Solutions Book Chapter 13, SCI 113 Spring 2008

(1) **Exercise 13.2** A has eigenvalues $\lambda = \dfrac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2}$, both are real numbers since $(a-c)^2 + 4b^2 \geq 0$. In case $(a-c)^2 + 4b^2 = 0$, we get repeated eigenvalue which equals in this case $\dfrac{a+c}{2}$.

(2) **Exercise 13.3** Eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 9$. $A^{-1} = \dfrac{1}{36}\begin{pmatrix} 7 & -3 \\ -2 & 6 \end{pmatrix}$.
Eigenvalues of A^{-1} are $\lambda_1 = \dfrac{4}{36}$ and $\lambda_2 = \dfrac{9}{36}$. So dividing each eigenvalue of A by the determinant of A (which is 36), we get all the eigenvalues of A^{-1}. Eigenvalues of A^2 are $\lambda_1 = 4^2 = 16$ and $\lambda_2 = 9^2 = 81$. So the eigenvalues of A^2 are the squares of the eigenvalues of A.

(3) **Exercise 13,4** (a) Eigenvalues of A are $\lambda_1 = 4$, $\lambda_2 = 1$ and $\lambda_3 = -1$. Eigenvectors corresponding to $\lambda_1 = 4$, $\lambda_2 = 1$ and $\lambda_3 = -1$ are given respectively by $t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, where t is any real number.

(4) **Exercise 13.5** Eigenvalues are $\lambda_1 = 4$ (repeated eigenvalue), $\lambda_2 = -1$ and $\lambda_3 = 2$. Eigenvectors corresponding to $\lambda_1 = 4$, $\lambda_2 = -1$ and $\lambda_3 = 2$ are given respectively by $t\begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $t\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $t\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$, where s and t are any real numbers.

(5) **Exercise 13.6** $\lambda_1 = 1$ is a repeated eigenvalue (the other eigenvalue is $\lambda_2 = 6$). The eigenvectors corresponding to $\lambda_1 = 1$ have the form $t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$. Thus, the eigenvectors are linear combination of two vectors: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, two linearly independent vectors.

(6) **Exercise 13.8** First notice that if λ is an eigenvalue of A with eigenvector X (i.e. $AX = \lambda X$ and X is non-zero), then

$$A^2X = A(AX) = A(\lambda X) = \lambda(AX) = \lambda^2 X.$$

Thus λ^2 is an eigenvalue of A^2 with eigenvector X. So, if $A = A^2$, and if $AX = \lambda X$, then from above $A^2X = \lambda^2 X$ so that $\lambda X = \lambda^2 X$. Thus $(\lambda^2 - \lambda)X = 0$ which implies that $\lambda^2 - \lambda = 0$ (notice that $X \neq 0$) or that λ is 0 or 1.
With the given 3×3 matrix A, it is easy to check that $A = A^2$. The eigenvalues of $A = A^2$ are $\lambda_1 = 0$ and $\lambda_2 = 1$ (repeated eigenvalue). The

eigenvectors are $t \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ (corresponding to $\lambda_1 = 0$), and $t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} +$ $s \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$ (corresponding to $\lambda_2 = 1$), where s and t are any real numbers.

(7) **Exercise 13.11** s_1, s_2 and s_3 are linearly dependent because $s_3 = 2s_1 + s_2$ or $2s_1 + s_2 - s_3 = 0$ (so we have found constants $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = -1$ non-zero such $\alpha_1 s_1 + \alpha_1 s_2 + \alpha_3 s_3 = 0$ (see p. 255 in your book)).