Assignment 3, SCI 113 Spring 2008

due date: April 17,2008

- Write your name and student number on each page you hand in.
- You are allowed to use the book Mathematical Techniques by Jordan and Smith and the lecture notes by Frits Beukers.
- You should explain how you have calculated your answers
- You should do this assignment individually. You are not allowed to work in groups.

(1) Let $S = \{(x+1)^3, (x+1)^2, x+1, 1\}$.
   (a) Show that the set S is linearly independent in P_3, the vector space of all polynomials of degree at most 3.
   (b) Express the polynomial $p(x) = x^3 + x^2 - x - 1$ as a linear combination of elements of S.

(2) Consider the vectors $\mathbf{a} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$, and $\mathbf{c} = 2\mathbf{j} + 3\mathbf{k}$ in $\mathbb{R}^3$.
   (a) Find a vector perpendicular to both **a** and **b**.
   (b) Find the volume of the parallelepiped with vertex at the origin determined by the vectors **a**, **b**, and **c**. Sketch the parallelepiped.

(3) In this problem you will be solving a system of linear differential equations. From calculus, we know that if $y = f(t)$ satisfies $y'(t) = \dfrac{dy}{dt} = ky(t)$, then the funcion y must have the form $y = f(t) = Ce^{kt}$, where $C = f(0)$. Now suppose we have a system of linear differential equations

$$\begin{cases} y_1' = \dfrac{dy_1}{dt} = k_1 y_1 \\ \\ y_2' = \dfrac{dy_1}{dt} = k_2 y_2. \end{cases}$$

Then, the solution to this system is $y_1 = f_1(t) = C_1 e^{k_1 t}$, $y_2 = f_2(t) = C_2 e^{k_2 t}$ with $C_1 = f_1(0)$ and $C_2 = f_2(0)$ (are constants). Notice that the above system can be written in matrix form as

$$\mathbf{y}' = D\mathbf{y},$$

where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\mathbf{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$, and $D = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$.

Summarizing, we see that if

$$\mathbf{y}' = D\mathbf{y},$$

with $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\mathbf{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$, and $D = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$, then the solution $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ has the form $y_1 = f_1(t) = C_1 e^{k_1 t}$, $y_2 = f_2(t) = C_2 e^{k_2 t}$, where the coefficients of t in the exponents are given by the diagonal elements of D.

Now consider the following system of linear differential equations

$$\begin{cases} y_1' = \dfrac{dy_1}{dt} = 3y_1 + 2y_2 \\ \\ y_2' = \dfrac{dy_1}{dt} = 6y_1 - y_2. \end{cases}$$

(a) Write the above system in the form

$$\mathbf{y}' = A\mathbf{y},$$

where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\mathbf{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$, and A a 2×2 matrix.

(b) Diagonalize the matrix A in part (a), i.e. find an invertible matrix C, and a diagonal matrix D such that $C^{-1}AC = D$.

(c) Define a vector (function) $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ by $\mathbf{y} = C\mathbf{w}$. Show that $\mathbf{w}' = D\mathbf{w}$, where $\mathbf{w}' = \begin{pmatrix} w_1' = \dfrac{dw_1}{dt} \\ w_2' = \dfrac{dw_2}{dt} \end{pmatrix}$, and D is the diagonal matrix in part (b). Solve this system for $\mathbf{w}$.

(d) Use the solution of part (c), and the relationship $\mathbf{y} = C\mathbf{w}$ to solve the original system for $\mathbf{y}$.