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Measure and Integration: Extra Exercises

1. Let $(E, \mathcal{B}, \mu)$ be a probability space, i.e. $\mu(E) = 1$. Let $f : E \to [0, 1)$ be a measurable function such that $\mu\left(f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}))\right) = \frac{1}{2^n}$ for $n \geq 1$ and $k = 0, 1, \cdots, 2^n - 1$. Show that $\int_E f^2 \, d\mu = \frac{1}{3}$.

2. Consider the measure space $([a, b], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[a, b]$, andλ is the restriction of the Lebesgue measure on $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be a bounded Riemann integrable function. Show that the Riemann integral of f on $[a, b]$ is equal to the Lebesgue integral of f on $[a, b]$, i.e.

$$(R) \int_a^b f(x)dx = \int_{[a,b]} f d\lambda.$$

3. Let $0 < a < b$. Prove with the help of Fubini's theorem that $\int_0^\infty (e^{-at} - e^{-bt})\frac{1}{t} dt = \log(b/a)$.

4. Let $(E, \mathcal{B}, \mu)$ be a measure space. Show that μ is σ-finite **if and only if** there exists a **strictly** positive measurable function $f \in L^1(\mu)$.