CHAPTER 8

The Seifert- van Kampen theorem

1. The statement and first explanations

The Seifert- van Kampen theorem allows us to compute the fundamental group of a space by breaking it into pieces.

THEOREM 8.1. *(Seifert- van Kampen theorem) Let X be a topological space and assume that $X = U \cup V$ with $U, V \subset X$ opens such that $U \cap V$ is path connected and let $x_0 \in U \cap V$. We consider the a commutative diagram*

$$
\begin{tikzcd}
& \pi(U, x_0) \arrow[dr, "j_1"] & \\
\pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi(X, x_0) \\
& \pi(V, x_0) \arrow[ur, "j_2"'] &
\end{tikzcd}
$$

where all the maps are induced by the inclusions. Then, for any group H and any group homomorphisms $\phi_1 : \pi(U, x_0) \longrightarrow H$, $\phi_2 : \pi(V, x_0) \longrightarrow H$ such that

$$\phi_1 i_1 = \phi_2 i_2,$$

there exists and is unique a group homomorphism $\phi : \pi(X, x_0) \longrightarrow H$ such that $\phi_1 = \phi j_1$, $\phi_2 = \phi j_2$.

$$
\begin{tikzcd}
& \pi(U, x_0) \arrow[dr, "j_1"] \arrow[drrr, "\phi_1"] & & & \\
\pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi(X, x_0) \arrow[rr, "\phi"] & & H \\
& \pi(V, x_0) \arrow[ur, "j_2"] \arrow[urrr, "\phi_2"'] & & &
\end{tikzcd}
$$

1.1. Explanation. Although it may not be clear from the statement, the theorem describes the way that $\pi(X, x_0)$ is determined by $\pi(U, x_0)$, $\pi(V, x_0)$ and $\pi(U \cap V, x_0)$. To understand this, we will look at the property described in the theorem in a slightly more general setting.

Starting data: Start with (N, G_1, G_2, i_1, i_2) where N, G_1 and G_2 are groups and $i_1 : H \longrightarrow G_1$, $i_2 : N \longrightarrow G_2$ are group homomorphisms:

(1.1)
$$\begin{tikzcd} & G_1 \\ N \arrow[ur, "i_1"] \arrow[dr, "i_2"] & \\ & G_2 \end{tikzcd}$$

Consider: triples (G, j_1, j_2) consisting of a group G and group homomorphisms $j_1 : G_1 \longrightarrow G$, $j_2 : G_2 \longrightarrow G$ such that $j_1 i_1 = j_2 i_2$, i.e. "commutative fillings" of the starting data (1.1):

$$\begin{tikzcd} & G_1 \arrow[dr, "j_1"] & \\ N \arrow[ur, "i_1"] \arrow[dr, "i_2"] & & G \\ & G_2 \arrow[ur, "j_2"] & \end{tikzcd}$$

Definition 8.2. *We say that a triple (G, j_1, j_2) as above satisfies the universal property (with respect to the starting data (1.1)) if for any other such triple (H, ϕ_1, ϕ_2), there exists and is unique a group homomorphism $\phi : G \longrightarrow G'$ such that $\phi_1 = \phi j_1$, $\phi_2 = \phi j_2$:*

$$\begin{tikzcd} & G_1 \arrow[dr, "j_1"] \arrow[drr, "\phi_1"] & & \\ N \arrow[ur, "i_1"] \arrow[dr, "i_2"] & & G \arrow[r, "\phi"] & H \\ & G_2 \arrow[ur, "j_2"] \arrow[urr, "\phi_2"] & & \end{tikzcd}$$

We now show that the universal property determines G uniquely up to isomorphism.

Lemma 8.3. *If (G, j_1, j_2) and (G', j_1', j_2') are two triples which satisfy the universal property with respect to the starting data (1.1), then there exists (and is unique) an isomorphism of groups*

$$\phi : G \xrightarrow{\sim} G'$$

such that $j_1' = \phi j_1$, $j_2' = \phi j_2$

Proof. Due to the universal property of G, there exists a unique homomorphism $\phi : G \longrightarrow G'$ satisfying the equations in the lemma. We have to prove that it is an isomorphism. Interchanging G and G' (and applying the universal property for G'), we find a unique homomorphism $\phi : G' \longrightarrow G$ satisfying $j_1 = \phi j_1'$, $j_2 = \phi j_2'$. We will prove that $\phi'\phi = \mathrm{Id}_G$. Due to the universal property of G (the uniqueness part!), it suffices to check that

$$(\phi'\phi) j_1 = j_1, (\phi'\phi) j_2 = j_2,$$

which is immediate from the identities defining ϕ and ϕ'. Hence $\phi'\phi = \mathrm{Id}_G$ and, similarly, $\phi\phi' = \mathrm{Id}_{G'}$, proving that ϕ is an isomorphism. $\square$

Definition 8.4. *The amalgamated free product associated to the starting data (1.1) is a triple (G, j_1, j_2) satisfying the universal property. We also say that G is the free product of G_1 and G_2 over N (but keep in mind the maps involved!).*

REMARK 8.5. Hence, what the Seifert- van Kampen tells us is that $\pi(X, x_0)$ is uniquely determined by $\pi(U, x_0)$, $\pi(V, x_0)$ and $\pi(U \cap V, x_0)$, as the amalgamated free product associated to the starting data

$$\begin{tikzcd} & \pi(U, x_0) \\ \pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & \pi(V, x_0) \end{tikzcd}$$

Back to the general discussion, we know from the previous lemma that the amalgamated free product is unique up to isomorphism. One can also prove that it exists (for any starting data (1.1)), but the explicit *general* construction is not allways useful. Instead, it willbe more interesting to look at particular cases first (Cases A-D below). But first, let us reformulate the Seifert- van Kampen theorem using this new language.

THEOREM 8.6. *(Seifert- van Kampen theorem reformulated) Let X be a topological space and assume that $X = U \cup V$ with $U, V \subset X$ opens such that $U \cap V$ is path connected and let $x_0 \in U \cap V$. We consider the a commutative diagram*

$$\begin{tikzcd} & \pi(U, x_0) \arrow[dr, "j_1"] & \\ \pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi(X, x_0) \\ & \pi(V, x_0) \arrow[ur, "j_2"'] & \end{tikzcd}$$

where all the maps are induced by the inclusions. Then $\pi(X, x_0)$, together with the maps j_1 and j_2, is the amalgamated free product associated to $\pi(U, x_0)$, $\pi(V, x_0)$ and $\pi(U \cap V, x_0)$ (with maps i_1, i_2).

2. The case $G_1 = G_2 = \{1\}$

Hence assume that G_1 and G_2 are trivial. Then all the maps involved (i_1, i_2, j_1 etc) are the constant maps and all the equations are automatically satisfied. Hence we are looking for a group G with the property that for any other group H, there exists a unique group homomorphism $\phi : G \longrightarrow H$. Of course, $G = \{1\}$ does the job.

COROLLARY 8.7. *(Corollary A) Let X, U, V, x_0 be as in the Seifert-van Kampen theorem. If $\pi(U, x_0)$ and $\pi(V, x_0)$ are trivial, then so is $\pi(X, x_0)$.*

In particular, we deduce

COROLLARY 8.8. $\pi(S^n, p) = \{1\}$ *for all $n \geq 2$ and all $p \in S^n$.*

PROOF. Choose $U = S^n - \{p_N\}$ and $V = S^n - \{p_S\}$, where p_N is the north pole, p_S is the south pole. We know that U and V are homeomorphic to $\mathbb{R}^n$ (by the stereographic projection), hence they have trivial fundamental groups. Also, since $n \geq 2$, $U \cap V$ is path connected, hence Corollary A applies. □

EXERCISE 8.1. Use the covering projection $S^n \longrightarrow \mathbb{P}^n$ to deduce that $\pi(\mathbb{P}^n, p)$ is isomorphic to $\mathbb{Z}_2$ for all $n \geq 2$ and $p \in \mathbb{P}^n$.

3. The case $N = G_2 = \{1\}$

Look now at the case when N and G_2 are trivial. Also in this case there is an obvious solution to the universal property: namely $G = G_1$, $j_1 = \mathrm{Id}$ (while j_2 can only be one thing: the trivial map).

Corollary 8.9. *(Corollary B) Let X, U, V, x_0 be as in the Seifert-van Kampen theorem. If $\pi(V, x_0)$ and $\pi(U \cap V, x_0)$ are trivial, then the map induced by inclusion*

$$i_1 : \pi(U, x_0) \longrightarrow \pi(X, x_0)$$

is an isomorphism of groups.

Using this, we will prove the following result whose importance comes from the fact that it shows that cells of dimension higher then three do not affect the fundamental group.

Proposition 8.10. *Let X be space which is obtained from a path connected subspace $A \subset X$ by adjoining an n-cell e with $n \geq 3$. Then, for any $a \in A$, the inclusion $i : A \longrightarrow X$ induces an isomorphism of groups:*

$$i_* : \pi(A, a) \longrightarrow \pi(X, a).$$

Before we prove the proposition let us mention the following consequence which follows after applying the proposition repeatedly to eliminate cells of dimension ≥ 3.

Corollary 8.11. *If X is a compact, path connected space then, for any cell decomposition of X, the inclusion of the 2-skeleton X_2 into X, $i : X_2 \longrightarrow X$, induces an isomorphism of groups (for any base point $x \in X_2$):*

$$i_* : \pi(X_2, x) \xrightarrow{\sim} \pi(X, x).$$

Proof. (of the Proposition 8.10) Let $h : D^n \longrightarrow X$ be the defining map for e, so that the restriction to the open ball is a homeomorphism onto e, and the restriction to $S^{n-1} = \partial D^n$ is the characteristic map of e, $\chi : S^{n-1} \longrightarrow A$. We consider

$$U = X - \{h(0)\}, V = e.$$

Remark that

(1) V is simply connected, because it is homeomorphic to $\overset{\circ}{D}{}^n$ (via h).
(2) $U \cap V$ is simply connected. Indeed, $U \cap V = h(\overset{\circ}{D}{}^n - \{0\})$ is homeomorphic to $\overset{\circ}{D}{}^n - \{0\}$ (via h) which, in turn, is homotopic equivalent to S^{n-1}. Since S^{n-1} has trivial fundamental group for $n - 1 \geq 2$, the same will be true for $U \cap V$.
(3) U is homotopic equivalent to A. More precisely, the inclusion
$$i : A \longrightarrow U$$
is a homotopy equivalence (Exercise 7.14). In particular, the map induced in the fundamental groups,
$$k_* : \pi(A, a) \longrightarrow \pi(U, a)$$
is an isomorphism for all $a \in A$.

Due to the first two remarks, we can apply Corollary B, hence

$$j_1^x : \pi(U, x) \longrightarrow \pi(X, x)$$

is an isomorphism for all $x \in U \cap V = e - \{h(0)\}$. We use the super-script "x" to indicate that we use x as a base point. We claim that the same holds also for base points $a \in A$ (i.e. for j_1^a). So, let $a \in A$. Choose $x \in U \cap V$. There exists a path α in U starting at a and ending at x. Indeed, since $\overline{e} - \{h(0)\} = h(D^n - \{0\})$, this space is path connected, hence we can join x with any point $y \in \partial e = \chi(S^{n-1}) \subset A$ by a path inside $\overline{e} - \{h(0)\} \subset U$. On the other hand, since

A is path connected, y can be connected with a by a path inside $A \subset U$. Using the path α to change the base point (from a to x), we have a commutative diagram:

$$\begin{tikzcd} \pi(U,a) \arrow[r, "j_1^a"] \arrow[d, "\widehat{\alpha}"'] & \pi(X,a) \arrow[d, "\widehat{\alpha}"'] \\ \pi(U,x) \arrow[r, "j_1^x"] & \pi(X,x) \end{tikzcd}$$

(the commutativity follows immediately from the definition of j_1 and $\widehat{\alpha}$. Since both vertical maps $\widehat{\alpha}$, as well as j_1^a, are isomorphisms, from $\widehat{\alpha} j_1^a = j_1^x \widehat{\alpha}$ it follows that

$$j_1^a = \widehat{\alpha}^{-1} j_1^x \widehat{\alpha}$$

is an isomorphism. Composing with the map k_* (see the third remark above), the statement follows. $\square$

4. The case $G_2 = \{1\}$

Let us now look at a slightly more general case, when only G_2 is assumed to be trivial (this is what we need if we want to have an analogue of Proposition 8.10 in the case of two-cells). Assume first that N is a subgroup of G_1 and

$$i_1 : N \hookrightarrow G_1$$

is the inclusion map. In this case j_2 must be trivial and the equation $j_1 i_1 = j_2 i_2$ becomes $j_1 i_1 = 1$, i.e.

$$j_1(n) = 1, \quad \forall\, n \in N.$$

Recall that for any group homomorphism j_1, its kernel is defined by

$$\mathrm{Ker}(j_1) := \{g \in g_1 : j_1(g) = 1\}.$$

Hence, what we are looking at are pairs (G, j_1), with $j_1 : G_1 \longrightarrow G$ a group homomorphism with the property that $N \subset \mathrm{Ker}(j_1)$. The universal property reads: for any other such pair (H, ϕ_1), there exists a unique group homomorphism $\phi : G \longrightarrow H$ such that $\phi_1 = \phi j_1$. The existence of an universal pair (G, j_1) brings us to some elementary constructions on groups which we now recall:

- For a subgroup N of G_1 and $g \in G_1$, the N-coset defined by g is
$$gN = \{gn : n \in N\} \subset G_1.$$
These subsets (when g varies) form a partition of G_1. The quotient of G_1 by N is the set
$$G_1/N := \{gN : g \in G_1\}.$$
Equivalently, the action of N on G_1 defines an equivalence relation on G_1:
$$g_1 \sim g_2 \Longleftrightarrow \exists \;\; n \in N : \; g_2 = g_1 n,$$
the orbit through $g \in G_1$ is precisely the coset gN, hence
$$G_1/N = G_1/\sim .$$
- A subgroup N of G_1 is called a normal subgroup if
$$gng^{-1} \in N, \quad \forall \;\; g \in G_1, n \in N.$$
The importance of this notion is that, in this case, G_1/N can be made into a group with the multiplication defined by:
$$(gN) \cdot (g'N) = (gg')N$$

(if N was not normal, this multiplication would not have been well defined). Denoting by j_1 the quotient map

$$j_1 : G_1 \longrightarrow G_1/N,\ g \mapsto gN,$$

one knows from group theory (and it is an easy exercise- see also below) that $(G_1/N, j_1)$ has precisely the universal property that we are looking for.

- Let's now go back to the case where N is a subgroup of G_1 which is not necessarily normal. Since arbitrary intersections of normal subgroups of G_1 is a normal subgroup, we can talk about the smallest normal subgroup of G_1 containing N (which is the intersection of all normal subgroups of G_1 containing N), which we will denote by $\overline{N}$. Since the kernel $\mathrm{Ker}(j_1)$ of any group homomorphism $j_1 : G_1 \longrightarrow G$ is a normal group, we have:

$$j_1 \circ i_1 = 1 \Longleftrightarrow N \subset \mathrm{Ker}(j_1) \Longleftrightarrow \overline{N} \subset \mathrm{Ker}(j_1).$$

Let us now consider the general case when $i_1 : N \longrightarrow G_1$ is a group homomorphism (not necessarily an inclusion). In this case $i_1(N)$ is a subgroup of G_1 and we can apply the discussion above to $i_1(N)$.

LEMMA 8.12. *Consider the starting data (1.1) with $G_2 = \{1\}$. Let $i_1(N) \subset G_1$ be the image of i_1 (a subgroup of G_1), let $\overline{i_1(N)}$ be the smallest normal subgroup of G_1 containing $i_1(N)$ and consider the quotient group and the associated quotient map*

$$\pi : G_1 \longrightarrow G_1/\overline{i_1(N)}.$$

Then $(G_1/\overline{i_1(N)}, \pi, 1)$ has the universal property with respect to the starting data (1.1) with $G_2 = 1$.

Another triple $(G, j_1, 1)$ has the universal property with respect to this data if an only if

$$j_1 : G_1 \longrightarrow G$$

is surjective with kernel equal to $\overline{i_1(N)}$. In particular j_1 will induce a group isomorphism

$$G \cong G_1/\overline{i_1(N)}.$$

PROOF. The universal property with respect to the starting data (1.1) is equivalent to the universal property with respect to:

$$\begin{tikzcd} & G_1 \\ i_1(N) \arrow[ur, "\mathrm{incl}"] \arrow[dr, "i_2"'] & \\ & \{1\} \end{tikzcd}$$

(where "incl" is the inclusion). Hence we may assume that N is a subgroup of G_1 and $i_1 : N \longrightarrow G_1$ is the inclusion.

Let (H, ϕ_1) be an arbitrary pair with $\phi_1 i_1 = 1$. We have remarked that this implies

$$\overline{N} \subset \mathrm{Ker}(\phi_1).$$

We have to show that there exists an unique homomorphism $\phi : G_1/\overline{N} \longrightarrow H$ such that $\phi_1 = \phi\pi$. Explicitly, this equation translates into:

$$\phi(g\overline{N}) = \phi_1(g)$$

for all $g \in G_1$. This equation determines ϕ uniquely: it proves that ϕ will be unique, and provides us with the formula defining ϕ. We still have to check that this ϕ is well defined, i.e. we still have to show that, if $g'\overline{N} = g''\overline{N}$ then $\phi_1(g') = \phi_1(g'')$. Taking $n = g'{-}1g''$, we have $n \in \overline{N}$ and we have to show that $\phi_1(n) = 1$, and this follows from the inclusion we mentioned at the beginning of the proof. The last part of the lemma follows from the uniqueness insured by Lemma 8.3. □

COROLLARY 8.13. *(Corollary C) Let X, U, V, x_0 be as in the Seifert-van Kampen theorem. If $\pi(V, x_0)$ is trivial, then the map*

$$j_1 : \pi(U, x_0) \longrightarrow \pi(X, x_0)$$

is surjective and it induces an isomorphism of groups

$$\pi(U, x_0)/\overline{Im(i_1)} \xrightarrow{\sim} \pi(X, x_0),$$

where $\overline{Im(i_1)}$ is the smallest normal subgroup of $\pi(U, x_0)$ containing the image of $i_1 : \pi(U \cap V, x_0) \longrightarrow \pi(U, x_0)$.

We can now prove the following version of Proposition 8.10 in the case of two-cells.

PROPOSITION 8.14. *Assume that X is obtained from A by adjoining a two-cell e with characteristic map $\chi_e : S^1 \longrightarrow A$. Consider $a = \chi_e(1)$, and denote by $\alpha \in \pi(A, a)$ the class of the path $t \mapsto \chi_e(e^{2\pi ti})$. Then the map induced by inclusion $i : A \longrightarrow X$,*

$$i_* : \pi(A, a) \longrightarrow \pi(X, x)$$

is surjective and has as kernel the smallest normal subgroup of $\pi(A, a)$ containing α, denoted by $\langle\alpha\rangle$. In particular, there is an isomorphism of groups:

$$\pi(X, a) \cong \pi(A, a)/\langle\alpha\rangle.$$

PROOF. We proceed like in the proof of Proposition 8.10. First of all, we use the same notations, and the Remarks 1 and 3 apply also to this case. In particular we can apply Corollary C to deduce that, for any $x \in U \cap V = e - \{h(0)\}$, we have a sequence of maps:

$$\pi(U \cap V, x) \xrightarrow{i_1^x} \pi(U, x) \xrightarrow{j_1^x} \pi(X, x)$$

with j_1^x- surjective with the kernel equal to the smallest normal subgroup containing the image of i_1^x. Take $x = h(v_0)$, where $v_0 \in \overset{\circ}{D}{}^2 - \{0\}$. Since $U \cap V$ is homeomorphic to $\overset{\circ}{D}{}^2 - \{0\}$ which is homotopic equivalent $D^2 - \{0\}$ (the last two spaces are both homotopic equivalent to the circle), we find that the image of i_1^x coincides with the image of the map induced by $h : D^2 - \{0\} \longrightarrow U$,

$$h_* : \pi(D^2 - \{0\}, v_0) \longrightarrow \pi(U, x).$$

We now proceed again as in the proof of Proposition 8.10. Take a path α_0 in $D^2 - \{0\}$ going from 1 to v_0, and consider the path $\alpha = h \circ \alpha_0$- a path in U going from $a == h(1)$ to $x = h(v_0)$. We then have a commutative diagram

$$\begin{tikzcd}
\pi(D^2 - \{0\}, 1) \arrow[r, "h_*"] \arrow[d, "\widehat{\alpha_0}"'] & \pi(U, a) \arrow[r, "j_1^a"] \arrow[d, "\widehat{\alpha}"'] & \pi(X, a) \arrow[d, "\widehat{\alpha}"'] \\
\pi(D^2 - \{0\}, v_0) \arrow[r, "h_*"] & \pi(U, x) \arrow[r, "j_1^x"] & \pi(X, x)
\end{tikzcd}$$

where all the vertical maps are isomorphisms. From the similar property of the bottom line, we deduce that the upper line has the property that j_1^a is surjective with the kernel equal to the smallest normal subgroup containing the image of h_*. But, since $S^1 \hookrightarrow D^2 - \{0\}$ is a homotopy equivalence, $\pi(D^2 - \{0\}, 1)$ is the free group in one generator $[\gamma_1]$, where $\gamma_1(t) = e^{2\pi ti}$. Hence the image of

$$h_* : \pi(D^2 - \{0\}, 1) \longrightarrow \pi(U, a)$$

coincides with the group generated by

$$h_*([\gamma_1]) = [h \circ \gamma_1] \in \pi(U, a).$$

Finally, by Remark 3,

$$k_* : \pi(A, a) \longrightarrow \pi(U, a)$$

is an isomorphism, and, from the definition of α in the statement, we have

$$k_*(\alpha) = [h \circ \gamma_1],$$

hence i_* is surjective and has as kernel the subgroup generated by α. □

Example 8.15. Consider the torus $T = X/\sim$, where X is the square and $\sim$ is the equivalence relation which identifies the opposite sides of ∂X (see Section 5). Recall (see Example 6.11) that T is obtained from a bouquet of two circles by attaching a 2-cell (see also the picture which comes with that example). The bouquet of two circles arises as $\partial X/\sim$, and we view it as a subspace of the torus as shown in the picture (Figure 1), where a labels the first circle $S^1 \vee \{p\} \subset T$ and b labels the second circle $\{p\} \vee S^1 \subset T$. The attaching map comes from the homeomorphism $\tilde{\chi} : S^1 \longrightarrow \partial X$ which transforms the circle to the boundary of the square (e.g. by taking 4 points on the circle and pulling them apart straightening the arcs between the points). The attaching map itself is $\chi = \pi \circ \tilde{\chi}$ where $\pi_0 : \partial X \longrightarrow \partial X/\sim = S^1 \vee S^1$ is the quotient map. In other words, when $x \in S^1$ goes one time around the circle in the counterclockwise direction starting at q, during the first quarter of the circle $\chi(x)$ goes around the first circle $S^1 \vee \{p\} \subset T$ (in the direction of a), during the next quarter $\chi(x)$ covers the second circle $\{p\} \vee S^1 \subset T$ (in the direction of b), during the third quarter $\chi(x)$ goes again around the first circle $S^1 \vee \{p\} \subset T$ but in the direction opposite to a, and in the last quarter it goes again around the second circle $\{p\} \vee S^1 \subset T$ (in the direction opposite to b). Symbolically, we write $\chi(S^1) = aba^{-1}b^{-1}$.

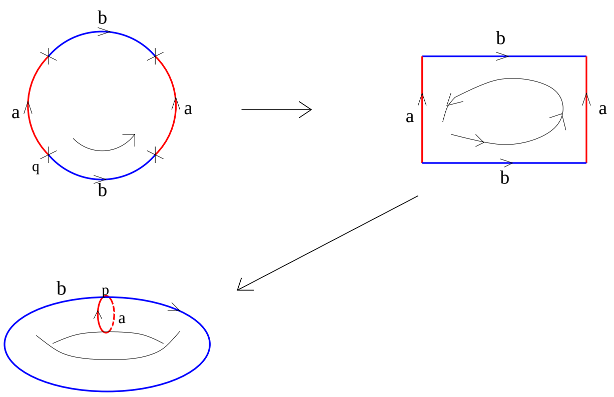


Figure 1.

We denote by the same letters a and b the elements

$$a, b \in \pi(T, p)$$

represented by the paths in T which go once around the two circles $S^1 \vee \{p\}$ and $\{p\} \vee S^1$, in the direction indicated in the picture. We see that the path defined by χ in $\pi(T, p)$ is precisely

$$\alpha = aba^{-1}b^{-1},$$

hence

$$\pi(T, p) = \pi(S^1 \vee S^1)/\langle aba^{-1}b^{-1}\rangle.$$

We will complete this computation in Example 8.20 below.

5. The case $N = \{1\}$

Assume now that $N = \{1\}$. The way to recognize the groups which have the universal property is described in the following lemma.

LEMMA 8.16. *Consider the starting data (1.1) with $N = \{1\}$. Then a triple (G, j_1, j_2) has the universal property if and only if, for any $g \in G$, there exists and are unique elements*

$$a_1, \dots, a_n \in G_1,\ b_1, \dots, b_n \in G_2$$

with $a_i \neq 1$ for all $i \geq 2$ and $b_j \neq 1$ for all $j \leq n-1$, such that

$$g = j_1(a_1)j_2(b_1)\dots j_n(a_n)j_n(b_n). \tag{5.1}$$

Before we prove this lemma, we show how to construct a group with this property. We start by considering "words in G_1 and G_2", i.e. sequences

$$w = (g_1)(g_2)\dots(g_n), \tag{5.2}$$

where each g_i is either in G_1 or in G_2. To avoid confusion and/or too complicated notations, we assume that G_1 and G_2 are disjoint (otherwise, if $a \in G_1 \cap G_2$, we would have to make the distinction between the word which contains a as an element of G_1, and the word which contains a as an element of G_2). One can realize this by taking a copy of G_2 which is disjoint from G_1.

We also allow the empty word

$$w_\emptyset = (\ ).$$

For any two words w and w', one can consider a new word, ww', which is made from w and w' put next to each other (juxtaposition).

A word (5.2) is called reduced if, for each i

- g_i is neither the unit of G_1, nor the unit of G_2.
- g_i and g_{i+1} are not in the same group.

Starting with an arbitrary word w, one can always produce a reduced word w_{red} by applying repeatedly the following operations

- If g_i is the unit of G_1 or of G_2, delete it (and the parenthesis around it) from the w.
- If g_i and g_{i+1} belong to the same group, then replace w by

$$(g_1)\dots(g_{i-1})(g_ig_{i+1})(g_{i+2})\dots(g_n).$$

We now define

$$G_1 * G_2 := \{w : w \text{ is a reduced word in } G_1 \text{ and } G_2\},$$

which is a group with:

- the multiplication of two words given by

$$w * w' := (ww')_{\mathrm{red}}.$$

- the unit element $w_\emptyset = (\ )$ (the empty word).

There are two group homomorphisms,

$$k_1 : G_1 \longrightarrow G, k_2 : G_2 \longrightarrow G$$

sending an element $g \in G_1$ or $g \in G_2$ to the word of length one (g).

We can now improve Lemma 8.16 as follows:

LEMMA 8.17. *Consider the starting data (1.1) with $N = \{1\}$. Then $(G_1 * G_2, k_1, k_2)$ has the universal property.*

For any other triple (G, j_1, j_2) which has the universal property with respect to this data, one has a group isomorphism

$$\phi : G_1 * G_2 \longrightarrow G,$$

uniquely determined by the condition that it sends $g_1 \in G_1$ to $j_1(g_1) \in G$ and $g_2 \in G_2$ to $j_2(g_2) \in G$.

PROOF. (of Lemma 8.16 and of Lemma 8.17). We first show that if (G, j_1, j_2) has the property described in Lemma 8.16, then it satisfies the universal property. This will apply in particular to $(G_1 * G_2, k_1, k_2)$ (which clearly has that property,). So, starting with (H, ϕ_1, ϕ_2), we have to find $\phi : G \longrightarrow H$ such that $\phi_1 = \phi j_1$, $\phi_2 = \phi j_2$. This means that

$$\phi(j_1(g_1)) = \phi_1(g_1), \phi(j_2(g_2)) = \phi_2(g_2)$$

for all $g_1 \in G_1$, $g_2 \in G_2$. For $g \in G$ arbitrary, consider its unique decomposition (5.1), we must have

$$\phi(g) = \phi_1(a_1)\phi_2(b_1)\ldots\phi_1(a_n)\phi_2(b_n).$$

This formula determines ϕ: it proves the uniqueness of ϕ, and also provides us with the defining formula. One still has to show that ϕ is a group homomorphism, which is left as an exercise.

The last part of the Lemma 8.17 follows from the uniqueness insured by Lemma 8.3. This also implies the remaining part of Lemma 8.16, namely that if (G, j_1, j_2) has the universal property, then any g has a unique decomposition as in the lemma. Indeed, using the isomorphism $\phi : G_1 * G_2 \longrightarrow G$, it suffices to prove the similar property for $(G_1 * G_2, k_1, k_2)$, which is clear from the construction of the free product. $\square$

COROLLARY 8.18. *(Corollary D) Let X, U, V, x_0 be as in the Seifert-van Kampen theorem. If $\pi(U \cap V, x_0)$ is trivial, then $\pi(X, x_0)$ is isomorphic to the free product of $\pi(U, x_0)$ and $\pi(V, x_0)$. More precisely, there is a unique group homomorphism*

$$\phi : \pi(U, x_0) * \pi(V, x_0) \longrightarrow \pi(X, x_0)$$

which sends $g_1 \in \pi(U, x_0)$ to $j_1(g_1)$ and $g_2 \in \pi(V, x_0)$ to $j_2(g_2)$, and ϕ is an isomorphism.

REMARK 8.19. (generators and relations): To be able to write down some of the groups we obtain, it is useful to represent groups by generators and relations.

First of all, starting with $G_1 = F(a_1)$-the free group (= the infinite cyclic group) with generator a_1:

$$F(a) = \{(a_1)^n : n \in \mathbb{Z}\},$$

and G_2-the free group with generator a_2, the free product $G_1 * G_2$ is the free group with two generators, denoted $F(a, b)$:

$$F(a_1, a_2) = F(a_1) * F(a_2).$$

The free group in three generators is defined similarly

$$F(a_1, a_2, a_3) = F(a_1, a_2) * F(a_3).$$

Inductively, one defines the free group generated by finite number of generators. More generally, for any set S, we can define the free group generated by S, denoted by

$$F(S).$$

This means that we have generators a_s, one for each $s \in S$, and each element of $g \in F(S)$ different from the identity element can be uniquely written as a product

$$g = a_{s_1}^{n_1} \ldots a_{s_p}^{n_p},$$

with $p \in \mathbb{Z}$ positive integer, $n_1, \dots, n_p \in \mathbb{Z}$ non-zero, and $s_1, \dots, s_p \in S$ with $s_i \neq s_{i+1}$ for all i.

Given a subset $R \subset F(S)$, we denote by

$$\langle R \rangle \subset F(S)$$

the smallest normal subgroup of $F(S)$ containing R (the intersection of all normal subgroups of $F(S)$ containing R), and we will consider the quotient:

$$F(S)/\langle R \rangle.$$

By considering this quotient, what we actually do is to force the elements of R to be trivial. To write a group as such a quotient $F(S)/\langle R \rangle$ is known as describing the group by generators (the elements of S) and relations (the elements of R).

A group can be represented by generators and relations in many ways. For instance, $(\mathbb{Z}^2, +)$ can be written as

$$F(a, b)/\langle aba^{-1}b^{-1} \rangle, \; F(a, b, c)/\langle aba^{-1}c, bc \rangle.$$

On the other hand, any group G can be realized in this way, i.e. there exist S and R such that G is isomorphic to $F(S)/\langle R \rangle$ (the problem is that there are many such choices). For instance, one could take $S = G$, $R = \{a_{gg'} a_{g'}^{-1} a_g^{-1} : g, g' \in G\} \cup \{a_1\}$.

Example 8.20. Let us consider a bouquet of two circles $X = S^1 \vee S^1$, and let $p \in X$ be the common point of the two circles. See Figure 2. Let U_0 be a small neighborhood of p in the

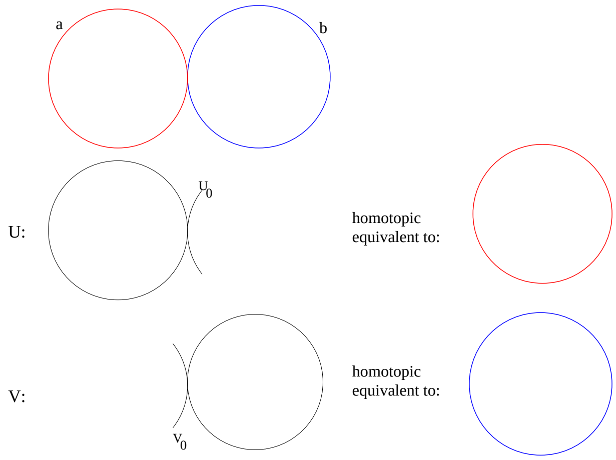


Figure 2.

second circle, let V_0 be a small neighborhood of p in the first circle, and let

$$U = S^1 \vee U_0, \; V = V_0 \vee S^1.$$

The intersection $U \cap V$ is contractible to p, hence we can apply the previous result. Note also that the inclusion

$$f_1 : S^1 = S^1 \vee \{p\} \hookrightarrow U$$

is a homotopy equivalence. Passing to fundamental groups, since $\pi(S^1)$ is free in one generator, we find that $\pi(U, p) = \langle a \rangle$ is the free group in one generator a, where a is the homotopy class of the path which starts and ends at p going around the first circle once. Similarly, $\pi(V, p) = \langle b \rangle$ is the free group in one generator b, where b is the homotopy class of the path which starts and ends at p going around the second circle once. We deduce from the previous corollary that $\pi(S^1 \vee S^1, p)$

$$\pi(S^1 \vee S^1, p) = F(a, b)$$

is the free group in two generators a and b shown in the picture.

We can now complete the computation of the fundamental group of the torus started in Example 8.15. We find

$$\pi(T^2, p) = F(a, b)/\langle aba^{-1}b^{-1}\rangle$$

which is the commutative group in two generators, hence isomorphic to $\mathbb{Z}^2$.

EXAMPLE 8.21. Consider $Y = S^1 \vee S^1 \vee S^1$ a bouquet of three circles. We can use the previous example in two ways: both the result obtained there, as well as the idea for the computation. First of all, take $X = S^1 \vee S^1 \vee \{p\}$ (p is the common point of the circles), $U = S^1 \vee S^1 \vee U_0$, $V = W_0 \vee V_0 \vee S^1$, with U_0, V_0, W_0-small neighborhoods of p. The intersection $U \cap V$ is contractible. The inclusion $X \hookrightarrow U$ is a homotopy equivalence, hence $\pi(U, p) = \langle a_1, a_2\rangle$ is the free group in two generators: a_1-represented by the path which goes once around the first circle, and a_2-represented by the path which goes once around the second circle. Similarly, $\pi(V, p) = \langle a_3\rangle$ is the free group in one generator with a_3-represented by the path which goes once around the third circle. Applying the previous proposition, we find $\pi(Y, p) = \langle a_1, a_2, a_3\rangle$. Applying repeatedly this argument, we find the fundamental group of any finite bouquet of circles:

$$\pi(\underbrace{S^1 \vee \ldots \vee S^1}_{n-\text{times}}, p) = F(a_1, \ldots, a_n),$$

the free group in n generators.

6. The general case

To explain the general case (i.e. when no restriction are made on the starting data (1.1)), we will represent our groups using generators and relations (explained in Remark 8.19). The Seifert- van Kampen theorem takes the following form.

COROLLARY 8.22. *Let X, U, V, x_0 be as in the Seifert-van Kampen theorem. Assume that*

$$\pi(U, x_0) = F(S_1)/\langle R_1\rangle,$$
$$\pi(V, x_0) = F(S_2)/\langle R_2\rangle$$
$$\pi(U \cap V, x_0) = F(S)/\langle R\rangle,$$

where S_1 and S_2 are chosen to be disjoint. For each $s \in S$, choose $f_s \in F(R_1)$ such that

$$i_1(a_s) = f_s\langle R_1\rangle \in F(S_1)/\langle R_1\rangle,$$

and similarly choose $g_s \in F(R_2)$ such that $i_2(a_s) = g_s\langle R_2$. Consider

$$R' = \{f_s g_s^{-1} : s \in S\} \subset F(S_1 \cup S_2).$$

Then

$$\pi(X, x_0) = F(S_1 \cup S_2)/\langle R_1 \cup R_2 \cup R'\rangle.$$

7. Some more exercises

Here are some more exercises about the fundamental group.

EXERCISE 8.2. Prove that

$$X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \neq 0\}$$

is not homeomorphic to $\mathbb{R}^3$. Also, compute the fundamental group of X.

EXERCISE 8.3. Compute the fundamental group of the spaces appearing in Figures 8 and 9 of Section 4, Chapter 6.

EXERCISE 8.4. Compute the fundamental group of the spaces appearing in Figures 6 of Chapter 7, Section 1.

EXERCISE 8.5. Compute the fundamental group of the spaces appearing in Figure 10 of Section 4, Chapter 6. Do the same for the spaces from Figure 13 of the same section.

EXERCISE 8.6. Compute the fundamental group of all the spaces appearing in Exercises 6.22, 4.21, 6.24, 6.25.

EXERCISE 8.7. Compute the fundamental group obtained by removing one point from the torus.

EXERCISE 8.8. Compute the fundamental group obtained by removing two points from the sphere. Similarly for the torus instead of the sphere.

EXERCISE 8.9. Go back to exercise 6.26, and compute the fundamental group of the double torus.

EXERCISE 8.10. Compute the fundamental group of the Klein bottle.

EXERCISE 8.11. Find a topological space whose fundamental group is isomorphic to G in each of the following cases:

(i) $G = \mathbb{Z}_n$.
(ii) $G = \mathbb{Z}_n \times \mathbb{Z}_m$.
(iii) $G = \mathbb{Z}_n * \mathbb{Z}_m$.

where n and m are arbitrary positive integers.
(Hint: look at exercise 6.27).

EXERCISE 8.12. Let x_0 be a point in the torus p. We know that any homeomorphism $f : T \longrightarrow T$ such that $f(x_0) = x_0$ induces a group isomorphism $f_* : \pi(T, x_0) \longrightarrow \pi(T, x_0)$. Show the converse: any group isomorphism arises in this way (for some f).

EXERCISE 8.13. Find a map $f : T \longrightarrow S^1$ which is not null-homotopic (i.e. is not homotopic to the constant map). Similarly for the Moebius band instead of the torus.

EXERCISE 8.14. Show that any map $f : S^n \longrightarrow S^1$ is null-homotopic, when $n \geq 2$.

CHAPTER 9

How to compute fundamental groups: a summary and examples

1. Summary and some example

So: how does one compute fundamental groups? Assume one wants to compute the fundamental group of a space X. Recall that, if X is path connected, one may choose any (convenient) point as the base point (cf. Theorem 7.22). Here are some possible steps.

1.1. Step 0:

1.1. Step 0: Try to see if your space is homotopic equivalent to a space X' which is simpler or whose fundamental group you already know. Then you can replace X by X' (cf. Theorem 7.23).

1.2. Step 1: Try to find a cell decomposition of X. Then you can throw away the n-cells with $n \geq 3$ (cf. Proposition 8.10 or Corollary 8.11).

Cell decompositions can be found by looking at the picture. Another useful way to find a cell decomposition is to realize your space as a quotient space obtained from the square or the disc, by identifying certain points on the boundary: then one can use (maybe repeatedly) Lemma 6.9 which we now recall (in slightly different notations): Let X be a Hausdorff space which is obtained from D^n (or any other space homeomorphic to D^n) by identifying certain points on $\partial D^n = S^{n-1}$ and let $A = \partial D^n / \sim$. We denote by

$$\chi : S^{n-1} = \partial D^n \longrightarrow \partial D^n / \sim = A$$

the quotient map. Then X is obtained from A by attaching an n-cell whose characteristic map is χ.

The case $n = 2$ is particularly important: many spaces X can be obtained from D^2 by identifying certain parts of $\partial D^2 = S^1$, and the identification can be shown on the picture by labeling by letters the parts that are to be identified. In the quotient $A = \partial D^2 / \sim$, each letter will appear only once (because we identified all the parts labeled by the same letter). When going once around the circle, we will meet various labels that will give us a word whose letters are labels. Reading this word in the space A describes the characteristic map $\chi : S^1 \longrightarrow A$. We have already seen this in the case of the torus (see Figure 1): in that case A was a bouquet of two circles A and B (two circles touching each other in one point), and the characteristic map $\chi : S^1 \longrightarrow A$ gave us the word $aba^{-1}b^{-1}$, which we can read in the picture of A to describe the characteristic map itself.

1.3. Step 2: You have reduced the problem to the computation of the fundamental group of the 2-skeleton X_2 (what remains after throwing away the n-cells with $n \geq 3$). Try to apply Step 0 to X_2. Otherwise, use Proposition 8.14 to get rid of the 2-cells. Applying the proposition repeatedly for each 2-cell, you will find that the fundamental group you are interested in is isomorphic to the quotient of $\pi(X_1)$ by the smallest normal subgroup generated by paths induced by the characteristic maps of the 2-cells.

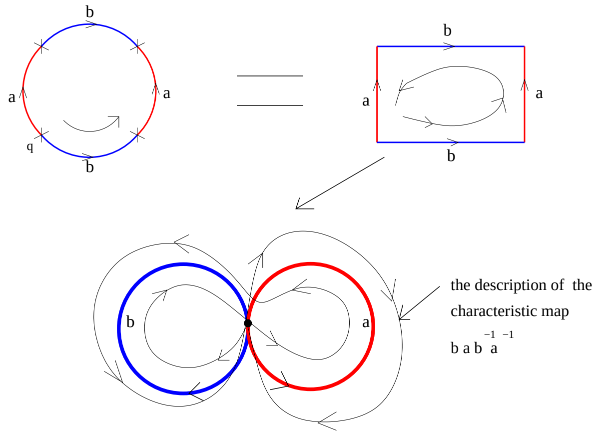


FIGURE 1.

1.4. Step 3: Hence you reduced the computation to the computation of the fundamental group of X_1. Try to apply Step 0 again. Otherwise, choose U and V inside X_1 so that you can apply Corollary D.

One should be aware that these steps should not be followed blindly- also think about simplifying them! For instance, the use of covering maps (e.g. of Exercise 7.23), of the fundamental group of the product (Exercise 7.19), or the trick of taking out points (Exercise 7.14) should not be underestimated.

Here are a few spaces whose fundamental group has been computed:

- The fundamental group of spaces like $\mathbb{R}^n$, D^n, $\mathring{D}^n$, or any other convex space in $\mathbb{R}^n$ is zero. In particular, since $S^n - \{p\}$ (a sphere minus a point) is homeomorphic to $\mathbb{R}^n$, its fundamental group is zero as well.
- The fundamental group of S^n is zero for all $n \geq 2$.
- The fundamental group of S^1 is the free group in one generator (hence isomorphic to $\mathbb{Z}$), where the generator is induced by the path $t \mapsto e^{2\pi t i}$. In particular, since spaces like
$$\mathbb{R}^2 - \{0\}, D^2 - \{0\}, \mathring{D}^2 - \{0\}$$
are all homotopic equivalent to S^1, the fundamental group of all such spaces is isomorphic to $(\mathbb{Z}, +)$.
- The fundamental group of a bouquet of n circles is the free group in n generators, each generator being defined by the path which goes once around one of the n circles.
- Various other spaces are homotopy equivalent to bouquets of circles. For instance -as we have seen- the torus T from which we remove one point is homotopic equivalent to a bouquet of two circles, hence its fundamental group is isomorphic to the free group in two generators.
- The fundamental group of the torus is isomorphic to $\mathbb{Z}^2$. This has been explained in this chapter using the Seifert van Kampen theorem.

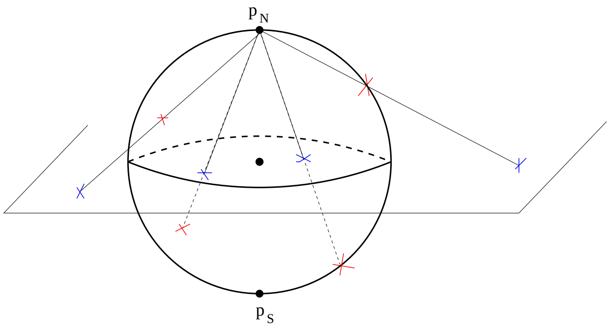
EXAMPLE 9.1. The fundamental group of $S^2 \times S^2$ can be computed using Exercise 7.19, and the result is the trivial group.

EXAMPLE 9.2. We have seen the computation of the fundamental group of the torus based on the Seifert van Kampen theorem. The advantage of this proof is that the same idea applies to many other examples. However, since the torus is homeomorphic to the product of two circles, it suffices to use Exercise 7.19 and immediately get the result.

Example 9.3. Consider X obtained from the sphere by removing two points:

$$X = S^2 - \{p_N, p_S\}.$$

Since S^2 minus a point is homeomorphic to $\mathbb{R}^2$, X is homeomorphic to $\mathbb{R}^2 - \{0\}$ (by stereographic projection, see Figure 2), hence it is homotopic equivalent to the circle $S^1 \subset X$ (homotopy that can be seen directly). Hence the fundamental group of X (let's say with base point $(1, 0, 0)$) is isomorphic to $(\mathbb{Z}, +)$, with the generator induced by the path that goes once around the middle circle.



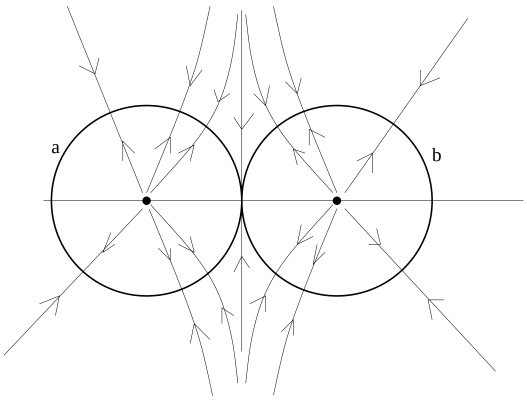
By stereographic projection (sending the red points to the blue ones), the spehere minus two points is homeomorphic to the plane minus the origin, hence it is homotopic equivalent to the midle circle

Figure 2.

Example 9.4. Consider the Euclidean plane from which we remove two points. For instance, take

$$X = \mathbb{R}^2 - \{(-1, 0), (0, 1)\}.$$

Similar to the fact that $\mathbb{R}^2$ minus a point is homotopic equivalent to the circle, we have already seen that X is homotopic equivalent to a bouquet of two circles (Figure 3). We conclude that $\pi(X, 0)$ is a free group in two generators a and b shown in the picture.



$\mathbb{R}^2$ minus two points is homotopic equivalent to a bouquet of two circles

Figure 3.

2. The Moebius band

Let us look at the Moebius band M. We have already remarked that M is homotopic equivalent to S^1 (with S^1 sitting inside M as the middle circle, hence the homotopy group of M is isomorphic to $(\mathbb{Z}, +)$.

Let us compute this group differently, using a cell-decomposition of M. Since M can be described as a quotient of the square, as mentioned above, we can use Lemma 6.9 to find a cell decomposition of M. But that is precisely what we have done in the second part of the Example 6.12. The conclusion is that M is obtained from the space B shown in Figure 4 by adjoining a 2-cell, with characteristic map described in the picture. Hence $\pi(M)$ is isomorphic to $\pi(B)/\langle\alpha\rangle$, where α is the path which the characteristic map follows ("$cab^{-1}a$"). On the other hand, by

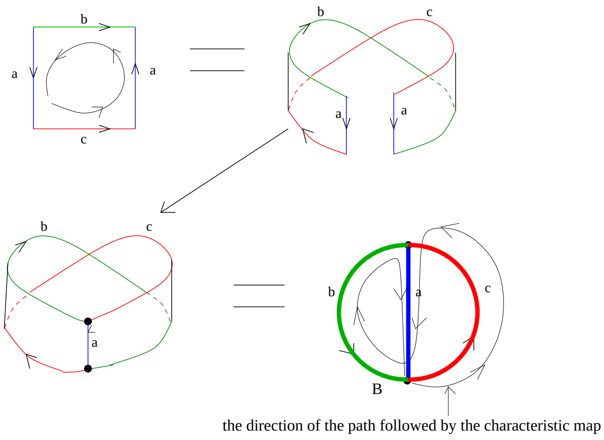


FIGURE 4.

collapsing "a" in the picture, we see that B is homotopic equivalent to a bouquet of two circles ("b" and "c"), and α is sent to cb^{-1}. Hence the group we are interested in is isomorphic to

$$F(c, b)/\langle cb^{-1}\rangle$$

the quotient obtained from the free group in two generators c and b by imposing $cb^{-1} = 1$, i.e. the free group in one generator.

3. The double torus

Consider the double torus T_2 (Figure 5). It can be obtained as a quotient of D^2 in the following way (indicated already in the last two exercises of Section 5). First cut the double torus by a circle in the middle. We obtain two "cut tori", where a cut torus is obtained from a torus from which we remove a small open ball. Going backward, i.e. identifying the two cut torus along their boundary circles as shown in the picture, gives us back T_2. This is the description of T_2 as the "connected sum $T\#T$" (to be explained below).

So, let's first look at the cut torus T_c. Since the torus is obtained from the square by identifying its opposite points, a cut torus can be obtained from a square from which we remove a small ball e as shown in the picture (Figure 6), followed by the identification of its opposite sides. In turn, $[0, 1] \times [0, 1] - e$ can be cut at the origin $(0, 0)$ to produce a pentagon. Going back from the pentagon to the cut-tours amounts to making the identifications described by the labeling in the picture (the c-side is not identified with any other segment!). This describes the cut-torus as obtained from te pentagon by identifying its sides according to the labeling.

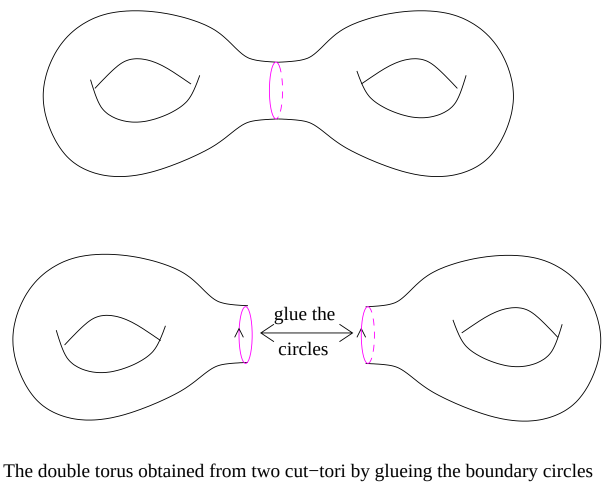


FIGURE 5.

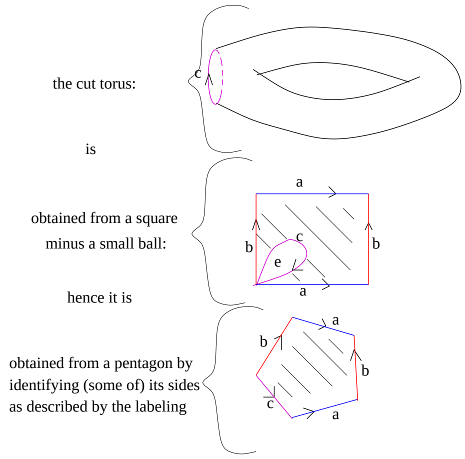


FIGURE 6.

Back to the double torus, we apply the previous construction to one of the cut torus, and we obtained a pentagon whose sides are labeled by a_1, b_1, c. We do the same for the other cut torus, and we obtain another pentagon whose sides are labelled by a_2, b_2, c. See Figure 22. The label c is the same because it represents the initial cut in the double torus; so, when glueing back, we also have to glue the two copies of c. However, one can start with gluing the two copies of c first, and leave the other identifications for later. This produces an octogone with the sides labelled as in the picture (Figure 7), and this gives a description of T_2 as obtained from an octagon by identifying its sides according to the labeling. To compute the fundamental group, we continue as before (e.g. as in the case of the torus, of the Moebius band, or of $\mathbb{P}^2$). We find that T_2 is obtained by adjoining to a bouquet of four circles (labeled a_1, b_1, a_2, b_2):

$$S^1 \vee S^1 \vee S^1 \vee S^1$$

a 2-cell with characteristic map $\chi : S^1 \longrightarrow S^1 \vee S^1 \vee S^1 \vee S^1$ described by

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}.$$

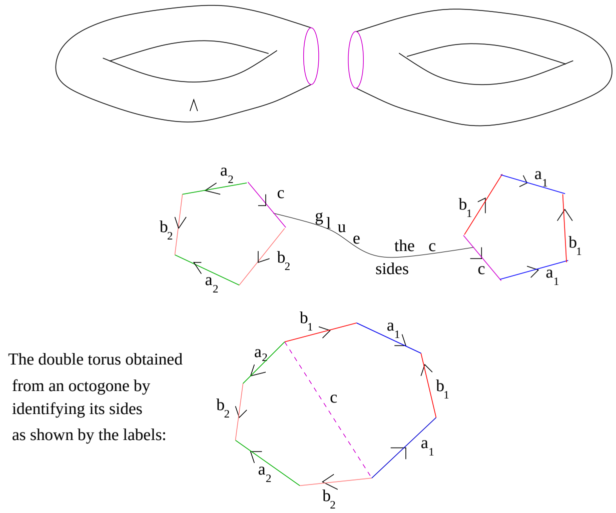


FIGURE 7.

Denoting by a_i, b_i the resulting elements in the fundamental group of $S^1 \vee S^1 \vee S^1 \vee S^1$, we find

$$\pi(T_2) \sim F(a_1, b_1, a_2, b_2)/\langle a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}\rangle.$$

More generally, consider

$$T_g = \underbrace{T\# \ldots \# T}_{g \text{ times}},$$

the "connected sum" of g copies of the torus, which can be visualized as the torus with g wholes. Repeating the arguments above, one obtains a description of T_g as obtained starting from a $4g$-sided polygon with the labeling of its sides:

$$c_g = a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1},$$

and making the identifications dictated by the labeling. The fundamental group of T_g will be the quotient

$$\pi(T_g) \sim F(a_1, b_1, \ldots, a_g, b_g)/\langle c_g \rangle.$$

4. The projective space

Let us look at the projective space $\mathbb{P}^n$. As in the case of the torus and the Moebius band, it is not really necessary to use the Seifert-van Kampen theorem (however, it is again a good illustration of the theorem).

First of all, recall that $\mathbb{P}^n$ admits a cell decomposition with 2-skeleton equal to $\mathbb{P}^2$ -see Example 6.29- hence we reduce the computation to the case $n = 2$ (cf. Proposition 8.10 or Corollary 8.11). But we have seen (see Section 7) that $\mathbb{P}^2$ can be obtained from the square $[0, 1] \times [0, 1]$ by identifying its opposite sides as shown in the picture, hence, as above, we can use Lemma 6.9. We can actually use the picture of the Moebius band because $\mathbb{P}^2$ can be obtained by the identifications made to obtain the Moebius band, plus one more: one also identifies b^{-1} and c in the picture (the notation "b^{-1}" refers to the fact that we change the orientation). Hence, denoting by $\sim$ the equivalence relation on B which identifies b^{-1} and c, $\mathbb{P}^2$ is obtained from $A = B/\sim$ by attaching a 2-cell. Hence A is obtained from B by folding it along the middle vertical line, and the result is clearly (homeomorphic to) a circle. See Figure 8. Hence $\pi(A) = F(u)$ (with generator $u = ca$ on the picture). Also, the path followed by the characteristic map (which was "$cab^{-1}a$" in B) becomes "$caca$", i.e. it goes twice around the circle. Hence

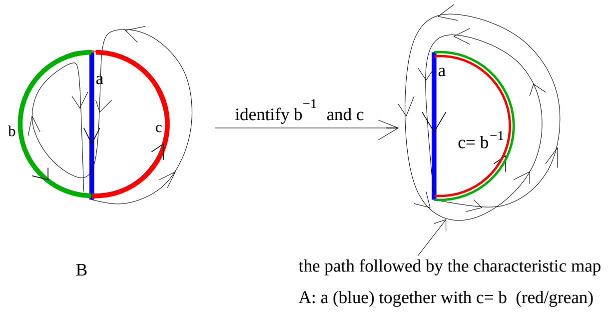


FIGURE 8.

the characteristic map induces in the fundamental group the element $u^2 \in \pi(A)$. Hence the fundamental group of $\mathbb{P}^2$ is $F(u)/\langle u^2 \rangle$, or, in additive notation, $\mathbb{Z}/2\mathbb{Z}$. Hence, for each $n \geq 2$,

$$\pi(\mathbb{P}^n) \cong \mathbb{Z}_2.$$

5. Surfaces

DEFINITION 9.5. *An n-dimensional topological manifold is a topological spaces M with the following properties:*

- *M is Hausdorff and second countable.*
- *it is locally homeomorphic to $\mathbb{R}^n$ i.e. for each $x \in M$, there exists a neighborhood U of x in M which is homeomorphic to $\mathbb{R}^n$.*

A (topological) surface is a 2-dimensional (topological) manifold.

There is a very basic operation which allows us to construct a new manifold out of two given one.

DEFINITION 9.6. *Given two topological manifolds M and N of the same dimension, define their connected sum, denoted $M\#N$ as follows: remove from M and N two "small balls" B_1 and B_2 and glue $M - B_1$ and $N - B_2$ along the sphere $\partial B_1 = \partial B_2$.*

For surfaces, it means that we remove two small disks and we glue the remaininig spaces along the bounday circles.

EXAMPLE 9.7. The connected sum $M\#S^2$ of any surface M with the sphere S^2 is homeomorphic to M itself.

EXAMPLE 9.8. The connected sum of two tori is the double torus. Repeating the operation of connected sum, one obtains all tori with arbitrary number of wholes:

$$T_g = \underbrace{T\# \ldots \#T}_{g \text{ times}}.$$

Similarly, one considers the connected sum of g copies of $\mathbb{P}^2$:

$$U_h = \underbrace{\mathbb{P}^2\# \ldots \#\mathbb{P}^2}_{h \text{ times}}.$$

EXAMPLE 9.9. Example 1.23 tells us that, after removing a small disk from $\mathbb{P}^2$, one obtaines a Moebius band. On the other hand, Exercise 1.21 tells us that the Klein bottle can be obtained by gluing two copies of the Moebius band along the boundary circle. In other words, the Klein bottle can be constructed out of $\mathbb{P}^2$, as the connected sum $\mathbb{P}^2\#\mathbb{P}^2$.

And here is one of the most beautiful theorems of topology:

THEOREM 9.10. *Any compact surface is homeomorphich to one, and only one, of the surfaces T_g with $g \geq 0$, or U_h with $h \geq 1$ (we define $T_0 = S^2$).*

What we are able to prove here is that any two of these spaces are not homeomorphic. The proof is based on the computation of the fundamental group. We have already done it for T_g. The computation for U_h is completely analogous. Starting from the fact that $U_1 = \mathbb{P}^2$ can be obtained from a square with its sides label led in the order

$$aabb = a^2b^2,$$

we find that U_h is obtained by starting from a $2h$-sided polygon with the labeling of its sides:

$$d_h = (a_1)^2 \dots (a_h)^2,$$

and making the identifications dictated by the labeling. Finally,

$$\pi(U_h) = F(a_1, \dots, a_h)/\langle (a_1)^2 \dots (a_h)^2 \rangle.$$

Since these fundamental groups are rather wild, in order to compare them, we will use one trick coming from group theory: their abelianization. Recall that, for a group G, one defines the abelianization of G as:

$$G_{\text{ab}} = G/\overline{\{ghg^{-1}h^{-1} : g, h \in G\}},$$

the quotient of G by the smallest normal subgroup containing all the elements of type $ghg^{-1}h^{-1}$ (in this way we force in the quotients all commutation relations). Two isomorphic groups have isomorphic abelianizations. Hence, to prove that the fundamental groups of the surfaces T_g and U_h are not isomorphic, it suffices to show that their abelianizations are not isomorphic. First of all, for T_g, we use the computation from Section 3: the abelianization is obtained by imposing the new relations which say that any two of the generators of $F(a_1, \dots, a_h)$ commute, and we obtain the abelian group in $2g$ generators

$$\pi(T_g)_{\text{ab}} \sim (\mathbb{Z}^{2g}, +).$$

In particular, T_g and $T_{g'}$ cannot be homeomorphic if $g \neq g'$. Similarly, for U_h, we will have $F(a_1, \dots, a_h)/\langle (a_1)^2 \dots (a_h)^2 \rangle$ to which we have to impose the new relations $a_i a_j = a_j a_i$ for all i and j. We find that $\pi(U_h)$ is isomorphic to the quotient of $\mathbb{Z}^h$ by the subgroup generated by

$$2a_1 + 2a_2 + \dots 2a_h$$

(note that we passed from the multiplicative notation for the group composition, to the additive notation- which is the one used fro $\mathbb{Z}^r$) where $a_i = (0, \dots, 0, 1, 0, \dots, 0)$ is the i-the generator (1 is on the i-th place). Hence we are talking about the subgroup

$$N_h = \{(2n, \dots, 2n) : n \in \mathbb{Z}\} \subset \mathbb{Z}^h.$$

It is not difficult to see that

$$\mathbb{Z}^h/N_h \mapsto \mathbb{Z}^{h-1} \times \mathbb{Z}_2, (n_1, \dots, n_h) \mapsto (n_1, \dots, n_{h-1}, \widehat{n_1 + \dots + n_{h-1}})$$

is an isomorphism, hence

$$\pi(U_h)_{\text{ab}} \sim \mathbb{Z}^{h-1} \times \mathbb{Z}_2.$$

This implies that U_h and $U_{h'}$ are not homeomorphic if $h \neq h'$ and that T_g is not homeomorphic to U_h for all g and h.

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