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Measure and Integration Exercises 8

1. Let $\mathcal{C} = \{(a, \infty) : a \in \mathbb{R}\}$, and let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra over $\mathbb{R}$.
   (a) Let $(E, \mathcal{B})$ be a measurable space. Suppose $f : E \to \mathbb{R}$ satisfies $f^{-1}(C) \in \mathcal{B}$ for all $C \in \mathcal{C}$. Show that f is measurable, i.e. $f^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{B}_{\mathbb{R}}$.
   (b) Suppose ν and μ are finite measures on $\mathcal{B}_{\mathbb{R}}$, and $\mu\left(f^{-1}(a, \infty)\right) = \nu\left((a, \infty)\right)$ for all $a \in \mathbb{R}$. Show that $\mu(f^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{B}_{\mathbb{R}}$.
2. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to [-\infty, \infty]$ a sequence of measurable functions. Show that $\sup_n f_n$ and $\inf_n f_n$ are measurable.
3. Let $(E, \mathcal{B}, \mu)$ be a measure space. Suppose $f : E \to [-\infty, \infty]$ is a function such that $f = \sum_{i=1}^{n} a_i 1_{A_i}$, where $a_1, \cdots, a_n$ are **distinct** elements of $[-\infty, \infty]$ and $A_1, A_2, \cdots, A_n$ are disjoint subsets of E. Show that f is measurable (i.e. $f^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{B}_{[-\infty,\infty]}$) **if and only if** $A_1, A_2, \cdots, A_n \in \mathcal{B}$.
4. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f : E \to [0, \infty]$ a measurable simple function such that $\int_E f \, d\mu < \infty$. Define $\lambda : \mathcal{B} \to [0, \infty]$ by

$$\lambda(B) = \int_B f \, d\mu.$$

   (a) Show that λ is a **finite** measure on $\mathcal{B}$.
   (b) Suppose that $\mu(f = 0) = 0$. Show that $\lambda(B) = 0$ **if and only if** $\mu(B) = 0$.