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Measure and Integration Exercises 12

1. Let $(E_1, \mathcal{B}_1, \mu_1)$ and $(E_2, \mathcal{B}_2, \mu_2)$ be σ-finite measure spaces. Let $\Gamma \in \mathcal{B}_1 \times \mathcal{B}_2$. For $x_1 \in E_1$, $x_2 \in E_2$, let $\Gamma(x_1) = \{x_2 \in E_2 : (x_1, x_2) \in \Gamma\}$ and $\Gamma(x_2) = \{x_1 \in E_1 : (x_1, x_2) \in \Gamma\}$. Show that the following are equivalent:
   (i) $(\mu_1 \times \mu_2)(\Gamma) = 0$,
   (ii) $\mu_1(\Gamma(x_2)) = 0$ for μ_2 almost every $x_2 \in E_2$,
   (iii) $\mu_2(\Gamma(x_1)) = 0$ for μ_1 almost every $x_1 \in E_1$.

2. $(E, \mathcal{B}, \mu)$ be a σ-finite measure space, and $f : X \to [0, \infty)$ measurable. Define
$$\Gamma(f) = \{(x, t) \in E \times [0, \infty) : t < f(x)\},$$
and
$$\overline{\Gamma}(f) = \{(x, t) \in E \times [0, \infty) : t \leq f(x)\}.$$
   (a) Show that the function $F : E \times [0, \infty) \to \mathbb{R}$ given by $F(x, t) = f(x) - t$ is measurable with respect to the product σ-algebra $\mathcal{B} \times \mathcal{B}_{[0,\infty)}$, where $\mathcal{B}_{[0,\infty)}$ is the restriction of the Borel σ-algebra on $[0, \infty)$.
   (b) Show that $\Gamma(f), \overline{\Gamma}(f) \in \mathcal{B} \times \mathcal{B}_{[0,\infty)}$, and
$$(\mu \times \lambda_{\mathbb{R}})(\Gamma(f)) = (\mu \times \lambda_{\mathbb{R}})(\overline{\Gamma}(f)) = \int_E f(x)\, d\mu(x).$$

3. Consider $(\mathbb{R}, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra, λ is Lebesgue measure and μ is counting measure (i.e. $\mu(A)$ = number of elements in A). Show that
$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x_1, x_2) d\lambda(x_1) d\mu(x_2) = 0$$
while
$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x_1, x_2) d\mu(x_2) d\lambda(x_1) = \infty.$$
Why does not this violate Tonelli's Theorem?

4. Let $E = \{(x, y) : 0 < x < \infty, 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x, y) = y \sin x\, e^{-xy}$.
   (a) Show that f is $\lambda \times \lambda$ integrable on E.
   (b) Applying Fubini's Theorem to the function f, show that
$$\int_0^\infty \frac{\sin x}{x} \left(\frac{1 - e^{-x}}{x} - e^{-x} \right) dx = \frac{1}{2} \log 2.$$