CHAPTER 4

Topological properties

1. Connectedness

DEFINITION 4.1. *We say that a topological space $(X, \mathcal{T})$ is* <u>*connected*</u> *if X cannot be written as the union of two disjoint non-empty opens $U, V \subset X$.*

We say that a topological space $(X, \mathcal{T})$ is <u>*path connected*</u> *if for any two points $x, y \in X$, there exists a path γ connecting x and y, i.e. a continuous map $\gamma : [0, 1] \longrightarrow X$ such that $\gamma(0) = x$, $\gamma(1) = y$.*

Given $(X, \mathcal{T})$, we say that a subset $A \subset X$ is connected (or path connected) if A, together with the induced topology, is connected (path connected).

EXERCISE 4.1. For a topological space $(X, \mathcal{T})$, show that the following are equivalent:

(1) X is connected.
(2) $\emptyset$ and X are the only subsets of X which are both open and closed.
(3) the only continuous functions $f : X \longrightarrow \{0, 1\}$ are the constant ones (where $\{0, 1\}$ is endowed with the discrete topology).

EXAMPLE 4.2.

(1) $X = \{0, 1\}$ with the discrete topology is not connected. Indeed, $U = \{0\}$, $V = \{1\}$ are be two disjoint non-empty opens (in X) whose union is X.
(2) $X = [0, 1) \cup [2, 3]$ is not connected. Indeed, $U = [0, 1)$, $V = [2, 3]$ are two disjoint non-empty opens (in X) whose union is X. More generally, if $X \subset \mathbb{R}$ is connected, then X must be an interval (the converse is true as well-see below). Indeed, if X is not interval, we find $r, s \in X$ and $t \in (r, s)$ such that $t \notin X$. But then
$$U = (-\infty, t) \cap X, V = (t, \infty) \cap X$$
are opens in X, nonempty (U contains r while V contains s), disjoint, and whose union is X (because $t \notin X$).
(3) Proving that a space is connected by checking directly the definition is not always straightforward. We will see however (see the theorem below) that if we can prove that a space is path connected, then connectedness follows. The use of this comes from the fact that, in contrast with connectedness, proving that a space is path connected can be done more directly. For instance, any interval $I \subset \mathbb{R}$ is path connected. Indeed, for $x, y \in I$, since I is an interval, $[x, y] \subset I$, hence the straight linear path from x to y is contained in I; in other words,
$$\gamma : [0, 1] \longrightarrow X, \gamma(t) = (1 - t)x + ty$$
is a path connecting x and y.
(4) By the same argument as above, any convex subset $X \subset \mathbb{R}^n$ is path connected (recall that X-convex means that for any $x, y \in X$, the whole segment $[x, y]$ is contained in X). An example which is not of this type, but is still path connected, is $X = \mathbb{R}^2 - \{0\}$: if $x, y \in X$ and the segment $[x, y]$ doe not contain the origin, we use the linear path

from x to y. But even if $[x, y]$ contains the origin, we can join them by a path going around the origin (see Figure 1).

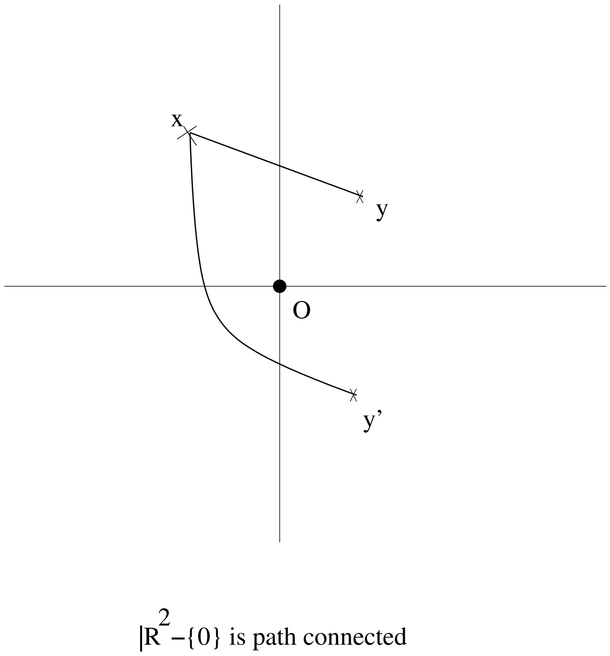

FIGURE 1.

Before proving that path connected spaces are connected we point out some simple properties of connected spaces.

PROPOSITION 4.3.
(i) *If $f : X \longrightarrow Y$ is a continuous map and X is connected, then $f(X)$ is connected.*
(ii) *Given $(X, \mathcal{T})$, if for any two points $x, y \in X$, there exists $\Gamma \subset X$ connected such that $x, y \in \Gamma$, then X is connected.*

PROOF. For (i), replacing Y by $f(X)$, we may assume that f is surjective, and we want to prove that Y is connected. If it is not, we find $U, V \subset Y$ disjoint nonempty opens whose union is Y. But then $f^{-1}(U), f^{-1}(V) \subset X$ are disjoint (since U and V are), nonempty (since U and V are and f is surjective) opens (because f is continuous) whose union is X- and this contradicts the connectedness of X. For (ii) we reason again by contradiction, and we assume that X is not connected, i.e. $X = U \cup V$ for some disjoint nonempty opens U and V. Since they are non-empty, we find $x \in U$, $y \in V$. By hypothesis, we find Γ connected such that $x, y \in \Gamma$. But then

$$U' = U \cap \Gamma, \ V' = V \cap \Gamma$$

are disjoint non-empty opens in Γ whose union is Γ- and this contradicts the connectedness of Γ. $\square$

THEOREM 4.4. *Any path connected space is connected.*

PROOF. It is enough to prove that the closed interval $[0, 1]$ is connected. Indeed, using (i) of the previous proposition, it follows that the image $\gamma([0, 1])$ of any path from x to y is a connected set containing x and y, and then we can use (ii) of the proposition to conclude connectedness. To prove that $[0,1]$ is connected, we assume the contrary, hence

$$[0,1] = U \cup V$$

with U, V-non-empty disjoint opens in $[0,1]$. Since $U = [0,1] - V$, U must be closed in $[0,1]$, hence

$$R := \text{sup}U \in U.$$

Since U is open this implies that $R = 1$ (if not, we find an interval $(R - \epsilon, R + \epsilon) \subset U$ and then $R + \frac{1}{2}\epsilon$ is an element in U strictly greater then its supremum-impossible). In conclusion, $1 \in U$. But exactly the same argument shows that $1 \in V$, and this contradicts the fact that $U \cap V = \emptyset$. □

The previous theorem and the remarks made in examples 2 and 3 in Example 4.2 imply the following:

Corollary 4.5. *The only connected subsets of* $\mathbb{R}$ *are the intervals.*

Combining with part (i) of Proposition 4.3 we deduce the following:

Corollary 4.6. *If* X *is a connected space and* $f : X \longrightarrow \mathbb{R}$ *is a continuous function, then* $f(X)$ *is an interval.*

Exercise 4.2. Show that the quotient of a connected spaces is connected. Similarly for path connectedness.

Example 4.7. There are a few more consequences that one can derive by playing with connectedness properties.

(1) $\mathbb{R}$ cannot be homeomorphic to S^1. If the two were homeomorphic, then so would be the spaces obtained by removing one point from each one of them. Indeed, if $f : \mathbb{R} \longrightarrow S^1$ is a homeomorphism, then the restriction of f to $\mathbb{R} - \{0\}$ defines a homeomorphism from $\mathbb{R} - \{0\}$ to $S^1 - \{p\}$, where $p = f(0)$ (the inverse is just the restriction of the inverse of f, hence continuous). But $\mathbb{S}^1 - \{f(0)\}$ is connected (it is clearly path connected) while $\mathbb{R} - \{0\}$ is not- hence they cannot be homeomorphic.
(2) $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}^2$. The argument is similar to the previous one (recall that $\mathbb{R}^2 - \{0\}$ is path connected, hence connected).
(3) The more general statement that $\mathbb{R}^n$ and $\mathbb{R}^m$ cannot be homeomorphic if $n \neq m$ is much more difficult to prove. One possible proof is a generalization of the argument given above (when $n = 1$, $m = 2$)- but that is based on "higher versions of connectedness", a notion which is at the core of algebraic topology.

Exercise 4.3. Prove that the following spaces are not homeomorphic:

(1) S^1 and $[0,1)$.
(2) $[0,1)$ and $\mathbb{R}$.
(3) S^1 and S^2.
(4) S^1 and a bouquet of two circles.

Exercise 4.4. In Exercise 1.8 you showed that four of the spaces in the picture are homeomorphic to each other, but they did not seem to be homeomorphic to the fifth one. You now have the tool to prove the last assertion (so do it!).

Exercise 4.5. Assuming that you know that $S^1 \times (0,1)$ is not homeomorphic to $\mathbb{R}^2$, show that the sphere S^2 is not homeomorphic to the space obtained from S^2 by gluing two antipodal points.

EXERCISE 4.6. Recall that by a circle we mean any topological space which is homeomorphic to S^1, and by a circle embedded in a topological space X we mean any subset $A \subset X$ which, when endowed with the induced topology, is a circle. Similarly, by a bouquet of two circle we mean any space which is homeomorphic to the space drawn in Figure 2, and we talk about embedded bouquets of circles. Let T be a a torus.

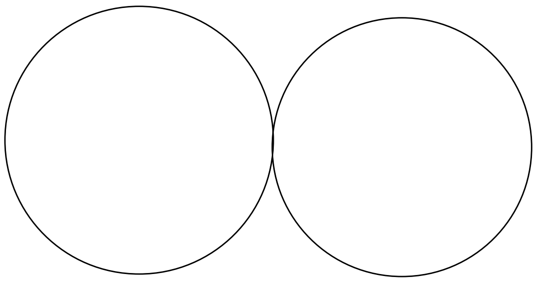

FIGURE 2.

(1) Describe a circle C embedded in T such that the complement of $T - C$ is connected.
(2) Describe a bouquet of two circle B embedded in T such that $T - B$ is connected.

EXERCISE 4.7. Show that the group $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices with real coeficients (see Exercise 3.3) is not connected.

EXERCISE 4.8. Prove that if $X \subset \mathbb{R}^n$ is connected the its closure is also connected but its interior may fail to be connected.

EXERCISE 4.9. Show that any two of the three spaces drawn in Figure 3 are not homeomorphic.

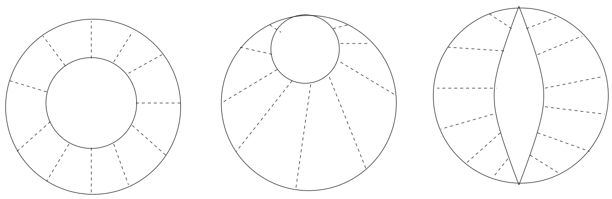

FIGURE 3.

DEFINITION 4.8. *Let $(X, \mathcal{T})$ be a topological space. A connected component of X is any maximal connected subset of X, i.e. any connected $C \subset X$ with the property that, if $C' \subset X$ is connected and contains C, then C' must coincide with C.*

PROPOSITION 4.9. *Let $(X, \mathcal{T})$ be a topological space. Then*

(i) *Any point $x \in X$ belongs to a connected component of X.*
(ii) *If C_1 and C_2 are connected components of X then either $C_1 = C_2$ or $C_1 \cap C_2 = \emptyset$.*
(iii) *Any connected component of X is closed in X.*

PROOF. To prove the proposition we will use the following

EXERCISE 4.10. If $A, B \subset X$ are connected and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

For (i) of the proposition, let $x \in X$ and define $C(x)$ as the union of all connected subsets of X containinig x. We claim that $C(x)$ is a connected component. The only thing that is not clear is the connectedness of $C(x)$. To prove that, we will use the criterion given by (ii) of Proposition 4.3. For $y, z \in C(x)$ we have to find $\Gamma \subset X$ connected such that $y, z \in \Gamma$. Due to the definition of $C(x)$, we find C_y and C_z- connected subsets of X, both containing x, such that $y \in C_y$, $z \in C_z$. The previous exercise implies that $\Gamma := C_y \cup C_z$ is connected containinig both y and z.

To prove (ii), we use again the previous exercise applied to $A = C_1$ and $B = C_2$, and the maximality property of connected components.

To prove (iii), due to the maximality of connected components, it suffices to prove that if $C \subset X$ is connected, then $\overline{C}$ is connected. Assume that $Y := \overline{C}$ is not connected. We find D_1, D_2 nonempty opens in Y, disjoint, such that $Y = D_1 \cup D_2$. Take $U_i = C \cap D_i$. Clearly, U_1 and U_2 are disjoint opens in C, with $C = U_1 \cup U_2$. To reach a contradiction (with the fact that C is connected) it suffices to show that U_1 and U_2 are nonempty. To show that U_i is non-empty ($i \in \{1, 2\}$), we use the fact that D_i is non-empty. We find a point x_i D_i. Since x_i is in the closure of C in Y, we have $U \cap C \neq \emptyset$ for each neighborhood U of x_i in Y. Choosing $U = D_i$, we have $C \cap D_i \neq \emptyset$. □

Exercise 4.11. Prove that if X and Y are homeomorphic, then they have the same number of connected components.

Remark 4.10. From the previous proposition we deduce that, for any topological space $(X, \mathcal{T})$, the family $\{C_i\}_{i \in I}$ of connected components of $(X, \mathcal{T})$ (where I is an index set) give a partition of X:

$$X = \cup_{i \in I} C_i, \quad C_i \cap C_j = \emptyset \ \forall\ i \neq j,$$

called the partition of X into connected components.

Exercise 4.12. Let X be a topological space and assume that $\{X_1, \ldots, X_n\}$ is a partition of X, i.e.

$$X = X_1 \cup \ldots \cup X_n, \quad X_i \cap X_j = \emptyset \quad \forall\ i \neq j.$$

Then the following are equivalent:

(i) All X_i's are closed.
(ii) All X_i's are open.

Moreover, if each X_i is connected, then $\{X_1, \ldots, X_n\}$ coincides with the partition of X into connected components.

Finally, one has the following partial converse of Theorem 4.4.

Theorem 4.11. [1] *Let $(X, \mathcal{T})$ be a topological space with the property that any point of X has a path connected neighborhood. Then X is path connected if and only if it is connected.*

Proof. We still have to prove that, if X is connected, then it is also path connected. We define on X the following relation $\sim$: $x \sim y$ if and only if x and y can be joined by a (continuous) path. This is an equivalence relation. Indeed, for $x, y, z \in X$:

- $x \sim x$: consider the constant path.
- if $x \sim y$ then $y \sim x$: if γ is a path from x to y, then $\gamma^-(t) = \gamma(1 - t)$ is a path from y to x.

[1] this has not been discussed in the lectures

- if $x \sim y$ and $y \sim z$, then $x \sim z$. Indeed, if γ_1 is a path from x to y, while γ_2 from y to z, then
$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } x \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$
is a path from x to z.

For $x \in X$, we denote by $C(x)$ the equivalence class of x:
$$C(x) = \{y \in X : y \sim x\}.$$
Note that each $C(x)$ is path connected. We claim that $C(x) = X$ for any $x \in X$. The fact that $\sim$ is an equivalence relation implies that
$$\{C(x) : x \in X\}$$
is a partition of X: if $C(x) \cap C(x') \neq \emptyset$, then $C(x) = C(x')$. What we want to prove is that this partition consists of one set only. Since X is connected, it is enough to prove that $C(x)$ is open for each $x \in X$. Fixing x, we want to prove that for any $y \in C(x)$, there is an open U such that $y \in U \subset C(x)$. To see this, we use the hypothesis and we choose any path connected neighborhood V of y. Since $y \sim x$ and $z \sim y$ fro any $z \in V$, we deduce that $z \sim x$ fro any $z \in V$, hence $V \subset C(x)$. Since V is a neighborhood of y, we find U open such that $y \in U \subset V$, and this U clearly has the desired properties. $\square$

2. Compactness

2.1. The definition of compactness. Probably many of you have seen the notion of compact space in the context of subsets of $\mathbb{R}^n$, as sets which are closed and bounded. Although not obviously at all, this is a topological property (it can be defined using open sets only).

Definition 4.12. *Given a topological space $(X, \mathcal{T})$ an <u>open cover of X</u> is a family $\mathcal{U} = \{U_i : i \in I\}$ (I-some index set) consisting of open sets $U_i \subset X$ such that*
$$X = \cup_{i \in I} U_i.$$
A <u>subcover</u> is any cover $\mathcal{V}$ with the property that $\mathcal{V} \subset \mathcal{U}$.

We say that a topological space $(X, \mathcal{T})$ is compact if <u>from any</u> open cover $\mathcal{U} = \{U_i : i \in I\}$ of X one can extract a finite open subcover, i.e. there exist $i_1, \ldots, i_k \in I$ such that
$$X = U_{i_1} \cup \ldots \cup U_{i_k}.$$

Remark 4.13. Given a topological space $(X, \mathcal{T})$ and $A \subset X$, the compactness of A (viewed as a topological space with the topology induced from X, cf. subsection 1) can be expressed using "open coverings of A in X", i.e. families $\mathcal{U} = \{U_i : i \in I\}$ (I-some index set) consisting of open sets $U_i \subset X$ such that
$$A \subset \cup_{i \in I} U_i.$$
A subcover is any cover (of A in X) $\mathcal{V}$ with the property that $\mathcal{V} \subset \mathcal{U}$. With these, A is compact if and only if from any open cover of A in X one can extract a finite open subcover. This follows immediately from the fact that the opens for the induced topology on A are of type $A \cap U$ with $U \in \mathcal{T}$ and from the fact that, for a family $\{U_i : i \in I\}$, we have
$$\bigcup_i (A \cap U_i) = A \Longleftrightarrow A \subset \bigcup_i U_i,$$

Exercise 4.13. Show that the set
$$\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$$
is compact in $\mathbb{R}$.

EXERCISE 4.14. Let X be a topological space and $A, B \subset X$. If A and B are compact, show that $A \cup B$ is compact.

EXAMPLE 4.14.

(1) $(X, \mathcal{T}_{\text{discr}})$ is compact if and only if X is finite (use the cover of X by the open-point opens).

EXERCISE 4.15. Given a set X, when is $(X, \mathcal{T}_{\text{cf}})$ compact?

(2) $\mathbb{R}$ is not compact. Indeed,
$$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (-n, n)$$
is an open cover from which we cannot extract a finite open subcover. Generalizing a bit this argument, we see that if a set $A \subset \mathbb{R}$ is compact, then A must be bounded (write $A \subset \cup_n (-n, n)$).

(3) $[0, 1)$ is not compact. Indeed,
$$[0,1) \subset \bigcup_n (-\infty, 1 - \frac{1}{n})$$
defines an open cover of $[0, 1)$ in $\mathbb{R}$, from which we cannot extract a finite open subcover. Again, this argument can be extended to show that if a set $A \subset \mathbb{R}$ is compact, then $\sup A$ belongs to A (and similarly $\inf A$). Indeed, if $\sup A$ was not in A, for each $b \in A$ we would find a bigger element in A, i.e. there would exist $a \in A$ such that $b \in (-\infty, a)$. Hence
$$A \subset \bigcup_{a \in A} (-\infty, a),$$
and this is an open cover of A in $\mathbb{R}$ which does not admit a finite open subcover. Indeed, if
$$(-\infty, a_1), \ldots, (-\infty, a_n)$$
was a finite subcover, taking $a = max\{a_1, \ldots, a_n\}$ we would have
$$A \subset \bigcup_{i=1}^{n} (-\infty, a_i) = (-\infty, a)$$
which is impossible because $a \in A$.

(4) $[0, 1]$ is compact.

PROOF. Assume the contrary, i.e. there exists a "bad cover of $[0, 1]$", i.e. an open cover $\mathcal{U}$ of $[0, 1]$ in $\mathbb{R}$ which does not admit any finite open subcover (of $[0, 1]$ in $\mathbb{R}$). Divide $[0, 1]$ into two intervals of equal length. The $\mathcal{U}$ will be a bad cover for at least one of the two intervals, call it I_1. Divide now I_1 into two intervals of equal length and, again, choose one of them, call it I_2, so that $\mathcal{U}$ is a bad cover of I_2. Continuing this process, we find intervals I_k with
$$I_{k+1} \subset I_k, \text{diam}(I_k) = \frac{1}{2^k},$$
(where the diameter of an interval $[a, b]$ is $(b - a)$) and such that $\mathcal{U}$ is a bad open cover of I_k in $\mathbb{R}$. Choosing $x_k \in I_k$, we have
$$d(x_k, x_{k+1}) \leq \frac{1}{2^k}$$
for all k, where d denotes the Euclidean metric ($d(a, b) = |a - b|$). We claim that the sequence of real numbers $(x_k)_{k \geq 1}$ is a Cauchy sequence, i.e. $d(x_k, x_{k+n})$ can be made

arbitrarily small for k big enough and all n. To see this, we use the triangle inequality and the previous inequality repeatedly:

$$d(x_k, x_{k+n}) \leq \sum_{i=k}^{k+n-1} d(x_i, x_{i+1}) \leq \sum_{i=k}^{k+n-1} \frac{1}{2^i} = \frac{1}{2^{k-1}}(1 - \frac{1}{2^n}),$$

and deduce

$$d(x_k, x_{k+n}) < \frac{1}{2^{k-1}}$$

for all $k, n \geq 0$ integers. This shows that $(x_k)_{k\geq 1}$ is a Cauchy sequence so, from the known properties of the real line, the sequence will be convergent. Let x be its limit. Since $x_k \in [0,1]$ for all k, we deduce that $x \in [0,1]$. Taking $n \to \infty$ in the previous inequality we also deduce that

$$d(x_k, x) \leq \frac{1}{2^{k-1}}.$$

We deduce that, for each k,

$$I_k \subset [x - \frac{3}{2^k}, x + \frac{3}{2^k}].$$

Indeed, if $y \in I_k$, since $x_k \in I_k$ and the diameter of I_k is $1/2^k$, we have $d(y, x_k) \leq 1/2^k$, hence

$$d(y, x) \leq d(y, x_k) + d(x_k, x) \leq \frac{1}{2^k} + \frac{1}{2^{k-1}} = \frac{3}{2^k}.$$

We now use the cover $\mathcal{U}$. Let $U \in \mathcal{U}$ such that $x \in U$. Since U is open, we find $r > 0$ such that

$$(x - r, x + r) \subset U.$$

But then, choosing k such that $\frac{3}{2^k} < r$, we will have $I_k \subset (x - r, x + r) \subset U$, i.e. the open cover $\mathcal{U}$ of I_k in $\mathbb{R}$ will have a finite subcover- namely the one consisting of the open set U alone. This contradicts the fact that $\mathcal{U}$ was a bad cover for I_k. □

EXERCISE 4.16. Look at the examples (2)-(3) above.

(1) We have seen that any compact $A \subset \mathbb{R}$ must be bounded. Prove the same for subsets of $\mathbb{R}^n$.
(2) We have seen that if $A \subset \mathbb{R}$ is compact, then sup$A$ and inf$A$ must belong to A. Use the same type of argument to prove that if $A \subset \mathbb{R}$ is compact, then A is closed.

The notion of open cover is relevant for various other topological constructions. For instance, we have:

EXERCISE 4.17. Let $(X, \mathcal{T})$ be a topological space with the property that the topology $\mathcal{T}$ is the topology induced by a topology basis $\mathcal{B}$ which is countable. Show that:

(i) From any open cover of X one can extract a subcover which is countable.
(ii) X is compact if and only if for any decreasing sequence of nonempty closed subsets

$$\ldots \subset F_3 \subset F_2 \subset F_1 \subset X,$$

one has $\cap_{n=1}^{\infty} F_n \neq \emptyset$.

2.2. Topological properties of compact spaces. In this section we point out two topological properties of compact spaces. On short, they say:

- closed inside compact is compact.
- compact inside Hausdorff is closed.

PROPOSITION 4.15. *If $(X, \mathcal{T})$ is a compact space, then any closed subset $A \subset X$ is itself compact.*

PROOF. If $\mathcal{U}$ is an open cover of A in X, then adding $X - A$ to $\mathcal{U}$ (which is open since A is closed), we get an open cover of X. Extracting a finite subcover (which may or may not contain $X - A$), denoting by $U_1, \ldots, U_n$ the elements of this finite subcover which are different from $X - A$, this will define a finite subcover of the original cover of A in X. $\square$

THEOREM 4.16. *In a Hausdorff space $(X, \mathcal{T})$, any compact set is closed.*

REMARK 4.17. These will imply that $A \subset \mathbb{R}$ is compact if and only if it is closed and bounded. For the direct implication, we have already remarked that A must be bounded (Example 4.14 (2)) while Theorem 4.16 implies that A must be closed. For the converse, since A is bounded, we find $m < M$ real numbers such that $A \subset [m, M]$. But $[m, M]$ is compact (it is homeomorphic to $[0, 1]$) and then we can use Proposition 4.15 to conclude that A is compact.
Note also that the same argument works for subsets of $\mathbb{R}^n$, provided we prove that $[0, 1]^n$ is compact. But this will follow from a general property of compact spaces (product of two compacts is compact) and will be discussed in the next section.

EXERCISE 4.18. Find a topological space $(X, \mathcal{T})$ which is compact but is not Hausdorff. In the example that you found, exhibit a compact set $A \subset X$ such that A is not closed.

Theorem 4.16 will follow easily from the following

PROPOSITION 4.18. *In a Hausdorff space $(X, \mathcal{T})$, any two disjoint compact sets A and B can be separated topologically, i.e. there exist opens $U, V \subset X$ such that*

$$A \subset U, \ B \subset V, \ U \cap V = \emptyset.$$

PROOF. (of Proposition 4.18): We introduce the following notation: given $Y, Z \subset X$ we write $Y|Z$ if Y and Z can be separated, i.e. if there exist opens U and V (in X) such that $Y \subset U$, $Z \subset V$ and $U \cap V = \emptyset$. We claim that, if Z is compact and $Y|\{z\}$ for all $z \in Z$, then $Y|Z$.

PROOF. (of the claim) We know that for each $z \in Z$ we find opens U_z and V_z such that $Y \subset U_z$, $z \in V_z$ and $U_z \cap V_z = \emptyset$. Note that $\{V_z : z \in Z\}$ is an open cover of Z in X: indeed, any $z \in Z$ belongs at least to one of the opens in the cover (namely V_z). By compactness of Z, we find a finite number of points $z_1, \ldots, z_n \in Z$ such that

$$Z \subset V_{z_1} \cup \ldots \cup V_{z_n}.$$

Denoting by V the last union, and considering

$$U = U_{z_1} \cup \ldots \cup U_{z_n},$$

V is an open containing Z, U is an open containing Y. Moreover, $U \cap V = \emptyset$: indeed, if x is in the intersection, since $x \in V$ we find k such that $x \in V_{z_k}$; but $x \in U$ hence $x \in U_z$, which is impossible since $U_{z_k} \cap V_{z_k} = \emptyset$. In conclusion, U and V show that Y and Z can be separated. $\square$

Back to the proof of the proposition, let $a \in A$ arbitrary. Now, since X is compact, we have $\{a\}|\{b\}$ for all $b \in B$. Since B is compact, the claim above implies that $\{a\}|B$, or, equivalently, $B|\{a\}$. This holds for all $a \in A$, hence using again the claim (and the fact that A is compact) we deduce that $A|B$. $\square$

PROOF. (of Theorem 4.16) For Theorem 4.16, assume that $A \subset X$ is compact, and we prove that $\overline{A} \subset A$: if $x \in \overline{A}$, then, for any neighborhood U of x, $U \cap A \neq \emptyset$, and this shows that x cannot be separated from A; using the proposition, we conclude that $x \in A$. $\square$

EXERCISE 4.19. Let $(X, \mathcal{T})$ be a Hausdorff space, $A, B \subset X$. If A and B are compact, show that $A \cap B$ is compact. Is the Hausdorffness assumption on X essential?

EXERCISE 4.20. If X is a Hausdorff space, and $A \subset X$ is compact, show that the quotient X/A (obtained from X by collapsing A to a point) is Hausdorff.

2.3. Compactness of products, and compactness in $\mathbb{R}^n$. Next, we are interested in the compactness of the product of two compact spaces. To prove the main result, we will need the following lemma which is interesting on its own.

LEMMA 4.19. *(the tube lemma)* Let X and Y be two topological spaces, $x_0 \in X$, and $U \subset X \times Y$ open (in the product topology) such that

$$\{x_0\} \times Y \subset U.$$

If Y is compact, then there exists $W \subset X$ open containing x_0 such that

$$W \times Y \subset U.$$

PROOF. Due to the definition of the product topology, for each $y \in Y$, since $(x_0, y) \in U$, there exist opens $W_y \subset X$, $V_y \subset Y$ such that

$$W_y \times V_y \subset U.$$

Now, $\{V_y : y \in Y\}$ will be an open cover of Y, hence we find $y_1, \ldots, y_n \in Y$ such that

$$Y = V_{y_1} \cup \ldots \cup V_{y_n}.$$

Choose

$$W = W_{y_1} \cap \ldots \cap W_{y_n}.$$

This is a finite intersection of open neighborhoods of x_0 hence it is itself an open neighborhood of x_0. To see that $W \times Y \subset U$, let $(x, y) \in W \times Y$. Since the V_{y_i}'s cover Y, we find i such that $y \in V_{y_i}$. But $x \in W$ implies $x \in W_{y_i}$, hence $(x, y) \in W_{y_i} \times V_{y_i}$. But $W_z \times V_z \subset U$ for all $z \in Y$, hence $(x, y) \in U$. $\square$

EXERCISE 4.21. Find an example which shows that the previous lemma fails if Y is not compact.

THEOREM 4.20. *If X and Y are compact spaces, then $X \times Y$ is compact.*

PROOF. Let $\mathcal{U}$ be an open cover of $X \times Y$. For each $x \in X$,

$$\{x\} \times Y \subset X \times Y$$

is compact (why?), hence we find a $\mathcal{U}_x \subset \mathcal{U}$ finite such that

$$\{x\} \times Y \subset \bigcup_{U \in \mathcal{U}_x} U. \tag{2.1}$$

Using the previous lemma, we find W_x open containing x such that

$$W_x \times Y \subset \bigcup_{U \in \mathcal{U}_x} U. \tag{2.2}$$

Now, $\{W_x : x \in X\}$ is an open cover of X, hence we find a finite subcover

$$X = W_{x_1} \cup \ldots \cup W_{x_p}.$$

Then

$$\mathcal{V} = \mathcal{U}_{x_1} \cup \ldots \cup \mathcal{U}_{x_p}$$

is finite union of finite collections, hence finite. Moreover, $\mathcal{V}$ still covers $X \times Y$: given (x, y) arbitrary, using (2.2), we find i such that $x \in W_{x_i}$. Hence $(x, y) \in W_{x_i} \times Y$, and using (2.1) we find $U \in \mathcal{U}_{x_i}$ such that $(x, y) \in U$. Hence we found $U \in \mathcal{V}$ such that $(x, y) \in U$, □

As explained in Remark 4.17, we now have:

COROLLARY 4.21. *A subset $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

EXAMPLE 4.22. In particular, spaces like the spheres S^n, the closed disks D^n, the Moebius band, the torus, etc, are compact.

2.4. Compactness and continuous functions. As we have already pointed out, if $f : X \longrightarrow Y$ is a continuous map, it may fail to send open subsets U of X to open subsets $f(U)$ of Y, and the same remark holds for closed sets as well. Compact spaces however are much better behaved from this point of view.

THEOREM 4.23. *If $f : X \longrightarrow Y$ is a continuous function and $A \subset X$ is compact, then $f(A) \subset Y$ is compact.*

PROOF. If $\mathcal{U}$ is an open cover of $f(A)$ in Y, then $f^{-1}(\mathcal{U}) := \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of A in X, hence it has a finite subcover $\{f^{-1}(U_i) : 1 \leq i \leq n\}$ with $U_i \in \mathcal{U}$. But then $\{U_i\}$ will be a finite subcover of $\mathcal{U}$. □

We have also seen that a bijective continuous map $f : X \longrightarrow Y$ between two topological spaces may fail to be a homeomorphism- f^{-1} may fail to be continuous. The role of compactness for this kind of "problem" is the following.

THEOREM 4.24. *If $f : X \longrightarrow Y$ is continuous and bijective, and if X is compact and Y is Hausdorff, then f is a homeomorphism.*

PROOF. We have to show that the inverse g of f is continuous. For this we show that if U is open in X then the pre-image $g^{-1}(U)$ is open in Y. Since the open (or closed) sets are just the complements of closed (respectively open) subsets, and $g^{-1}(Y - U) = X - g^{-1}(U)$, we see that the continuity of g is equivalent to: if A is closed in X then the pre-image $g^{-1}(A)$ is closed in Y. To prove this, let A be a closed subset of X. Since g is the inverse of f, we have $g^{-1}(A) = f(A)$, hence we have to show that $f(A)$ is closed in Y. Since X is compact, Proposition 4.15 implies that A is compact. By the previous theorem, $f(A)$ must be compact. Since Y is Hausdorff, Theorem 4.16 implies that $f(A)$ is closed in Y. □

COROLLARY 4.25. *If X is connected and compact and $f : X \longrightarrow \mathbb{R}$ is continuous, then $f(X) = [m, M]$ for some $m, M \in \mathbb{R}$.*

EXAMPLE 4.26. Here is an example which shows the use of compactness. We will show that there is no injective continuous map $f : S^1 \longrightarrow \mathbb{R}$.

PROOF. Assume there was such a map. Since S^1 is both connected and compact, its image is a closed interval $[m, M]$, and f becomes a continuous bijection $f : S^1 \longrightarrow [m, M]$. By the previous theorem, f should be a homeomorphism. But S^1 and $[m, M]$ cannot be homeomorphic: otherwise, using the removing-point trick, $[m, M]$ minus its midlle point (which is not connected) would be homormophic to S^1 minus some point (which is connected). Contradiction. □

REMARK 4.27. In the previous paragraphs, we had several arguments which we did not complete because Theorem 4.24 was not yet available. This was the case e.g. in

1. our discussion on the abstract torus, abstract Moebius band, etc (see Example **??**), and the fact that they are homeomorphic to their realizations in $\mathbb{R}^3$.
2. Example 3.24, when identifying D^n with the cone of S^{n-1}, etc.

In all these cases, we had to prove that certain spaces X and Y are homeomorphic, and we were able to produce a continuous bijection $f : X \longrightarrow Y$; however, the very last step- that f is actually a homeomorphism- was missing. All these arguments can now be completed by using Theorem 4.24.

Example 4.28. For clarity, let's now have a complete discussion on the Moebius band, putting together all the pieces we have seen so far. At the beginning (chapter I, section 4), we defined the Moebius band intuitively, as a subspace of $\mathbb{R}^3$ obtained from a square by gluing two of its opposite sides (Figure 14, Chapter I). The result was a subspace of $\mathbb{R}^3$ (hence a topological space with the induced topology). With the argument that the explicit models in $\mathbb{R}^3$ depend on various choices, in Example 3.17 of the previous chapter, we described the abstract Moebius band M_{abs} (defined intrinsically, with no choices involved, and no embedding in $\mathbb{R}^3$ being used). In the same example we also produced one (of the many possible) continuous injections

$$f : M_{\text{abs}} \longrightarrow \mathbb{R}^3$$

whose image coincides with (one of the) Moebius bands in $\mathbb{R}^3$. What we did not show yet is that f is an embedding. Well, that can be done now without much effort: f as a map from M_{abs} to $f(M_{\text{abs}})$, is a continuous bijection from a compact space (M_{abs} is the quotient of $[0,1]\times[0,1]$, hence compact) and a Hausdorff space ($f(M_{\text{abs}})$ is with the topology induced from the Euclidean topology on $\mathbb{R}^3$, hence Hausdorff). By theorem 4.24, this must be a homeomorphism, hence f is an embedding.

Exercise 4.22. Do the same for the abstract torus. Then prove that the torus is homeomorphic to $S^1 \times S^1$ (see Exercise 1.16 in the first chapter).

Exercise 4.23. Have a similar discussion for the Klein bottle (see Exercise 1.20 in the first chapter).

Exercise 4.24. Show that $\mathbb{P}^2$ can be embedded in $\mathbb{R}^4$ (see Exercise 1.22 in the first chapter). What about $\mathbb{P}^n$ for general n?

Exercise 4.25. Show that the cone and the suspension of any compact topological space are compact.

2.5. Sequential compactness. When one deals with sequences, one often sees statements of type "we now consider a subsequence with this property". Compactness is related to the existence of convergent subsequences. In general, given a topological space $(X, \mathcal{T})$, one says that X is sequentially compact if any sequence $(x_n)_{n\geq 1}$ of elements of X has a convergent subsequence. Recall that a subsequence of $(x_n)_{n\geq 1}$ is a sequence $(y_k)_{k\geq 1}$ of type

$$y_k = x_{n_k}, \text{ with } n_1 < n_2 < n_3 < \ldots .$$

However, we have already mentioned (and seen in various other cases) that topological properties involving sequences usually require the axiom of first countability.

Theorem 4.29. *Any first countable compact space is sequentially compact.*

Proof. [2] Let X be first countable and compact, and assume that $(x_n)_{n\geq 1}$ is an arbitrary sequence in X. For each integer $n \geq 1$ we put

$$U_n = X - \overline{\{x_n, x_{n+1}, \ldots\}}.$$

[2]not discussed in detail in the lectures

Note that these define an increasing sequence of open subsets of X:

$$U_1 \subset U_2 \subset U_3 \subset \dots.$$

We now claim that $\cup_n U_n \neq X$. If this is not the case, $\{U_n : n \geq 1\}$ is an open cover of X hence we find a finite set F such that $\{U_i : i \in F\}$ covers X. Since our cover is increasing, we find that $U_p = X$ where $p = \max F$, and this is clearly impossible. In conclusion, $\cup_n U_n \neq X$. Hence there exists $x \in X$ such that, for all $n \geq 1$, $x \notin U_n$. Choose a countable basis of neighborhoods of x

$$V_1, V_2, V_3, \dots.$$

Since $x \notin U_1$, we have $V \cap \{x_1, x_2, \dots\} \neq \emptyset$ for all neighborhoods V of x. Choosing $V = V_1$, we find n_1 such that

$$x_{n_1} \in V_1.$$

Next, we use the fact that $x \notin U_n$ for $n = n_1 + 1$. This means that $V \cap \{x_n, x_{n+1}, \dots\} \neq \emptyset$ for all neighborhoods V of x. Choosing $V = V_2$, we find $n_2 > n_1$ such that

$$x_{n_2} \in V_2.$$

We continue this process inductively (e.g. the next step uses $x \notin U_n$ for $n = n_1 + n_2 + 1$) and we find $n_1 < n_2 < \dots$ such that

$$x_{n_k} \in V_k$$

for all $k \geq 1$. Then $(x_{n_k})_{k \geq 1}$ is a subsequence of $(x_n)_{n \geq 1}$ converging to x. □

3. Local compactness and the one-point compactification

DEFINITION 4.30. *A topological space X is called locally compact if any point of X admits a compact neighborhood.*

We will be mainly interested in locally compact spaces which are Hausdorff.

EXAMPLE 4.31.

(0) Any compact Hausdorff space $(X, \mathcal{T})$ is locally compact Hausdorff.

(1) $\mathbb{R}^n$ is locally compact. Indeed, for any $x \in \mathbb{R}^n$, the closed unit ball

$$B[x, 1] = \{y \in \mathbb{R}^n : d(x, y) \leq 1\},$$

is compact (closed and bounded!).

(2) any open subset $U \subset \mathbb{R}^n$ is locally compact. Indeed, if $x \in U$, then there exists ϵ such that

$$B(x, \epsilon) \subset U,$$

and then $B[x, \epsilon/2]$ is a compact neighborhood of x in U. More generally, an open subset of a locally compact Hausdorff space is locally compact.

(3) any closed subset $A \subset \mathbb{R}^n$ is locally compact. Indeed, for any $a \in A$, $B[a, 1] \cap A$ is a neighborhood of a (in A) which is compact (because A is closed- use again that closed inside compact is compact). More generally, the same type of argument shows that a closed subset of a locally compact Hausdorff space has the same property.

(1) although the interval $(0, 1]$ is neither open nor closed (in $\mathbb{R}$), it is locally compact (combine the arguments from (2) and (3)).

(4) $\mathbb{Q} \subset \mathbb{R}$ is not locally compact. To see this, assume that the origin has a compact neighborhood N in $\mathbb{Q}$. Hence $N \subset \mathbb{Q}$ and there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \cap \mathbb{Q} \subset N$. Since N is compact, it must be closed in $\mathbb{R}$. Passing to closures in the previous inclusion, we obtain $[-\epsilon, \epsilon] \subset N$ which is impossible because $N \subset \mathbb{Q}$.

EXERCISE 4.26. Prove that, in a locally compact Hausdorff space, any point has a basis of compact neighborhoods.

REMARK 4.32. The property insured by the previous exercise provides us with the most natural description of local compactness- which fits into the general idea of what it means for a property (compactness in our case) to hold locally. This property immediately implies:

(1) for each $x \in X$, the collection of *all* compact neighborhoods of x is a basis of neighborhoods of x.
(2) for each $x \in X$ and any open neighborhood U of x, there exists a compact neighborhood N of x such that $N \subset U$.
(3) for each $x \in X$ and any open neighborhood U of x, there exists an open neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

3.1. One-point compactification. Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Roughly speaking, the one-point compactification is a compact Hausdorff space obtained from X by adding "the infinity", and which contains X as a subspace.

DEFINITION 4.33. *Let $(X, \mathcal{T})$ be a topological space. A <u>one-point compactification</u> of X is a compact Hausdorff space $(\tilde{X}, \tilde{\mathcal{T}})$ with an embedding*

$$i : X \longrightarrow \tilde{X}$$

such that $\tilde{X} - X$ consists of one point only.

EXAMPLE 4.34.

(1) If $X = (0, 1]$, then X^+ is (homeomorphic to) $[0, 1]$. Indeed, $\tilde{X} = [0, 1]$ is compact, and the inclusion $i : (0, 1] \longrightarrow [0, 1]$ satisfies the properties from the previous proposition.
(2) If $X = (0, 1)$, then X^+ is (homeomorphic to) the circle S^1. Indeed, $i : (0, 1) \longrightarrow S^1$ as in Figure 4 (e.g. $i(t) = (cos(2\pi t), sin(2\pi t))$ has the properties from the previous proposition.

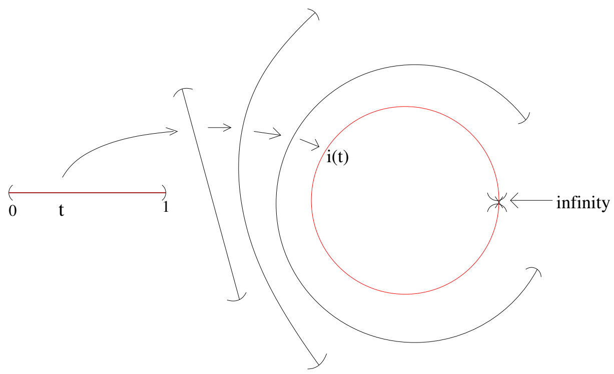

FIGURE 4.

(3) If $X = [-1, 0) \cup (1, 2) \subset \mathbb{R}$, then X^+ is homeomorphic with the subset of $\mathbb{R}^2$ drawn in Figure 5.
(4) If $X = \overset{\circ}{D}{}^n$ the one-point compactification is S^n.

THEOREM 4.35. *(existence and uniqueness of the one-point compactification)*

(i) *For any locally compact Hausdorff space X, there exists a one-point compactification.*
(ii) *If $\tilde{X}$ and $\tilde{\tilde{X}}$ are one-point compactifications of the same space X, then they are homeomorphic.*

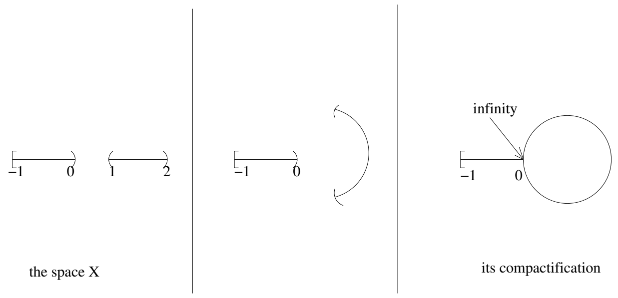

FIGURE 5.

PROOF. For the existence, we will use an element which is not a point of X, and which we denote by the symbol ∞ for convenience. We consider

$$X^+ = X \cup \{\infty\}.$$

Note that $X \subset X^+$ and any subset of X can be viewed as a subset of X^+. We consider the family of subsets of X^+:

$$\mathcal{T}^+ = \mathcal{T} \cup \mathcal{T}(\infty),$$

where

$$\mathcal{T}(\infty) = \{X^+ - K : K \subset X, K - \text{compact}\}.$$

We first have to show that $\mathcal{T}^+$ is a topology on X^+. Clearly $X^+, \emptyset \in \mathcal{T}^+$. Next, we show that the intersection of two sets U and V from $\mathcal{T}^+$ is in $\mathcal{T}^+$. Since $\mathcal{T}^+ = \mathcal{T} \cup \mathcal{T}(\infty)$, we have three cases. If $U, V \in \mathcal{T}$, we know that $U \cap V \in \mathcal{T}$. If U and V are both of type $X - K$ with K-compact, then so is their intersection, because union of two compacts is compact (show this!). Finally, if U is of type $U = X^+ - K$ with $K \subset X$ compact, while V is open in X, then $U \cap V = U - K = U \cap (X - K)$ is open in X because U and $X - K$ are (we use again that compact is closed).

Next, we show that arbitrary union of sets from $\mathcal{T}^+$ is in $\mathcal{T}^+$. Note that we already know that the same statement holds for $\mathcal{T}$, and also for $\mathcal{T}(\infty)$ since intersections of compacts is compact (why?). We deduce that we only have to show that, if $U \in \mathcal{T}$ and $V \in \mathcal{T}(\infty)$, then $U \cup V \in \mathcal{T}^+$. Writing $V = X^+ - K$ with $K \subset X$ compact, we have

$$U \cap V = X^+ - K',$$

where $K' = K \cap (X - U)$. Since $K - K' = K \cap U$, $K - K'$ is open in K, i.e. K' is closed in the compact K, hence K' is compact (Proposition 4.15 again!). This shows that $U \cap V \in \mathcal{T}^+$. This ends the proof of the fact that $\mathcal{T}^+$ is a topology.

Next, we show that X^+ is compact. Hence assume that we have an open cover $\mathcal{U}$ of X^+. Since a set in $\mathcal{T}$ contains ∞, $\mathcal{U}$ must contain a set of type $U = X^+ - K$ with $K \subset X$ compact. Let $\mathcal{U}' = \{V \cap X : V \in \mathcal{U}, V \neq U\}$. Then $\mathcal{U}'$ is an open covering of K in X^+ hence, from the compactness of K, we find a finite subcollection $\mathcal{V} \subset \mathcal{U}'$ which covers K. But then

$$\mathcal{V} \cup \{X^+ - K\}$$

is a subcollection of $\mathcal{U}$ which covers X^+. This proves that X^+ is compact.

Next, we show that X^+ is Hausdorff. So, let $x, y \in X^+$ distinct, and we are looking for $U, V \in \mathcal{T}^+$ such that $U \cap V = \emptyset$, $x \in U, y \in V$. If $x, y \in X$ we are done because X is Hausdorff and $\mathcal{T} \subset \mathcal{T}^+$. Hence we may assume that $x \in X$, $y = \infty$. We have to find an open $U \subset X$ containing x and a compact $K \subset X$ such that

$$U \cap (X^+ - K) = \emptyset.$$

For that, we choose any compact neighborhood K of x and we consider $U \subset X$ open such that $x \in U \subset K$. This proves that X^+ is compact.

Next, we have to show that the obvious map (inclusion) $i : X \longrightarrow X^+$ is an embedding, i.e. that the restriction of $\mathcal{T}^+$ to X coincides with $\mathcal{T}$. From the definition of $\mathcal{T}^+$ we have

$$\mathcal{T}^+|_X = \mathcal{T} \cup \{U \cap X : U \in \mathcal{T}(\infty)\}.$$

But for $U = X - K \in \mathcal{T}(\infty)$ we have $U \cap X = X - K \in \mathcal{T}$ (again, we use that compacts in Hausdorff are closed); hence $\mathcal{T}^+|_X = \mathcal{T}$.

This concludes the proof of (i). For (ii), we show that any other one-point compactification $\tilde{X}$ of X is homeomorphic to X^+. Let $\tilde{x} \in \tilde{X}$ be the point with the property that

$$\tilde{X} = i(X) \cup \{\tilde{x}\}.$$

We define

$$f : \tilde{X} \longrightarrow X^+$$

as follows. For $y \in \tilde{X}$, we have two cases

(1) $y \in i(X)$, and then we write $y = i(x)$ with $x \in X$ and we put $f(y) = x$.
(2) $y = \tilde{x}$ and then we put $f(y) = \infty$.

We check that f is continuous, i.e. $f^{-1}(U) \in \tilde{\mathcal{T}}$ for any $U \in \mathcal{T}^+$. We have two cases. If $U \in \mathcal{T}$, then $f^{-1}(U) \subset i(X)$ and

$$f^{-1}(U) = \{y = i(x) : f(y) \in U\} = \{i(x) : x \in U\} = i(U)$$

which is open in $i(X)$ since i is an embedding. However, $i(X)$ is open in $\tilde{X}$ (its complement consists of one point only, hence it is closed because $\tilde{X}$ is Hausdorff). We deduce that $f^{-1}(U)$ is open in $\tilde{X}$. The second case is when $U = X^+ - K$ with $K \subset X$ compact. Then $f^{-1}(U)$ consist of $\tilde{x}$ and $f^{-1}(X - K) = i(X) - i(K)$, i.e.

$$f^{-1}(U) = X^+ - i(K).$$

Since $i(K)$ is compact (the image of a compact is compact), it is closed in X^+, hence $f^{-1}(U)$ is open. This proves that f is continuous. Since f is clearly bijective and we know that $\tilde{X}$ is compact and X^+ is Hausdorff, we deduce (see Theorem 4.24) that f is a homeomorphism. $\square$

Exercise 4.27. What is the one-point compactification of $X = (0, 1) \cup (2, 3)$?

Exercise 4.28. What is the one-point compactification of $\mathbb{R}^2$?

Exercise 4.29. Show that the following subspace of $\mathbb{R}^2$ is locally compact and find its one-point compactification:

$$X = S^1 \cup ((0, 2) \times \{0\}) \subset \mathbb{R}^2.$$

Exercise 4.30. For any continuous map $f : S^1 \longrightarrow T^2$ we define

$$X_f := T^2 - f(S^1).$$

(i) Is it true that, for any continuous function f, X_f compact? But locally compact? But metrizable? But connected?
(ii) Describe two embeddings $f_1, f_2 : S^1 \longrightarrow T^2$ such that X_{f_1} and X_{f_2} are not homeomorphic.
(iii) Describe the one-point compactifications $X^+_{f_1}$ and $X^+_{f_2}$.
(iv) Describe $f : S^1 \longrightarrow T^2$ continuous such that X^+_f is homeomorphic to S^2.

EXERCISE 4.31. Consider
$$X = [0,1] \times [0,1)$$
with the topology induced from $\mathbb{R}^2$. Prove that X is a locally compact Hausdorff space and describe its one-point compactification (use a picture). What happens if we replace X by $Y = X \cup \{(1,1)\}$?

EXERCISE 4.32. Show that, for any circle C embedded in the Klein bottle K, $K - C$ is locally compact. Then describe such a circle C such that $(K - C)^+$ is homeomorphic to $\mathbb{P}^2$.

EXERCISE 4.33. Show that there exists a Hausdorff space X and an embedded circle C in X, such that the one-point compactification of $X - C$ is homeomorphic to X.

EXERCISE 4.34. Let X be a Hausdorff compact space and $A \subset X$ be a closed subset. Show that

(1) $X - A$ is locally compact.
(2) X/A is compact and Hausdorff.
(3) X/A is (homeomorphic to) the one-point compactification of $X - A$.

EXERCISE 4.35. Let X be a connected, locally compact, Hausdorff space. Show that X is compact if and only if X^+ is not connected.

4. Compactness in metric spaces

[3]

As we have already mentioned, a subspace $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. This is a result which re-writes topological property (compactness) in terms of another topological property (closedness) and a metric property (boundedness). We would like not only to prove this result, but also generalize it to general metric spaces (X, d). Let's look closer at the two properties we mentioned (boundedness and closedness).

Boundedness: We first have a closer look at the boundedness condition. Given the metric (X, d), the notion of boundedness (with respect to d) of a subset $A \subset X$ is clear: one requires that there exists $r > 0$ such that $d(a, b) < r$ for all $a, b \in r$ or, equivalently, that A can be included into a ball $B(x, R)$ (for some $x \in X$, $R > 0$). However, this notion is not very useful because it depends too much on the metric (see the next exercise). One needs to replace this "obvious condition of boundedness in a metric spaces" with another condition which does not depend so strongly on the metric.

DEFINITION 4.36. *Given a metric space (X, d), we say that $A \subset X$ is totally bounded in X (or precompact in X) if for any $\epsilon > 0$, there exist a finite number of balls of radius ϵ covering A.*

If $A = X$, we say that (X, d) is totally bounded (or precompact).

EXERCISE 4.36. For a metric space (X, d) we define $d_1 : X \times X \longrightarrow \mathbb{R}$ by

$$\overline{d}(x, y) = \min\{d(x, y), 1\}. \tag{4.1}$$

Show that

(1) $\overline{d}$ is a metric inducing the same topology on X as d.
(2) any subset $A \subset X$ is bounded with respect to $\overline{d}$.
(3) A subset $A \subset X$ is totally bounded with respect to $\overline{d}$ if and only if it is totally bounded with respect to d.

[3]not discussed in the lectures

(4) When $X = \mathbb{R}^n$ and d is the Euclidean metric, then $A \subset \mathbb{R}^n$ is totally bounded with respect to d if and only if it is bounded (with respect to d).

Closedness: We would like to understand what do we have to replace the condition "$A \subset \mathbb{R}^n$ is closed" by, in the case of metric spaces. This is a more subtle. First of all, it is worth remarking that the notion of totally boundedness is not really a relative notion: a subset A of a metric space (X, d) is totally bounded if and only if the metric space (A, d_A) itself (where d_A is the restriction of d to A) is totally bounded. Recall also that the same phenomena happened for the notion of compactness (see Remark 4.13). However, a statement like "A is closed in X" clearly does not have the same property. The key to this problem is the notion of completeness of metric spaces. The point is that

- similar to compactness and totally boundedness, completeness is not a relative notion.
- $A \subset \mathbb{R}^n$ is closed if and only if it is complete.

(see the next exercise). Recall the notion of completeness.

DEFINITION 4.37. *Given a metric space (X, d) and a sequence $(x_n)_{n\geq 1}$ in X, we say that $(x_n)_{n\geq 1}$ is a Cauchy sequence if*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0,$$

i.e., for each $\epsilon > 0$, there exists an integer n_ϵ such that

$$d(x_n, x_m) < \epsilon$$

for all $n, m \geq n_\epsilon$. One says that (X, d) is complete if any Cauchy sequence is is convergent. We say that $A \subset X$ is complete if A, together with the restriction of d to A, is complete.

EXERCISE 4.37. Let (X, d) be a metric space. Show that

(1) Any $A \subset X$ which is complete must be closed.
(2) If X is complete, then $A \subset X$ is complete if and only if A is closed in X.

We are now able to state the general compactness criteria for metric spaces.

THEOREM 4.38. *For a metric space (X, d), the following are equivalent:*

(i) *X is compact.*
(ii) *X is sequentially compact.*
(iii) *X is complete and totally bounded.*

Before we turn to the proof, let us mention the following consequence

COROLLARY 4.39. *A subset A of a complete metric space (X, d) is compact if and only if it is closed (in X) and totally bounded (with respect to d).*

PROOF. (of Theorem 4.38 and Corollary 4.39) First the theorem.

(i)$\Longrightarrow$ (ii): Since metric spaces are first countable, this follows from Theorem 4.29.

(ii)$\Longrightarrow$ (iii): Assume now that X is sequentially compact. We first prove that X is complete. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence. By hypothesis, we find a convergent subsequence $(x_{n_k})_{k\geq 1}$. Let x be its limit. We prove that the whole sequence (x_n) converges to x. Let $\epsilon > 0$. We look for an integer N_ϵ such that $d(x_n, x) < \epsilon$ for all $n > N_\epsilon$. Since (x_n) is Cauchy we find N'_ϵ such that

$$d(x_n, x_m) < \epsilon/2$$

for all $n, m \geq N'_\epsilon$. Since $(x_{n_k})_{k\geq 1}$ converges to x, we find k_ϵ such that

$$d(x_{n_k}, x) < \epsilon/2$$

for all $k \geq k_\epsilon$. Put

$$N_\epsilon = \max\{N'_\epsilon, n_{k_\epsilon}\}.$$

Then, for $n > N_\epsilon$, choosing k such that $n_k > n$ (such a k exists since $n_1 < n_2 < \ldots$ is a sequence that tends to ∞), we must have $k > k_\epsilon$ and $n_k > N'_\epsilon$, hence

$$d(x_n, x_{n_k}) < \epsilon/2, \ d(x_{n_k}, x) < \epsilon/2.$$

Using the triangle inequality, we obtain $d(x_n, x) < \epsilon$, and this holds for all $n \geq N_\epsilon$. This proves that (x_n) converges to x. We now prove that X is totally bounded. Assume it is not. Then we find $r > 0$ such that X cannot be covered by a finite number of balls of radius r. Construct a sequence $(x_n)_{n\geq 1}$ as follows. Start with any $x_1 \in X$. Since $X \neq B(x_1, r)$, we find $x_2 \in X - B(x_1, r)$. Since $X \neq B(x_1, r) \cup B(x_2, r)$, we find $x_3 \in X - B(x_1, r) \cup B(x_2, r)$. Continuing like this we find a sequence with $x_n \notin B(x_m, r)$ for $n > m$. Hence $d(x_n, x_m) > r$ for all $n \neq m$. But by hypothesis, (x_n) has a subsequence $(x_{n_k})_{k\geq 1}$ which converges to some $x \in X$. But then we find N such that $d(x_{n_k}, x) < r/2$ for all $k \geq N$, hence

$$d(x_{n_k}, x_{n_l}) \leq d(x_{n_k}, x) + d(x, x_{n_l}) < r$$

for all $k, l \geq N$, and this contradicts the condition "$d(x_n, x_m) > r$ for all $n \neq m$".

(iii)$\Longrightarrow$ (i): Assume that (X, d) is complete and totally bounded. The last condition ensures that for each integer $n \geq 0$, there is a finite set $F_n \subset X$ such that

$$X = \bigcup_{x\in F_n} B(x, \frac{1}{2^n}).$$

Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of X, and we want to prove that we can extract a finite subcover of $\mathcal{U}$. Assume this is not possible. We construct a sequence $(x_n)_{n\geq 1}$ inductively as follows. Since $\cup_{x\in F_1} B(x, \frac{1}{2}) = \cup_i U_i$, and the first union is a finite union (F_1 is finite), we find $x_1 \in F_1$ such that $B(x_1, \frac{1}{2})$ cannot be covered by of finite number of opens from $\mathcal{U}$. Now, since

$$B(x_1, \frac{1}{2}) = \bigcup_{x\in F_2} (B(x_1, \frac{1}{2}) \cap B(x, \frac{1}{4}))$$

we find $x_2 \in F_2$ such that

$$B(x_1, \frac{1}{2}) \cap B(x_2, \frac{1}{4}) \neq \emptyset$$

and $B(x_2, \frac{1}{4})$ cannot be covered by a finite numbers of opens from $\mathcal{U}$. Continuing this, at step n we find $x_n \in F_n$ such that

$$B(x_{n-1}, \frac{1}{2^{n-1}}) \cap B(x_n, \frac{1}{2^n}) \neq \emptyset$$

and $B(x_n, \frac{1}{2^n})$ cannot be covered by of finite number of opens from $\mathcal{U}$. Note that, choosing an element y in the (non-empty) intersection above, the triangle inequality implies that

$$d(x_{n-1}, x_n) < \frac{1}{2^{n-1}} + \frac{1}{2^n} = \frac{3}{2^n},$$

from which we deduce that $(x_n)_{n\geq 1}$ is a Cauchy sequence (why?). By hypothesis, it will converge to en element $x \in X$. Choose $U \in \mathcal{U}$ such that $x \in U$. Since U is open, we find $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Since $x_n \to x$, we find n_ϵ such that $d(x_n, x) < \epsilon/2$ for all $n > n_\epsilon$. Using the triangle inequality, we deduce that $B(x_n, \epsilon/2) \subset U$ for all $n \geq n_\epsilon$. Choosing n so that also $1/2^n < \epsilon/2$, we deduce that $B(x_n, 1/2^n) \subset U$, which contradicts the fact that $B(x_n, \frac{1}{2^n})$ cannot be covered by of finite number of opens from $\mathcal{U}$.

This ends the proof of the theorem. We are left with proving Corollary 4.39. But this follows immediately from the equivalence between (i) and (iii) in the theorem (applied to the metric space (A, d_A)) and the fact that a subset of a completeabstract-stuff metric space is complete if and only if it is closed (see Exercise 4.37). $\square$

5. More exercises

Exercise 4.38. *(Exam A, 2007)*

(i) Is it true that any continuous surjective map $f : S^1 \longrightarrow S^1$ is a homeomorphism? (0.75p)
(ii) Show that S^1 cannot be embedded in $\mathbb{R}$. (1p)
(iii) Show that any continuous injective map $f : S^1 \longrightarrow S^1$ is a homeomorphism. (0.75p)

Exercise 4.39. *(Exam A, 2007)* For any continuous map $f : S^1 \longrightarrow T^2$ we define

$$X_f := T^2 - f(S^1).$$

(i) Is it true that, for any continuous function f, X_f compact? But locally compact? But metrizable? But normal? But connected? (0.5p)
(ii) Describe two embeddings $f_1, f_2 : S^1 \longrightarrow T^2$ such that X_{f_1} and X_{f_2} are not homeomorphic. (.75p)
(iii) Describe the one-point compactifications $X_{f_1}^+$ and $X_{f_2}^+$. (0.75p)
(iv) Describe $f : S^1 \longrightarrow T^2$continuous such that X_f^+ is homeomorphic to S^2. (0.5p)

Exercise 4.40. *(Exam A, 2009)* Let $\mathcal{T}_l$ be the topology on $\mathbb{R}$ induced by the topology basis

$$\mathcal{B}_l := \{[a, b) : a, b \in \mathbb{R}, a < b\}.$$

(i) With the topology induced from $(\mathbb{R}, \mathcal{T}_l)$, is $[0, 1)$ connected? But compact? (*1p*)
(ii) The same questions for $(0, 1]$. (*1p*)
(iii) Are $[0, 1)$ and $(0, 1]$ (with the induced topologies) homeomorphic? (*0.25p*)
(iv) Compute the interior, the closure and the boundary of

$$A = (0, 1] \times [0, 1)$$

in the topological space $X = \mathbb{R} \times \mathbb{R}$ endowed with the product topology $\mathcal{T}_l \times \mathcal{T}_l$. (*1p*)

Exercise 4.41. *(Exam A, 2009)* Describe an embedding of the cylinder $S^1 \times [0, 1]$ into the space X, show that its complement is locally compact, and find the one-point compactification of the complement, in each of the cases:

(i) X is a torus. (*1.5p*)
(ii) X is the plane $\mathbb{R}^2$. (*1p*)

Exercise 4.42. *(Exam A, 2009)*

(i) Write the Moebius band as a union of two subspaces M and C where M is itself a Moebius band, C is a cylinder (i.e. homeomorphic to $S^1 \times [0, 1]$) and $M \cap C$ is a circle. (*1p*)
(ii) Similarly, decompose $\mathbb{P}^2$ as the union of a Moebius band M and another subspace Q, such that Q is a quotient of the cylinder and $M \cap Q$ is a circle. (*1p*)
(iii) Deduce that $\mathbb{P}^2$ can be obtained from a Moebius band and a disk D^2 by gluing them along their boundary circles. (*0.5p*)

EXERCISE 4.43. *(Exam A, 2009)* Let X be a topological space and let $\mathrm{Cone}(X)$ be its cone. Show that:

(i) If X is compact then so is $\mathrm{Cone}(X)$. (*1p*)
(ii) If X is path connected then so is $\mathrm{Cone}(X)$. (*0.5p*)
(iii) If X is connected then so is $\mathrm{Cone}(X)$. (*0.25p*)