CHAPTER 7

The fundamental group

1. Homotopies and homotopy equivalences

DEFINITION 7.1. *Let X and Y be topological spaces. We denote by* Cont(X, Y) *the set of all continuous maps from X to Y. Given $f_0, f_1 \in \mathrm{Cont}(X, Y)$, a homotopy between f_0 and f_1 is a continuous map*

$$H : X \times [0, 1] \longrightarrow Y$$

with the property that

$$H(x, 0) = f_0(x),\ H(x, 1) = f_1(x),$$

for all $x \in X$. We say that f_0 and f_1 are homotopic, and we write

$$f_0 \sim f_1,$$

if there exists a homotopy H between f_0 and f_1.

REMARK 7.2. A homotopy between f_0 and f_1 should be viewed as a "continuous deformation" of f_0 into f_1: the homotopy H defines for each $t \in [0, 1]$ a continuous map $H_t \in \mathrm{Cont}(X, Y)$,

$$H_t : X \longrightarrow Y,\ x \mapsto H(t, x),$$

and the family $\{H_t\}$ goes from $H_0 = f_0$ to $H_1 = f_1$.

PROPOSITION 7.3. *Given topological spaces X, Y, Z,*

(i) *The homotopy relation $\sim$ is an equivalence relation on* Cont(X, Y).
(ii) *If $f_0, f_1 \in \mathrm{Cont}(Y, Z)$ are homotopic, and $g_0, g_1 \in \mathrm{Cont}(X, Y)$ are homotopic, then $f_0 g_0$ and $f_1 g_1$ are homotopic.*

PROOF. We first check (i). For any $f : X \longrightarrow Y$ we have $f \sim f$ via the homotopy $h(x, t) = f(x)$. If $f \sim g$ via the homotopy H, then $g \sim f$ via the homotopy H^- defined by $H^-(x, t) = H(x, 1 - t)$. Assume now that $f \sim g$ via a homotopy H and $g \sim h$ via a homotopy H'. Then we define H'' by

$$H''(x, t) = \begin{cases} H(x, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ H'(x, 2t - 1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases} .$$

Clearly, it satisfies $H''_0 = f$, $H''_1 = h$ (why is it continuous?). □

EXAMPLE 7.4. If $Y = \mathbb{R}^n$, any two continuous maps $f_0, f_1 : X \longrightarrow \mathbb{R}^n$ are homotopic. Indeed,

$$H(x, t) = (1 - t) f_0(x) + t f_1(x)$$

is a homotopy between f_0 and f_1. The same happens if $\mathbb{R}^n$ is replaced by any $C \subset \mathbb{R}^n$ which is convex (i.e. $tx + (1 - t)y \in C$ for all $x, y \in C$ and all $t \in [0, 1]$).

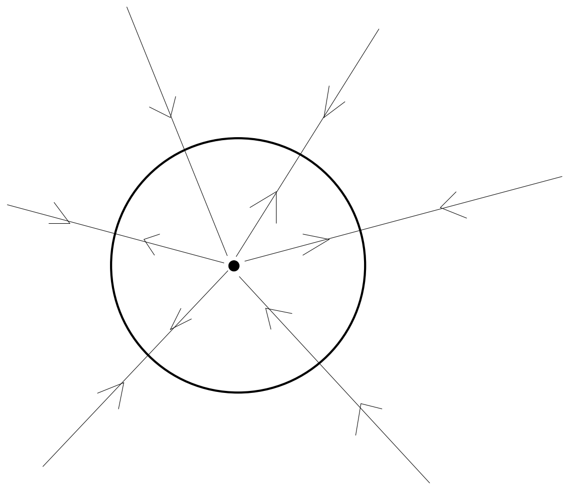
EXAMPLE 7.5. The map

$$f : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}, x \mapsto \frac{x}{||x||}$$

is homotopic to the identity map. For this we consider

$$H(x,t) = (1-t)\frac{x}{||x||} + tx$$

(which takes values in $\mathbb{R}^2 - \{0\}$!). See Figure 1.



|R^2 minus a point is homotopic equivalent to the circle

FIGURE 1.

DEFINITION 7.6. *We say that a continuous function $f : X \longrightarrow Y$ is a* <u>*homotopy equivalence*</u> *if there exists a continuous function $g : Y \longrightarrow X$ such that*

$$g \circ f \sim Id_X, \ \ f \circ g \sim Id_Y.$$

The map g will be called a <u>*homotopy inverse*</u> *of f. Given two topological spaces X and Y, we say that they are* <u>*homotopic equivalent*</u> *if there exists a homotopy equivalence $f : X \longrightarrow Y$.*

We say that a space X is <u>*contractible*</u> *if it is homotopic equivalent to the one-point space (a topological space with only one point).*

EXAMPLE 7.7. Example 7.4 implies that $\mathbb{R}^n$ is contractible. Indeed, taking an arbitrary $X = \{x_0\}$ with $x_0 \in \mathbb{R}^n$ arbitrary, $f : \mathbb{R}^n \longrightarrow X$ the constant map and $g : X \longrightarrow \mathbb{R}^n$ the inclusion, $f \circ g$ is the identity, while $g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is homotopic to (any other map hence also to) the identity. The same applies to any convex subset $C \subset \mathbb{R}^n$.

EXERCISE 7.1. *Give an example of a subspace $C \subset \mathbb{R}^n$ which is not convex but is contractible.*

Regarding contractibility, one has the following characterization.

EXERCISE 7.2. *For a topological space C, show that the following are equivalent:*

(i) *C is contractible.*
(ii) *The identity map $Id_C : C \longrightarrow C$ is homotopic to a constant map.*
(iii) *For any other topological space X, any two continuous maps $f, g : X \longrightarrow C$ are homotopic.*

EXAMPLE 7.8. Similar to the previous example, $\mathbb{R}^2 - \{0\}$ is homotopic equivalent to S^1. To see this, we consider the inclusion $i : S^1 \longrightarrow \mathbb{R}^2$ and the map

$$r : \mathbb{R}^2 - \{0\} \longrightarrow S^1, r(x) = \frac{x}{||x||}.$$

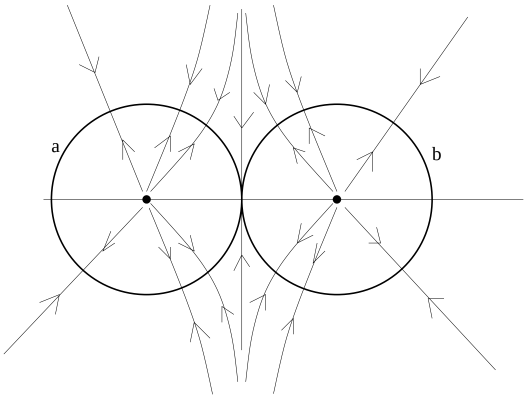
Clearly, $r \circ i = \mathrm{Id}_{S^1}$. We still have to check that

$$i \circ r : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\},$$

is homotopic to the identity map, but this is precisely the map from Example 7.5.

Note that the same argument shows that $D^2 - \{0\}$ is homotopic equivalent to S^1. Similarly, $\mathbb{R}^n - \{0\}$ and to $D^n - \{0\}$ are homotopic equivalent to S^{n-1} for any $n \geq 1$ integer, with the homotopy equivalences given by the inclusion of S^{n-1}.

EXAMPLE 7.9. Similarly, the space obtained by removing two points from $\mathbb{R}^2$ is homotopic equivalent to a bouquet of two circles, as indicated in Figure 2.



$\mathbb{R}^2$ minus two points is homotopic equivalent to a bouquet of two circles

FIGURE 2.

EXERCISE 7.3. *Show that the cylinder $C = S^1 \times [0,1]$ is homotopic equivalent to a circle. Similarly, the Moebius band is homotopic equivalent to a circle. Try to write down the explicit formulas.*

EXERCISE 7.4. *More generally, show that for any space X, the associated cylinder $Cyl(X)$ is homotopic equivalent to X.*

EXERCISE 7.5. *Show that, for any space X, the cone of X, $C(X)$, is contractible.*

EXERCISE 7.6. *Show that the spaces in Figure 3 are homotopic equivalent.*

EXERCISE 7.7. *Go back to Exercise 6.14. Show that the spaces from Figure 8 and Figure 9 are both homotopic equivalent to a bouquet of two circles. Then show that the two spaces drawn in Figure 10 are homtopic equivalent.*

EXERCISE 7.8. *Consider the cylinder $C = S^1 \times [0,1]$. Show that*

(1) *For any $p = (x,t) \in C$ with $t \notin \{0,1\}$, $C - \{p\}$ is homotopic equivalent to $S^1 \vee S^1$.*
(2) *For any $p = (x,t) \in C$ with $t \in \{0,1\}$, $C - \{p\}$ is homotopic equivalent to S^1.*

EXERCISE 7.9. *Show that, after removing two points from the sphere, the resulting space is homotopic equivalent to a circle.*

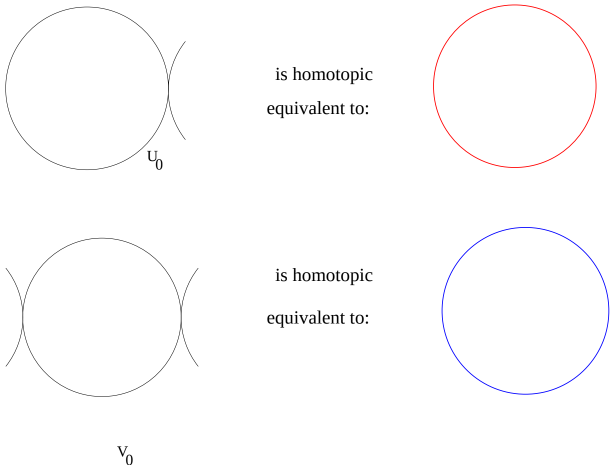


FIGURE 3.

EXAMPLE 7.10. Similar to Example 7.8, the space obtained from the square by removing one point from its interior is homotopic equivalent to its boundary, hence (also) to S^1. See Figure 4. This also follows directly from Example 7.8 because the square is homeomorphic to the disk. More generally, for any convex subspace $X \subset \mathbb{R}^n$ and any interior point $x_0 \in \overset{\circ}{X}$, one can prove

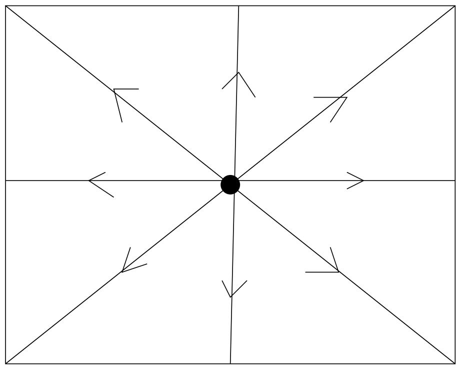


FIGURE 4.

that $X - \{x_0\}$ is homotopic equivalent to S^{n-1}.

EXERCISE 7.10. *Show that the space after removing three points from the plane is homotopic equivalent to a bouquet of three circles.*

EXAMPLE 7.11. The torus minus a point is homotopic equivalent to a bouquet of two circles. To see this, recall first that the torus can be obtained from the square by gluing its opposite sides. Hence the torus minus a point can be obtained from the same procedure, but starting with X which a square minus a point in its interior. We have remarked that this last space is homotopic equivalent to a circle. Onthe other hand, this homotopy equivalence preserves the boundary of the square, hence it induces a homotopy equivalence between the space X/R (X is the square minus a point, and R is the equivalence relation encoding the gluing) and S^1/R- the space obtained from S^1 by performing the induced gluings. The result is $S^1 \vee S^1$. See Figure 5.

EXERCISE 7.11. *What if we remove a small disk from the torus?*

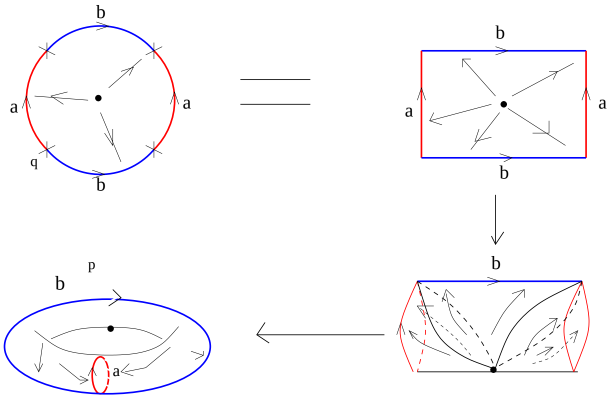


FIGURE 5.

EXERCISE 7.12. *Prove that the space obtained by removing a point from $\mathbb{P}^2$ is homotopic equivalent to a circle.*

EXERCISE 7.13. *What happens for the Moebius band?*

More generally, one has the following.

EXERCISE 7.14. *Assume that X is obtained from A by adjoining an n-cell. Prove that, for any point $x \in X - A$, the inclusion*

$$i : A \longrightarrow X - \{x\}$$

is a homotopy equivalence. Write down the complete proof carefully.

EXERCISE 7.15. *Go back to Exercise 6.21. Which one of the spaces there do you think is homtopic equivalent to the space W from Figure 6?*

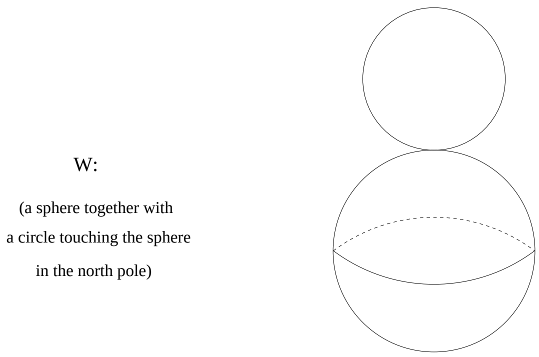


FIGURE 6.

REMARK 7.12 ((The homotopy category)). Some of the constructions here can be nicely interpreted using the language of "category theory". Precisely, a category $\mathcal{C}$ consists of

- certain objects of $\mathcal{C}$, which we will denote by capital letters (e.g. X, A, etc).
- for any two objects X and Y, a set of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ which are called morphisms (or arrows) from X to Y, and which we will denote by small letters. Given an arrow $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we also write $f : X \longrightarrow Y$.

- for any three objects X, Y and Z, a map
$$\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z),$$
which is called the composition of arrows, and which is denoted by $(a,b) \mapsto a \circ b$. The composition is required to satisfy:
$$(a \circ b) \circ c = a \circ (b \circ c),$$
(whenever the composition is defined).
- for any object X, there is a specified arrow $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, called the identity arrow of X. It is required that, for any $a \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$,
$$a \circ \mathrm{Id}_X = \mathrm{Id}_Y \circ a = a.$$

Each category $\mathcal{C}$ has its own notion of isomorphisms: we say that a morphism $a \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an isomorphism if there exists $b \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ such that
$$b \circ a = \mathrm{Id}_X,\ a \circ b = \mathrm{Id}_Y.$$

Standard examples are:

(i) The category of sets has as objects the sets, the morphisms in this category between two sets X and Y are the usual functions $f : X \longrightarrow Y$, and the composition is the usual composition of functions. The isomorphisms of this category correspond to bijective functions.

(ii) The category of groups has as objects all groups, the morphisms in this category between two groups G and H are the usual group homomorphisms $f : G \longrightarrow H$, and the composition is the usual composition of maps. The isomorphisms in this category correspond to the usual group isomorphisms.

(iii) The category of topological spaces has as objects the topological spaces, the morphisms between two space X and Y are the continuous maps $f : X \longrightarrow Y$ and, again, the composition is the usual composition of maps. The isomorphisms of this category are the homeomorphisms.

Back to the previous proposition, for any two spaces X and Y we consider the set of homotopy classes of maps from X to Y:
$$[X,Y] := \mathrm{Cont}(X,Y)/\sim .$$
For $f : X \longrightarrow Y$ continuous, we denote by $[f] \in [X,Y]$ the equivalence class of f. The proposition ensures that there is a well-defined map
$$[Y,Z] \times [X,Y] \longrightarrow [X,Z], ([f],[g]) \mapsto [f] \circ [g] := [f \circ g]$$
What we have really done was to define a new category, called the homotopy category of spaces: it has as objects all topological spaces, while the set of morphisms from X to Y is $[X,Y]$ with the composition that we have just described. In this language, two spaces are homotopic equivalent if and only if they are isomorphic in the homotopy category,

2. Path homotopies

Recall that a path in a topological space X is a continuous map
$$\gamma : [0,1] \longrightarrow X.$$
The point $x = \gamma(0)$ is called the initial (or start) point of γ, while the point $y = \gamma(1)$ is called the final (or end) point of γ. We also say that γ is a path (in X) from x to y. We denote by
$$P(X,x,y)$$
the set of all paths in X from x to y.

Definition 7.13. *Given a space X, $x, y \in X$, we say that two paths*
$$\gamma, \gamma' \in P(X,x,y)$$
are path homotopic, and we write
$$\gamma \sim_p \gamma'$$

if there exists a homotopy H between γ and γ' such that each $H_t(-) = H(-,t) : [0,1] \longrightarrow X$ is a path from x to y.

We say that a path $\gamma \in P(X,x,x)$ is null-homotopic if it is path homotopic to the constant path $c_x \in P(X,x,x)$ ($c_x(t) = t$ for all t).

REMARK 7.14. A path homotopy H should be viewed a "continuous deformation" of γ into γ' through paths from x to y. Explicitly,

$$H : [0,1] \times [0,1] \longrightarrow X$$

must satisfy

$$H(s,0) = \gamma(s), H(s,1) = \gamma'(s),$$
$$H(0,t) = x, H(1,t) = y.$$

for all $s, t \in [0,1]$.

PROPOSITION 7.15. *Let X be a space, $x, y \in X$. The path homotopy relation $\sim_p$ is an equivalence relation on $P(X,x,y)$.*

PROOF. One just remarks that all the homotopies in the proof of Proposition 7.3 are path-homotopies. $\square$

How can one "compose" paths?

DEFINITION 7.16. *Let X be a topological space, $x, y, z \in X$. For $\gamma \in P(X,x,y)$ and $\gamma' \in P(X,y,z)$, we define the concatenation of γ and γ' as the new path $\gamma * \gamma' \in P(x,z)$ defined by*

$$(\gamma * \gamma')(s) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma'(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} .$$

Note that, by the pasting lemma (Lemma **??**), $\gamma * \gamma'$ is continuous.

Note that the concatenation of paths does not behave "like a composition"- for instance, it is not associative. Things become much nicer is we pass to homotopy classes of paths.

DEFINITION 7.17. *Given a space X and $x, y \in X$, we define the set of homotopy classes of paths from x to y as*

$$\Pi(X,x,y) := P(X,x,y)/\sim_p .$$

For $\gamma \in P(X,x,y)$, the induced equivalence class is denote by $[\gamma] \in \Pi(X,x,y)$.

PROPOSITION 7.18. *Let X be a topological space. Then, for any $x, y, z \in X$, the map*

$$\Pi(X,x,y) \times \Pi(X,y,z) \longrightarrow \Pi(X,x,z),$$
$$([\gamma],[\gamma']) \mapsto [\gamma] * [\gamma'] := [\gamma * \gamma']$$

is well defined. Moreover

1. *For $x \in X$, denote by $c_x \in P(X,x,x)$ the constant path, and $1_x = [c_x] \in \Pi(X,x,x)$. Then for any $a \in \Pi(X,x,y)$,*
$$1_x * a = a * 1_y = a.$$
2. *Given $a = [\gamma] \in \Pi(X,x,y)$, define $a^{-1} \in \Pi(X,y,x)$ as follows: write $a = [\gamma]$ with $\gamma \in P(X,x,y)$ and put $a^{-1} = [\gamma^-]$ where $\gamma^- \in P(X,y,x)$ is given by $\gamma^-(t) = \gamma(1-t)$. Then a^{-1} is well-defined and*
$$a * a^{-1} = 1_x, \ a^{-1} * a = 1_y.$$
3. *The new operation $*$ is associative:*
$$(a * b) * c = a * (b * c),$$
for all $a \in \Pi(X,x,y)$, $b \in \Pi(X,y,z)$, $c \in \Pi(X,z,u)$ (with $x, y, z, u \in X$).

PROOF. We will use the following construction: for $a, b \in \mathbb{R}$, we consider the affine function $l_{a,b}$ which sends a to 0 and b to 1:

$$l_{a,b} : [a,b] \longrightarrow [0,1],\ l_{a,b}(s) = \frac{l-a}{b-a}.$$

With the help of this, if we have a path $\gamma : [0,1] \longrightarrow X$, and we want to reparametrize it to obtain a path defined on $[a,b]$ which has the same image as γ, we will consider

$$\gamma \circ l_{a,b} : [a,b] \longrightarrow X, \gamma_{a,b}(s) = \gamma(\frac{l-a}{b-a}).$$

For the first part of the theorem, write $a = [\gamma]$, and we have to show that $c_x * \gamma \sim_p \gamma$, $\gamma * c_y \sim_p \gamma$. We prove the first one (the second one being similar). To produce paths h_t with $h_0 = \gamma$, $h_1 = c_x * \gamma$, look first at $c_x * \gamma$:

$$(c_x * \gamma)(s) = \begin{cases} x & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}.$$

We will construct H_t by considering the points $p_t \in [0,1]$ and defining $H_t|_{[0,p_t]}$ to be constant equal to x, and $H_t|_{[p_t,1]}$ to be γ reparametrized (i.e. γ_{1,p_t} described above). We need to choose p_t so that $p_0 = 0$ (so that $H_0 = \gamma$) and $p_1 = \frac{1}{2}$ (so that $H_1 = c_x * \gamma$). The simples choice is the linear one: $p_t = \frac{t}{2}$, which produces the homotopy

$$H(s,t) = H_t(s) = \begin{cases} x & \text{if } 0 \leq s \leq \frac{t}{2} \\ \gamma(\frac{2s-t}{2-t}) & \text{if } \frac{t}{2} \leq s \leq 1 \end{cases}.$$

Next, for $\gamma \in P(X,x,y)$, we have to show that $\gamma * \gamma^- \sim_p c_x$ and $\gamma^- * \gamma \sim_p c_y$. We prove the first one (the second one being similar). The intuition is the following: $(\gamma * \gamma^-)(s)$ covers $\gamma([0,1])$ when s covers the first half of $[0,1]$, and then comes back covering the same path on the second half of the interval. To construct the homotopy H_t, we consider the path which, in the first half of the interval covers $\gamma([0,t])$ and then comes back. With this intuition, we define

$$H(s,t) = \begin{cases} \gamma(2st) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma(2(1-s)t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

which realizes the desired path homotopy.

For the last part of the theorem, assume that $a = [\gamma]$, $b = [\gamma']$, $c = [\gamma'']$, and consider the paths $H_0 = (\gamma * \gamma') * \gamma''$, $H_1 = \gamma * (\gamma' * \gamma'')$. Hence

$$H_0(s) = \begin{cases} \gamma(4s) & \text{if } 0 \leq s \leq \frac{1}{4} \\ \gamma'(4s-1) & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2} \\ \gamma''(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases},$$

$$H_1(s) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma'(4s-2) & \text{if } \frac{1}{2} \leq s \leq \frac{3}{4} \\ \gamma''(4s-3) & \text{if } \frac{3}{4} \leq s \leq 1 \end{cases}.$$

Remark that both H_0 and H_1 are of the following type. One divides the interval $[0,1]$ into three intervals, by using two numbers p and q $(0 < p < q < 1)$. On the first interval, i.e. on $[0,p]$, one consider γ reparametrized by $l_{0,p}$, (which, when s goes from 0 to p, will cover the whole path γ). Next, on $[p,q]$, one considers γ' reparametrized by $l_{p,q}$ and, on $[q,1]$, γ'' reparametrized by $l_{q,1}$. Putting together these three pieces, we find a curve constructed out of the curves γ, γ' and γ'', and of the numbers p and q. With these, H_0 is obtained for

$$p = p_0 = \frac{1}{4}, q = q_0 = \frac{1}{2},$$

while H_1 is obtained for

$$p = p_1 = \frac{1}{2},\ q = q_1 = \frac{3}{4}.$$

We are looking for a homotopy consisting of paths H_t (between H_0 and H_1). For this we consider (see Figure 7)

$$p_t = \frac{t+1}{4},\ q_t = \frac{t+2}{4}$$

(the affine maps which have the described values $p_0, p_1,\ q_0, q_1$), and we consider the path H_t given by the procedure we have just described, applied to $p = p_t,\ q = q_t$. We find the desired homotopy

$$H(s,t) = H_t(s) = \begin{cases} \gamma(\frac{4s}{t+1}) & \text{if } 0 \leq s \leq \frac{t+1}{4} \\ \gamma'(4s-t-1) & \text{if } \frac{t+1}{4} \leq s \leq \frac{t+2}{4} \\ \gamma''(\frac{4s-t-2}{2-t}) & \text{if } \frac{t+2}{4} \leq s \leq 1 \end{cases}.$$

□

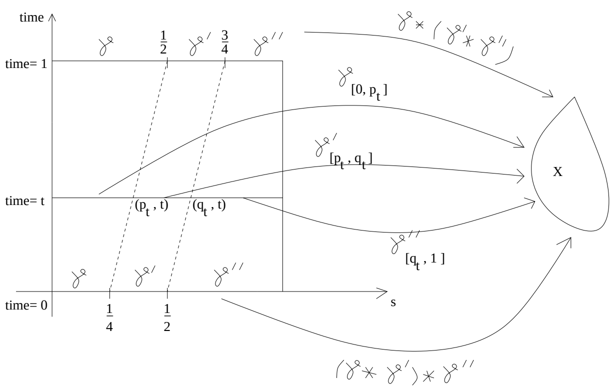


FIGURE 7.

EXERCISE 7.16. Let $\gamma : [0,1] \longrightarrow X$ be a path from x to y, and $0 = a_0 < a_1 < \ldots < a_p < a_{p+1} = 1$. For each $0 \leq i \leq p$ integer define

$$\gamma_i : [0,1] \longrightarrow X, \gamma_i(s) = \gamma((1-s)a_i + s a_{i+1}).$$

Show that

$$[\gamma] = [\gamma_p] * \ldots * [\gamma_1] * [\gamma_0].$$

REMARK 7.19 ((The homotopy groupoid)). The previous proposition too, should be viewed from the point of view of category theory. It says that, given the space X, there is an associated category $\Pi(X)$ with

- The objects of $\Pi(X)$ are the points of X.
- The set of morphisms from $x \in X$ to $y \in Y$ is $\Pi(X, x, y)$.
- The composition of morphisms comes from the concatenation of paths.

The last part of the proposition says that this category has one more property: any morphism is an isomorphism. Categories with this property are called groupoids. For this reason, $\Pi(X)$ is called the fundamental groupoid of X.

3. The fundamental group

DEFINITION 7.20. *The fundamental group of X with base point x, (X-a topological space, $x \in X$), is defined as*

$$\pi(X, x) := \Pi(X, x, x),$$

with the group structure is the one induced by the concatenation of paths.

We first discuss the functoriality of the fundamental group (what happens if we change the space?).

THEOREM 7.21. *If $f : X \longrightarrow Y$ is a continuous map and $x \in X$, then the map*

$$f_* : \pi(X, x) \longrightarrow \pi(Y, f(x)), [\gamma] \mapsto [f \circ \gamma]$$

is well defined and is a morphism of groups. If $g : Y \longrightarrow Z$ is another continuous map, then

$$g_* \circ f_* = (g \circ f)_*.$$

In particular, if $f : X \longrightarrow Y$ is a homeomorphism, then $\pi(X, x)$ is isomorphic to $\pi(Y, f(x))$, for all $x \in X$.

PROOF. The map $\gamma \mapsto f \circ \gamma$ (from paths in X to paths in Y) is compatible with path homotopies: if H is a path homotopy between γ and γ', then $f \circ f$ is a path homotopy between $f \circ \gamma$ and $f \circ \gamma'$. This implies that f_* is well defined. Next, using the definition of the concatenation of paths, we see that

$$f \circ (\gamma * \gamma') = (f \circ \gamma) * (f \circ \gamma')$$

or all $\gamma, \gamma' \in P(X, x, x)$, which implies that f_* is a group homomorphism. Given another function g, we have:

$$(g_* \circ f_*)([\gamma]) = g_*([f \circ \gamma]) = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma],$$

proving that $(g_* \circ f_*) = (g \circ f)_*$. Finally, if f is a homeomorphism, denoting by $g : Y \longrightarrow X$ its inverse and by $g_* : \pi(Y, y) \longrightarrow \pi(X, g(y))$ the induced map, where $y = f(x)$, we have $g_* \circ f_* = (g \circ f)_* = (Id_X)_* = Id$, and similarly $f_* \circ g_* = Id$, proving that f_* is an isomorphism. □

Next, what happens if we change the base point?

THEOREM 7.22. *Given a space X, and α a path in X from x to y, the map*

$$\widehat{\alpha} : \pi(X, x) \longrightarrow \pi(X, y),$$
$$\widehat{\alpha}([\gamma]) = [\alpha]^{-1} * [\gamma] * [\alpha]$$

is well defined and is an isomorphism of groups.

In particular, if X is path connected, then for any two points $x, y \in X$ the groups $\pi(X, x)$ and $\pi(X, y)$ are isomorphic.

PROOF. Let $a = [\alpha] \in \Pi(X, x, y)$. We have to show that

$$\widehat{a} : \pi(X, x) \longrightarrow \pi(X, y), \phi(u) = a^{-1} * \gamma * u$$

is a group isomorphism. First of all, note that if $b \in \Pi(Y, Z)$, then

$$\widehat{b} \circ \widehat{a} = \widehat{a * b}.$$

Indeed, for all $u \in \Pi(X, x, x)$,

$$\begin{aligned}
(\widehat{b} \circ \widehat{a})(u) &= b^{-1} * (a^{-1} * u * a) * b \\
&= (b^{-1} * a^{-1}) * u * (a * b) \\
&= (a * b)^{-1} * u * (a * b) = \widehat{a * b}(u).
\end{aligned}$$

In particular, this implies that $\widehat{a}$ is bijective with inverse $\widehat{b}$ with $b = a^{-1}$. Next, $\widehat{a}$ is a group homomorphism:

$$\begin{aligned} \widehat{a}(u) * \widehat{a}(v) &= (a^{-1} * u * a) * (a^{-1} * v * a) \\ &= a^{-1} * u * (a * a^{-1}) * v * a \\ &= a^{-1} * u * 1_y * v * a \\ &= a^{-1} * (u * v) * a = \widehat{a}(u * v). \end{aligned} \tag{3.1}$$

When X is path connected, then for any two points $x, y \in X$ we can find a path γ from x to y, and then $\hat{\gamma}$ will provide an isomorphism between $\pi(X, x)$ and $\pi(X, y)$. □

Finally, we show that two spaces which are homotopic equivalent (Definition 7.1) have isomorphic fundamental groups.

THEOREM 7.23. *If $f : X \longrightarrow Y$ is a homotopy equivalence then, for any $x_0 \in X$, the induced map*

$$f_* : \pi(X, x_0) \longrightarrow \pi(Y, f(x_0))$$

is an isomorphism of groups.

PROOF. Let $g : Y \longrightarrow X$ be a homotopy inverse of f (see Definition 7.1), $x_0 \in X$, and the maps

$$f_* : \pi(X, x_0) \longrightarrow \pi(Y, f(x_0)),\ \ g_* : \pi(Y, f(x_0)) \longrightarrow \pi(X, g(f(x_0)))$$

the maps induced in the fundamental groups. It suffices to show that $g_* f_*$ and $f_* g_*$ are isomorphism. Indeed, the injectivity of $g_* f_*$ implies that f_* is injective, while the surjectivity of $f_* g_*$ implies that f_* is surjective (show this!).

Due to the symmetry, it suffices to show that $g_* f_* = (gf)_*$ is bijective. Let $h = gf$. We are in the following situation: we have a continuous function $h : X \longrightarrow X$ which is homotopic to the identity map, $x_0 \in X$, and we want to prove that

$$h_* : \pi(X, x_0) \longrightarrow \pi(X, h(x_0))$$

is an isomorphism. Since h is homotopic to Id_X, we find

$$H : X \times [0, 1] \longrightarrow X$$

such that $H(x, 0) = x$, $H(x, 1) = h(x)$ for all $x \in X$. Consider

$$\alpha : [0, 1] \longrightarrow X, \alpha(s) = H(x_0, s).$$

This is a path from x_0 to $h(x_0)$, hence the previous theorem (Theorem 7.22) provides us with a group isomorphism

$$\widehat{\alpha} : \pi(X, x_0) \longrightarrow \pi(X, h(x_0)),\ \ [\gamma] \mapsto [\alpha]^{-1} * [\gamma] * [\alpha].$$

We will show that $h_* = \widehat{\alpha}$. Given $[\gamma] \in \pi(X, x_0)$, we have to check that

$$[h \circ \gamma] = [\alpha]^{-1} * [\gamma] * [\alpha].$$

Using the properties of the concatenation (see Proposition 7.18), we have to check that

$$[\alpha] * [h \circ \gamma] = [\gamma] * [\alpha],$$

i.e. that the paths $\gamma * \alpha$ and $\alpha * (h \circ \gamma)$ are path homotopic. For this, we define

$$G : [0, 1] \times [0, 1] \longrightarrow X,$$

$$G(s, t) = \begin{cases} H(\alpha(2st), 2s(1-t)) & \text{if } 0 \leq s \leq \frac{1}{2} \\ H(\alpha(1 - 2(1-s)(1-t)), 1 - 2(1-s)t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}.$$

One checks directly that G is a path homotopy between $\gamma * \alpha$ and $\alpha * (h \circ \gamma)$ (check it!). □

EXERCISE 7.17. Explain (on a picture) the formula for the homotopy G given in the previous proof.

EXAMPLE 7.24.

- If X is a contractible space, the $\pi(X, x)$ is trivial (i.e. consisting only of the identity element 1_x) for all $x \in X$. In particular,
$$\pi(\mathbb{R}^n, x) = 0.$$
- Consider the inclusion $i : S^{n-1} \longrightarrow \mathbb{R}^n - \{0\}$. Since this is a homotopy equivalence (Example 7.8), we deduce that we have an isomorphism
$$i_* : \pi_1(S^{n-1}, x) \xrightarrow{\sim} \pi_1(\mathbb{R}^n - \{0\}, x),$$
for all $x \in S^{n-1}$. We will see that the left hand side equals to $\mathbb{Z}$ if $n = 1$, and is trivial otherwise.
- Consider the Moebius band M. Inside it, one finds a copy of the circle S^1, and, as in the previous example, the inclusion $S^1 \longrightarrow M$ is a homotopy equivalence (Exercise 7.3). We deduce that the homotopy group of M is isomorphic to the homotopy group of S^1.

DEFINITION 7.25. *We say that a space X is simply connected if $\pi(X, x)$ is trivial for any $x \in X$. We say that X is 1-connected if it is path connected and simply connected.*

EXERCISE 7.18. *We are not quite ready to prove that the sphere S^2 (and S^n for all $n \geq 2$) is simply connected. However, using the stereographic projection, try to explain this result.*

EXERCISE 7.19. Show that, for any two topological spaces X and Y, and $x \in X$, $y \in Y$, one has an isomorphism of groups:
$$\pi(X \times Y, (x, y)) \cong \pi(X, x) \times \pi(Y, y).$$

4. Covering spaces and the homotopy group of S^1

So far, we know that the homotopy group of the spaces $\mathbb{R}^n$ (or any contractible space) is trivial. In this section we will compute the fundamental group of the circle; more precisely, we will show that it isomorphic to the cyclic group in one generator, i.e. to $(\mathbb{Z}, +)$, and we will produce an explicit isomorphism
$$\deg : \pi(S^1, 1) \longrightarrow \mathbb{Z}.$$
The idea behind this map is quite simple: it associates to the (equivalence class) of a path γ (starting and ending at $1 \in S^1$) its rotation number, i.e. the total number of rotation of S^1, counted in a the counterclockwise direction (so that a path that goes once around the circle, but on the clockwise direction, has degree -1). There are some standard paths: for each n there is the path γ_n which rotates n times around the circle (and has degree n). In general, any path γ in S^1 satrting end nding at $1 \in S^1$ is path homotopic to the path γ_n for $n = \deg(\gamma)$. Intuitively this can be explained as follows. Interpret it as a rope which goes around a rigid S^1. It may go for some time in one direction, then turn back for a while, then change direction again, etc. See Figure 8 for a path of degree two. Hold now the piece of rope by its ends, and start pulling it, as long as it is possible. Note that the pulling process does not change the degree. We can keep on pulling untill all the "turnings" of the rope will be smoothened out, and we will end up with γ_n. The process of pulling defines the homotopy between γ and γ_n.

The remaining part of this section will make these idea more precise. In particular, we will introduce some tools (covering spaces) which may be used in various other situations. The basic idea is to relate S^1 to $\mathbb{R}$, by the map
$$p : \mathbb{R} \longrightarrow S^1, p(t) = (cos(2\pi t), sin(2\pi t)).$$

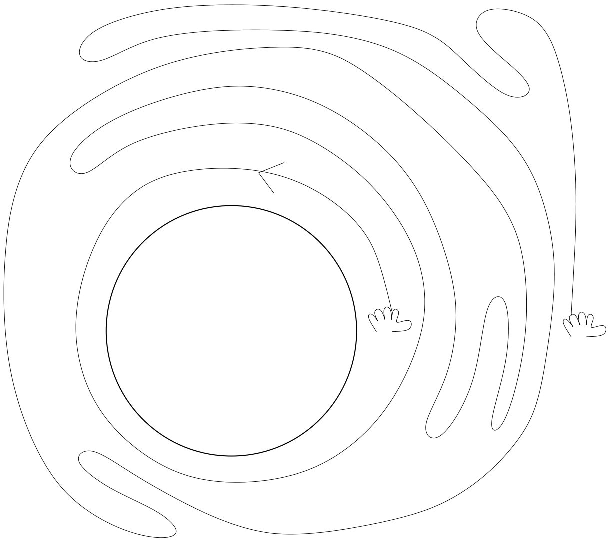


FIGURE 8.

We will use the base point $1 \in S^1$. Note that $p^{-1} = \mathbb{Z}$. It is useful to picture this map as follows: spiral $\mathbb{R}$ above the circle (on a cylinder), as shown in Figure 9, and then p is just the projection into the circle. In particular, we see that the pre-image p^{-1} of small opens $U \subset S^1$ will consist of a disjoint family of copies of U. It is precisely this property that allows us to compute $\pi(S^1, 1)$. What we will actually do is to look at more general maps which have this property (which

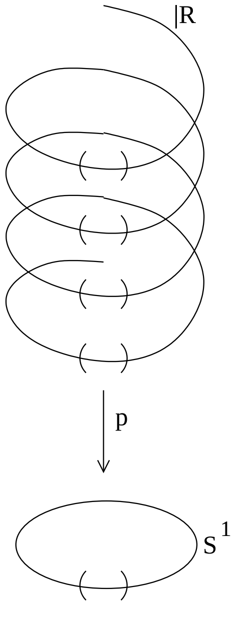


FIGURE 9.

will be called covering maps), and extract the relevant information about fundamental groups. Returning to our example, we will be able to conclude the computation of $\pi(S^1, 1)$.

DEFINITION 7.26. *Let $p : E \longrightarrow B$ be a continuous map. Given $U \subset B$ an open subset, a partition of $p^{-1}(U)$ into slices is a family $\{V_i\}_{i \in I}$ of opens in E such that*

- *$\{V_i\}$ is a partition of $p^{-1}(U)$;*
- *$p|_{V_i} : V_i \longrightarrow U$ is a homeomorphism for each $i \in I$.*

A covering map is a continuous surjective map $p : E \longrightarrow B$ with the property that each point $b \in B$ has an open neighborhood U such that $p^{-1}(U)$ admits a partition into slices.

We also say that p is a covering of B, or that E is a covering space of B with covering projection p.

Exercise 7.20. Show that $p : \mathbb{R} \longrightarrow S^1$ is a covering map.

Exercise 7.21. *View the unit circle as*

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

Show that for any integer $n \geq 1$, the map

$$f_n : S^1 \to S^1, f_n(z) = z^n.$$

is a covering map.

Proposition 7.27. *Let $p : E \longrightarrow B$ be a covering map, and let $b_0 \in B$. Consider*

- *a path $\gamma : [0, 1] \longrightarrow B$ starting at b_0.*
- *a lift e_0 of b_0, i.e. $e_0 \in p^{-1}(b_0)$.*

Then there exists an unique path $\tilde{\gamma} : [0, 1] \longrightarrow E$ such that

- *$\tilde{\gamma}$ starts at e_0;*
- *$\tilde{\gamma}$ is a lift of γ, i.e. $p \circ \tilde{\gamma} = \gamma$.*

Proof. First we prove the uniqueness. Assume that $\tilde{\gamma}$ and $\tilde{\tilde{\gamma}}$ are two lifts of γ starting at the same point e_0. We consider

$$S = \{s \in [0, 1] : \tilde{\gamma}|_{[0,s]} = \tilde{\tilde{\gamma}}\tilde{\tilde{\gamma}}\},$$

and we prove that S is both closed and open in $[0, 1]$this will imply that $S = [0, 1]$, proving the uniqueness.

We first show that S is open. Let $s_0 \in S$. Choose an open neighborhood U of $\gamma(s_0)$ such that $p^{-1}(U) = \cup V_i$ is a partition into slices. Choose i such that $\tilde{\gamma}(s_0) \in V_i$. Since $\tilde{\gamma}$ and $\tilde{\tilde{\gamma}}$ are continuous, we find a neighborhood $D = (s_0 - \epsilon, s_0 + \epsilon) \cap [0, 1]$ of s_0 in $[0, 1]$ such that

$$\tilde{\gamma}(D) \subset V_i,\ \tilde{\tilde{\gamma}}(D) \subset V_i.$$

Since $p|_{V_i} : V_i \longrightarrow U$ is a homeomorphism and $p \circ \tilde{\gamma} = p \circ \tilde{\tilde{\gamma}}$, we deduce that $\tilde{\gamma}|_D = \tilde{\tilde{\gamma}}|_D$, which implies that $D \subset S$.

We now show that S is closed. Let $s_0 \in \overline{S}$, and we show that $s_0 \in S$. For any $s < s_0$, $(s, s_0) \cap S \neq \emptyset$, which implies that $\tilde{\gamma}(s) = \tilde{\tilde{\gamma}}(s)$ (for all $s < s_0$). We still have to show that this equality also holds for $s = s_0$. Assume it does not, and let $e = \tilde{\gamma}(s_0)$, $e' = \tilde{\tilde{\gamma}}(s_0)$. Let U be as above, with $p^{-1}(U) = \cup V_i$, and choose i and j such that $e \in V_i$, $e' \in V_j$. Note that, due to the assumption $e \neq e'$ and the fact that $p|_{V_i}$ is a homeomorphism, we must have $i \neq j$, hence $V_i \cap V_j = \emptyset$. On the other hand, due to continuity, we find a neighborhood D of s_0 such that

$$\tilde{\gamma}(D) \subset V_i,\ \tilde{\tilde{\gamma}}(D) \subset V_j.$$

But this contradicts the fact that $\tilde{\gamma}(s) = \tilde{\tilde{\gamma}}(s)$ for all $s < s_0$. This concludes the proof of the fact that S is closed.

Before we prove the existence part, let us point out the following consequence.

COROLLARY 7.28. *Let $p : E \longrightarrow B$ be a covering map, and let $f : X \longrightarrow B$ be a continuous function defined on a space X. A lift of f is a continuous map $\tilde{f} : X \longrightarrow E$ such that $p\tilde{f} = f$.*

If X is path connected, then any two lifts of f which coincide in the same point of X must coincide everywhere.

PROOF. Let $x_0 \in X$ such that $\tilde{f}(x_0) = \tilde{\tilde{f}}(x_0)$. For $x \in X$ arbitrary, choose a path α from x_0 to x. Then $\tilde{f} \circ \alpha$ and $\tilde{\tilde{f}} \circ \alpha$ are two lifts of $f \circ \alpha$ which coincide at the initial point. By the uniqueness proven above, they coincide everywhere; in particular, the end points (i.e. $\tilde{f}(x)$ and $\tilde{\tilde{f}}(x)$) must coincide. □

We now return to the proof of the proposition (existence part). Let $S' \subset [0,1]$ be the set of those s with the property that $\gamma|_{[0,s]}$ admits a lift starting at e_0. One can proceed as above and prove that S' is open and closed in $[0,1]$. We give here a slightly different argument. Remark first that if $s \in S'$, then all $t < s$ are in S'. This implies that $S' = [0, s_0]$ or $S' = [0, s_0)$ for some $s_0 \in [0,1]$. Also, we can find a lift $\tilde{\gamma} : [0, s_0) \longrightarrow E$ of $\gamma|_{[0,s_0)}$. To see this, we may assume $s_0 \neq 0$, and then $s_0 - \frac{1}{n} \in S'$ for n large enough. Choose $\tilde{\gamma}_n : D_n \longrightarrow E$ lifts of $\gamma|_{D_n}$ starting at e_0 defined on $D_n = [0, s_0 - \frac{1}{n}]$. The uniqueness proven above implies that $\tilde{\gamma}_n$ and $\tilde{\gamma}_m$ coincide on $D_n \cap D_m$ for all n and m, hence we can define $\tilde{\gamma}$ on $D_0 = [0, s_0)$ so that $\tilde{\gamma}|_{D_n} = \tilde{\gamma}_n$ for all n, and this will be the desired lift.

Hence $S' = [0, s_0]$ or $S' = [0, s_0)$ for some $s_0 \in [0,1]$, and γ has a lift $\tilde{\gamma}$ on $[0, s_0)$, starting at e. Let U be an open neighborhood of $\gamma(s_0)$ such that $p^{-1}(U) = \cup V_i$ is a partition into slices. Choose ϵ such that $D = (s_0 - \epsilon, s_0 + \epsilon) \cap [0,1]$ is sent by γ inside U. Consider also $D' = (s_0 - \epsilon, s_0)$. Since continuous maps send connected spaces to connected spaces, $\tilde{\gamma}(D')$ must be connected; but it lies in the disjoint union of the V_i's, hence we find i such that

$$\tilde{\gamma}(D') \subset V_i.$$

Since $p|_{V_i}$ is a homeomorphism, we must have

$$\tilde{\gamma}(s) = (p|_{V_i})^{-1}(\gamma(s))$$

for all $s \in D'$. But we can take this formula as the definition of $\tilde{\gamma}$ also for $s \in D$, and we obtain a lift $\tilde{\gamma}$ defined on D. Hence

$$D = (s_0 - \epsilon, s_0 + \epsilon) \cap [0,1] \subset S'.$$

Recalling that $S' = [0, s_0]$ or $S' = [0, s_0)$, we see that the only possibility is when $S = [0,1]$. □

REMARK 7.29. In particular, we obtain a map

$$\phi_{e_0} : P(B, b_0, b_0) \longrightarrow p^{-1}(b_0) \tag{4.1}$$

which associates to a path γ the end-point $\tilde{\gamma}(1)$ of its lift evaluated at the end point.

EXAMPLE 7.30. Apply the previous lemma to the covering map $p : \mathbb{R} \longrightarrow S^1$ and $e_0 = 0$. For any path in S^1 starting and ending at $1 \in S^1$ we find a path $\tilde{\gamma}$ in $\mathbb{R}$ starting at 0 such that

$$\gamma(t) = (cos(2\pi\tilde{\gamma}(t)), sin(2\pi\tilde{\gamma}(t))).$$

Note that $\tilde{\gamma}(1) \in p^{-1}(0) = \mathbb{Z}$.

DEFINITION 7.31. *Given a path γ in S^1 starting and ending in $1 \in S^1$, we define the degree of γ by*

$$deg(\gamma) := \tilde{\gamma}(1) \in \mathbb{Z}.$$

For instance, for each $n \in \mathbb{Z}$, the path which goes around the circle n times:

$$\gamma_n(t) = (cos(2\pi nti), sin(2\pi nti)),$$

has as lift

$$\tilde{\gamma}_n(t) = nt,$$

hence

$$\deg(\gamma_n) = n.$$

Our next aim is to show that the degree (or, more generally, the map (4.1)) only depends on path homotopy classes of paths.

LEMMA 7.32. *Let $p : E \longrightarrow B$ be a covering map, $b_0 \in B$, $e_0 \in E$ a lift of b_0. Let $\gamma, \gamma' \in P(B, b_0, b_0)$, and consider their lifts $\tilde{\gamma}$, $\tilde{\gamma}'$ which start at e_0. If γ and γ' are path homotopic, then $\tilde{\gamma}$ and $\tilde{\gamma}'$ have the same end-point and they are path homotopic.*

PROOF. Let $H : [0,1] \times [0,1]$ be a path homotopy between γ and γ'. Let $\gamma_t(s) = H(s,t)$ (so that $\gamma_0 = \gamma$, $\gamma_1 = \gamma'$). For each t we consider the lift $\tilde{\gamma}_t$ of γ_t starting at e_0 (which, by the previous proposition, exists and is unique). Put $\tilde{H}(s,t) = \tilde{\gamma}_t(s)$. Locally, H is the inverse of the restriction of p to a slice, composed with H, from which one deduces that H is continuous (fill in the details!). On the other hand, $t \mapsto \tilde{H}(1,t)$ is a lift of the constant path $H(1,t) = b_0$, hence will be constant (use the uniqueness part of previous proposition). In conclusion, $\tilde{H}$ is a path homotopy between $\tilde{\gamma}_0 = \tilde{\gamma}$ and $\tilde{\gamma}_1 = \tilde{\gamma}'$. □

REMARK 7.33. Continuing the previous remark, the map (4.1) will define a map

$$\phi_{e_0} : \pi(B, b_0) \longrightarrow p^{-1}(b_0).$$

In particular, for the covering map $p : \mathbb{R} \longrightarrow S^1$ we find that the notion of degree induces a map

$$\deg : \pi(S^1, 1) \longrightarrow \mathbb{Z}.$$

THEOREM 7.34. *$deg : \pi(S^1, 1) \longrightarrow \mathbb{Z}$ is an isomorphism of groups.*

PROOF. We have already seen that $\deg(\gamma_n) = n$, where $\gamma_n(t) = p(nt)$ for $t \in [0,1]$. Hence deg is surjective. To show it is injective, assume that

$$\deg(\gamma) = \deg(\gamma').$$

This means that, choosing lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ as above (starting at 0), $\tilde{\gamma}(1) = \tilde{\gamma}'$. But we know that $\mathbb{R}$ is simply connected- any two paths which start and end at the same point are path homotopic. Hence $\tilde{\gamma}$ and $\tilde{\gamma}'$ are path homotopic, which implies that γ and γ' are path homotopic (in explicit formulas, $\tilde{H}(s,t) = (1-t)\tilde{\gamma}(s) + t\tilde{\gamma}'(s)$ is a path homotopy between $\tilde{\gamma}$ and $\tilde{\gamma}'$, while $H(s,t) = p(H(s,t))$ is a path homotopy between γ and γ'.

Hence deg is bijective. To see it is a group homomorphism it suffices to check that $\deg(\gamma_n * \gamma_n) = \deg(\gamma_n) + \deg(\gamma_m)$ for all $n, m \in \mathbb{Z}$. By the definition of concatenation, we have

$$(\gamma_n * \gamma_m)(t) = \begin{cases} p(2nt) & \text{if } 0 \le t \le \frac{1}{2} \\ p(m(2t-1)) & \text{if } \frac{1}{2} \le t \le 1 \end{cases},$$

which has as lift

$$t \mapsto \begin{cases} 2nt & \text{if } 0 \le t \le \frac{1}{2} \\ m(2t-1) + n & \text{if } \frac{1}{2} \le t \le 1 \end{cases},$$

hence

$$\deg(\gamma_n * \gamma_m) = m(2 \cdot 1 - 1) + n = m + n = \deg(\gamma_n) + \deg(\gamma_m).$$

□

EXERCISE 7.22. *Consider again* $f_n : S^1 \longrightarrow S^1$, $f_n(z) = z^n$ *and consider the induced map*

$$(f_n)_* : \pi(S^1, 1) \to \pi(S^1, 1).$$

After identifying $\pi(S^1, 1)$ *with* $(\mathbb{Z}, +)$, *show that* $(f_n)_*$ *is identified with*

$$mult_n : \mathbb{Z} \to \mathbb{Z}, \quad r \mapsto nr.$$

EXERCISE 7.23. *If* $p : E \longrightarrow B$ *is a covering map defined on a simply connected space* E, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, *then*

$$\phi_{e_0} : \pi(B, b_0) \longrightarrow p^{-1}(b_0)$$

is a bijection.

EXERCISE 7.24. *Let* $f : [0, 1] \longrightarrow S^1$ *be a continuous map with* $f(0) = f(1) = 1$. *If* $deg(f) = n$, *prove that the equation*

$$f(x) = 1$$

has at least $n + 1$ *solutions.*

COROLLARY 7.35. *There are no continuous retractions of* D^2 *into* S^1 *(i.e. continuous maps* $r : D^2 \longrightarrow S^1$ *such that* $r(x) = x$ *for all* $x \in S^1$*).*

PROOF. Assume such a retraction $r : D^2 \longrightarrow S^1$ exists. Let $i : S^1 \longrightarrow D^2$ be the inclusion. Consider the maps induced in the fundamental groups

$$\pi(S^1, 1) \xrightarrow{i_*} \pi(D^2, 1) \xrightarrow{r_*} \pi(S^1, 1).$$

Since $r \circ i = \mathrm{Id}_{S^1}$, we have $r_* \circ i_* = \mathrm{Id} : \pi(S^1, 1) \longrightarrow \pi(S^1, 1)$. But $\pi(D^2, 1) = 0$ (trivial) because D^2 is contractible, hence $\mathrm{Id} = r_* i_*$ will be constant. But this can only happen if $\pi(S^1, 1)$ was trivial- which is not. Hence r does not exist. $\square$

COROLLARY 7.36. *(Brouwer fixed point theorem) Any continuous function* $f : D^2 \longrightarrow D^2$ *has at least one fixed point.*

PROOF. Assume that $f : D^2 \longrightarrow D^2$ has no fixed point. For each $x \in D^2$, since $f(x) \neq x$, we can talk about the line through x and $f(x)$. This line intersects S^1 in two points, and let $r(x)$ be the point with the property that x lies between $r(x)$ and $f(x)$. Then $r : D^2 \longrightarrow S^1$ will be a retraction of of D^2 into S^1- contradiction. $\square$

Related to the last corollary, here is one exercise.

EXERCISE 7.25. *In general, given a topological space* X *and a subset* $A \subset X$, *a retraction of* X *into* A *is a continuous map* $r : X \longrightarrow A$ *with the property that* $r(a) = a$ *for all* $a \in A$.

Prove that if r *is a retraction of* X *into* A, *then the map induced in the fundamental groups:*

$$r_* : \pi(X, a) \longrightarrow \pi(A, a)$$

is surjective for all $a \in A$. *What about the map induced by the inclusion of* A *into* X*?*

EXERCISE 7.26. *Prove that the Moebius band cannot be retracted into its boundary.*