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Measure and Integration Exercises 16

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra. Define σ on $\mathcal{B}(\mathbb{R})$ by $\sigma(\Gamma) = \sum_{n\in\mathbb{Z}\setminus\{0\}\cap\Gamma} \frac{1}{n^2}$.

   (a) Show that σ is a measure on $\mathcal{B}(\mathbb{R})$ such that $\sigma \perp \lambda$, where λ is Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

   (b) Let $f \in L^1(\lambda)$ be non-negative, and define μ on $\mathcal{B}(\mathbb{R})$ by $\mu(\Gamma) = \int_\Gamma f\,d\lambda$. Let $\nu = \mu + \sigma$. Find the Lebesgue decomposition of ν with respect to λ.

2. Let $(E, \mathcal{B}, \nu)$ be a measure space, and $h : E \to \mathbb{R}$ a non-negative measurable function. Define a measure μ on $(E, \mathcal{B})$ by $\mu(A) = \int_A h d\nu$ for $A \in \mathcal{B}$. Show that for every measurable function $F : E \to \mathbb{R}$ one has

$$\int_E F\,d\mu = \int_E Fh\,d\nu$$

   in the sense that if one integral exists, then the other integral also exists, and they are equal.

3. Suppose that μ_i, ν_i are finite measures on $(E, \mathcal{B})$ with $\mu_i \ll \nu_i$, $i = 1, 2$. Let $\nu = \nu_1 \times \nu_2$ and $\mu = \mu_1 \times \mu_2$.

   (a) Show that $\mu \ll \nu$.

   (b) Prove that $\frac{d\mu}{d\nu}(x, y) = \frac{d\mu_1}{d\nu_1}(x) \cdot \frac{d\mu_2}{d\nu_2}(y)$ ν a.e.

4. Let $(E, \mathcal{B})$ be a measurable space, μ a finite measures on $(E, \mathcal{B})$ and ν a σ-finite measure on $(E, \mathcal{B})$. Show that $\mu \ll \nu$ **if and only if** for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{B}$ with $\nu(A) < \delta$, then $\mu(A) < \epsilon$.