Universiteit Utrecht Universiteit Utrecht Boedapestlaan 6

Mathematisch Instituut 3584 CD Utrecht

Measure and Integration Exercises 14

1. $(E, \mathcal{B}, \mu)$ be a σ-finite measure space, and $f : E \to [0, \infty)$ measurable. Define

$$\Gamma(f) = \{(x, t) \in E \times [0, \infty) : t < f(x)\},$$

and

$$\overline{\Gamma}(f) = \{(x, t) \in E \times [0, \infty) : t \leq f(x)\}.$$

   (a) Show that the function $F : E \times [0, \infty) \to \mathbb{R}$ given by $F(x, t) = f(x) - t$ is measurable with respect to the product σ-algebra $\mathcal{B} \times \mathcal{B}_{[0,\infty)}$, where $\mathcal{B}_{[0,\infty)}$ is the restriction of the Borel σ-algebra on $[0, \infty)$.

   (b) Show that $\Gamma(f), \overline{\Gamma}(f) \in \mathcal{B} \times \mathcal{B}_{[0,\infty)}$, and

$$(\mu \times \lambda_{\mathbb{R}})(\Gamma(f)) = (\mu \times \lambda_{\mathbb{R}})(\overline{\Gamma}(f)) = \int_E f(x)\, d\mu(x).$$

2. Let $E = \{(x, y) : 0 < x < \infty, 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x, y) = y \sin x \, e^{-xy}$.

   (a) Show that f is $\lambda \times \lambda$ integrable on E.

   (b) Applying Fubini's Theorem to the function f, show that

$$\int_0^\infty \frac{\sin x}{x} \left(\frac{1 - e^{-x}}{x} - e^{-x} \right) dx = \frac{1}{2} \log 2.$$

3. Let $(L, (\ ,\ ))$ be an inner product space, and let $||x||_L = (x, x)^{1/2}$. $x \in L$.

   (a) Let $(x_n) \subseteq L$, and $x \in L$. Show that if $\lim_{n\to\infty} ||x_n - x||_L = 0$, then $\lim_{n\to\infty} ||x_n||_L = ||x||_L$.

   (b) Prove that the inner product $(\ ,\ )$ is jointly continuous, i.e. if $\lim_{n\to\infty} ||x_n - x||_L = 0$ and $\lim_{n\to\infty} ||y_n - y||_L = 0$, then $\lim_{n\to\infty} (x_n, y_n) = (x, y)$.

4. Let $(E, \mathcal{B}, \mu)$ be a measure space, and let $\{f_n\} \subseteq L^2(\mu)$ be such that

$$\lim_{m\to\infty} \sup_{n \geq m} ||f_n - f_m||_{L^2(\mu)} = 0.$$

Show that there exists a function $f \in L^2(\mu)$ such that $\lim_{n\to\infty} ||f_n - f||_{L^2(\mu)} = 0$. In other words $(L^2(\mu), ||\ ||_{L^2(\mu)})$ is a complete metric space.