit in X.

PROOF. The direct implication was already proven without any assumption on X. Let's now assume that X is first countable and each sequence in X has at most one limit, and we prove that X is Hausdorff. Assume it is not. We find then $x \neq y$ two elements of X such that $V \cap W \neq \emptyset$ for all $V \in \mathcal{V}(x)$ and $W \in \mathcal{V}(y)$. Choose $\{V_n : n \geq 1\}$ and $\{W_n : n \geq 1\}$ bases of neighborhoods of x and y, which we may assume to be decreasing (cf. Exercise 2.32). For each n, we find an element $x_n \in V_n \cap W_n$. As in the previous proofs, this implies that x_n converges both to x and to y- which contradicts the hypothesis. □

Here is an exercise which is another good illustration of the fact that the first-countability condition is when "sequences contain all the information about the topology".

EXERCISE 2.33. We say that two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X have the same convergence of sequences if, for a sequence $(x_n)_{n \geq 1}$ of elements of X, and $x \in X$, one has:

$$x_n \xrightarrow{\mathcal{T}_1} x \Longleftrightarrow x_n \xrightarrow{\mathcal{T}_2} x.$$

Show that

- If $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the first countability axiom and have the same convergence of sequences, then $\mathcal{T}_1 = \mathcal{T}_2$.
- This is no longer true if one gives up the first countability axiom.

(Hint for (ii): try $\mathcal{T}_{\text{discr}}$ and $\mathcal{T}_{\text{cf}}$. If it does not work, try to change one of them).