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Measure and Integration Exercises 6

1. Let E be an uncountable set, and $\mathcal{B} = \{A \subseteq E : A \text{ or } A^c \text{ is countable}\}$. Show that $\mathcal{B}$ is a σ-algebra over E.

2. Suppose E is a set, $\mathcal{C}$ a π-system over E and $\mathcal{B} = \sigma(E; \mathcal{C})$ (the smallest σ-algebra over E containing $\mathcal{C}$). Let μ and ν be two measures on $(E, \mathcal{B})$ such that (i) $\mu(E) = \nu(E) < \infty$, and (ii) $\mu(C) = \nu(C)$ for all $C \in \mathcal{C}$. Let $\mathcal{H} = \{A \in \mathcal{B} : \mu(A) = \nu(A)\}$.

   (a) Show that $\mathcal{H}$ is a λ-system over E.

   (b) Show that $\mathcal{B} = \mathcal{H}$, and conclude that $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}$.

3. A collection $\mathcal{M}$ of sets is said to be a *monotone class* if it satisfies the following two properties:

   (i) if $\{A_n\} \subseteq \mathcal{M}$ with $A_1 \subseteq A_2 \subseteq \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$, and

   (ii) if $\{B_n\} \subseteq \mathcal{M}$ with $B_1 \supseteq B_2 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} B_n \in \mathcal{M}$.

   (a) Show that the intersection of an arbitrary collection of monotone classes is a monotone class.

   (b) Let E be a set, and $\mathcal{B}$ a collection of subsets of E. Show that $\mathcal{B}$ is a σ-algebra if and only if $\mathcal{B}$ is an algebra and a monotone class.

   (c) Let $\mathcal{A}$ be an algebra over E, and $\mathcal{M}$ the smallest monotone class containing $\mathcal{A}$, i.e. $\mathcal{M}$ is the intersection of all monotone classes containing $\mathcal{A}$. Show that $\mathcal{M}$ is an algebra.

   (d) Using the same notation as in part (c), show that $\mathcal{M} = \sigma(E, \mathcal{A})$, where $\sigma(E, \mathcal{A})$ is the smallest σ-algebra over E containg the algebra $\mathcal{A}$.