Basic Mathematics

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Contents

Chapter 1

Taylor series

1.1 Derivatives

Suppose we have a function $f(x)$ which is defined on a domain $V \subset \mathbb{R}$. This domain can be an interval, such as $(2, 3)$ (endpoints not included) or $[3, 4]$ (endpoints included), halflines, such as $[2, \infty)$ (all numbers ≥ 2), or the set of real numbers itself. Let us assume that $f(x)$ can be differentiated at every point of the domain V. The derivative is a new function on V for which we have various notations, such as

$$f'(x), \qquad \frac{df}{dx}(x), \qquad Df(x).$$

Suppose that the derivative can again be differentiated. The resulting second derivative function can be denoted by

$$f''(x), \qquad \frac{d^2 f}{dx^2}(x), \qquad D^2 f(x).$$

Similarly, the n-th derivative function is denoted by

$$f^{(n)}(x), \qquad \frac{d^n f}{dx^n}(x), \qquad D^n f(x).$$

1.2 Computation of functions

We know how to compute a function like $f(x) = x^2$, simply take x and multiply with itself. But what about the sine function, i.e. how to compute

sin(0.3) for example? One might grab a pocket calculator and compute the value. But then the question arises how the calculator gets its value. Or, suppose we want sin(0.3) to a higher precision than the one provided by the pocket calculator, what should we do then?
The difference between the functions x^2 and $\sin(x)$ is that the first is a so-called *rational function* and the second is a *transcendental function*. Rational functions can be computed directly, transcendental functions are defined by limit procedures. Limit procedures are computations where we get better and better approximations of the actual value as we continue our computation. An example of a limit procedure is the use of *Taylor approximations*.
Here is the general theorem we like to use. In it we have a function $f(x)$ which is defined in an interval $(-p, p)$ around $x = 0$ and which can be differentiated as many times as we like.

Theorem 1.2.1 (Taylor) *Let $x \in (-p, p)$ and let $n = 1, 2, 3, \ldots$. Then*

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + Error$$

where

$$|Error| \le \frac{|x|^{n+1}}{(n+1)!}\max_{t\in I}|f^{(n+1)}(t)|.$$

Here I is the interval between 0 and x (i.e. $[0, x]$ if $x > 0$ and $[x, 0]$ if $x < 0$).

The application of this Theorem lies in the observation that if we take x not too big, like $x = 0.1$, then the power $x^{n+1} = 0.1^{n+1}$ in the Error term will be quite small, i.e. the *Error* term itself is small. In that way we can compute approximations of values of $f(x)$ by evaluating the summation in our Theorem (assuming we know $f(0), f'(0), f''(0), \ldots$).
Let us take the example $f(x) = \sin(x)$. We take $n = 5$. Here is a table of the first 6 derivatives of $\sin(x)$ and their values in $x = 0$.

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0
5	$\cos(x)$	1
6	$-\sin(x)$	

Application of our Theorem now tells us that

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + Error$$

with

$$|Error| \le \frac{|x|^6}{720} \times \max_{t\in[0,x]} |\sin(t)| \le \frac{|x|^6}{720}.$$

Check this yourself!! Taking $x = 0.3$ we find that

$$\sin(0.3) \approx 0.3 - \frac{0.3^3}{6} + \frac{0.3^5}{120} = 0.29552$$

with an error estimate

$$|Error| \le \frac{0.3^6}{720} \le 0.0000011.$$

If we want $\sin(0.3)$ with a better precision, we simply choose a bigger value of n and compute more terms. The expression

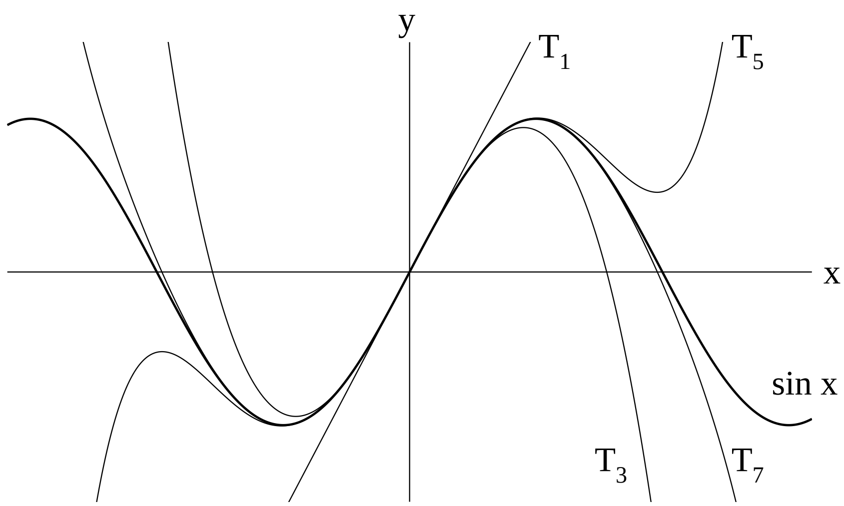
$$T_n(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n$$

is called the *n-th order Taylor approximation* of $f(x)$ around $x = 0$ and we shall denote it by $T_n(x)$. We have seen above that we can compute values of $\sin(x)$ with any precision we like by taking higher and higher order Taylor approximations. The construction of a Taylor approximation of $\sin(x)$ is based on the fact that we do know explicit values of $\sin(x)$ and its derivatives at the point $x = 0$.

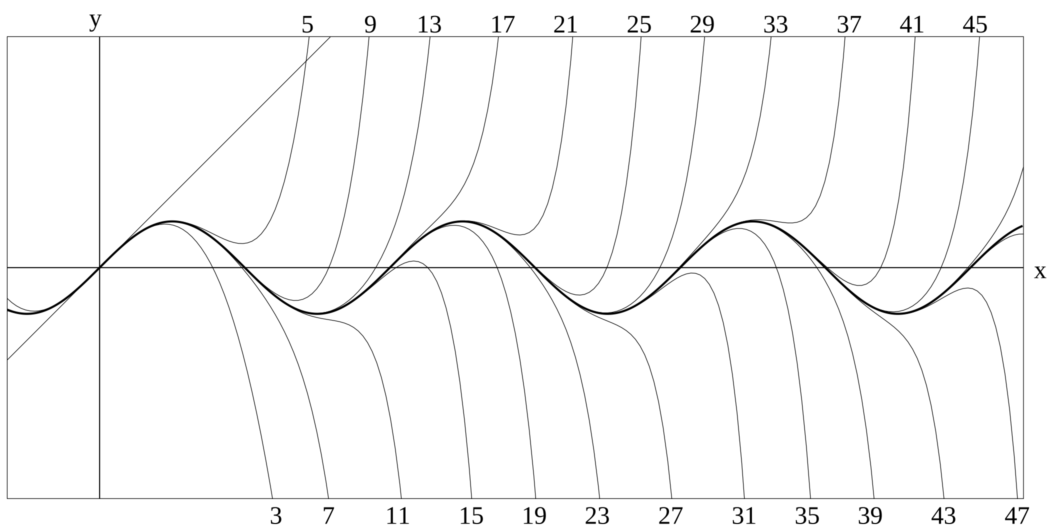
1.3 Philosophy of Taylor approximations

You may wonder where the expressions for the Taylor approximations come from. Consider the n-th order Taylor approximation $T_n(x)$. Take its value in $x = 0$. We get $T_n(0) = f(0)$ (Check this!). Now take the first derivative of $T_n(x)$ and then set $x = 0$. We get $T_n'(0) = f'(0)$ (Check!). Similarly for the second derivative, $T_n''(0) = f''(0)$. This holds all the way up to the n-the derivative, $T_n^{(n)}(0) = f^{(n)}(0)$. In other words, $T_n(x)$ is a so-called polynomial function, whose derivatives in the point 0 are the same as those of $f(x)$ all the way up to the n-th derivative.

The philosophy is that if two functions have the same derivatives at the point 0 up to a high order, then their function values near 0 will also have a tendency to be close together. Of course this is only a philosophy and it remains to be seen how it works in practice. As a matter of fact, in many cases it does work quite well. As an illustration you see here a combined plot of $\sin(x)$ and its Taylor approximations $T_1(x), T_3(x), T_5(x)$ and $T_7(x)$,



Observe that the approximations become better and better as the order increases. As a further illustration you see here a plot of $\sin(x)$ with $T_1(x), \ldots, T_{47}(x)$,



We will not give a proof of Taylor's theorem. However, in the Appendix to this chapter we do give a proof for the first order taylor approximation, i.e. the case $n = 1$. This approximation is a very important one, also known as the *linear approximation* of $f(x)$ at the point $x = 0$.

1.4 Taylor approximations at arbitrary points

There is no reason why we should restrict ourselves to approximations near the point $x = 0$. Suppose we have a function $f(x)$, which can be differentiated any number of times, and which is defined in an interval $(a-p, a+p)$ around some point a. We now like to choose our values of x close to a and we suggest this by writing $x = a+h$, where h is considered as a small displacement from a. We have the following Theorem,

Theorem 1.4.1 (Taylor) *Let $h \in (-p, p)$ and let $n = 1, 2, 3, \ldots$. Then*

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \frac{1}{6}f'''(a)h^3 + \cdots + \frac{1}{n!}f^{(n)}(a)h^n + Error$$

where

$$|Error| \leq \frac{|h|^{n+1}}{(n+1)!} \max_{t \in I} |f^{(n+1)}(t)|.$$

Here I is the interval between a and $a+h$ (i.e. $[a, a+h]$ if $h > 0$ and $[a+h, a]$ if $h < 0$).

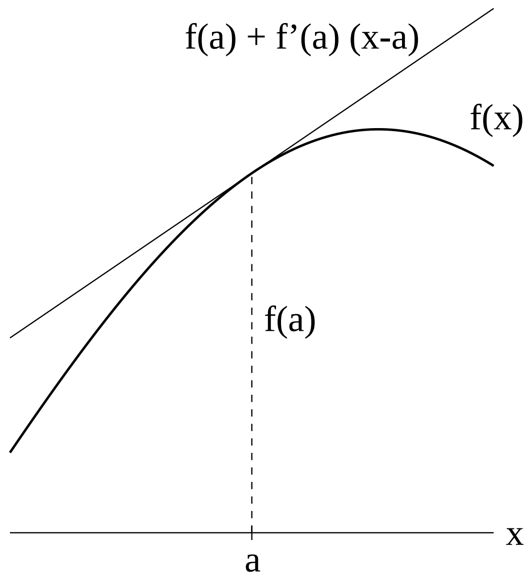
You may be able to notice that this Theorem follows from the previous Theorem applied to the function $g(x) = f(a+x)$ and then x replaced by h. As before, the most important approximation is the *linear approximation*

$$T_1(a+h) = f(a) + f'(a)h$$

or, if we replace h by $x - a$,

$$T_1(x) = f(a) + f'(a)(x-a).$$

The graph of this linear approximation is simply the straight line which is *tangent* to the graph of $f(x)$ above $x = a$.



From the expression $f(a) + f'(a)(x-a)$ we see that the slope ($= \tan(\alpha)$ where α is the angle between the line and the x-axis) of the tangent is equal to $f'(a)$. Since the tangent line and the graph of $f(x)$ have the same slope, the graph of $f(x)$ has also a slope of $f'(a)$ above the point $x = a$.

1.5 Exercises

Exercise 1.5.1 *Compute the fifth order Taylor approximation of* e^x *around* $x = 0$. *Using this Taylor approximation, compute an approximation of* $e^{0.1}$. *To how many digits is this value correct? Using your approximation, compute an appriximation of* e. *(Use of a pocket calculator is encouraged, without touching the* e^x *button however).*

Exercise 1.5.2 *Write down the Taylor approximation for* e^x *for any* n. *Do the same for* $\sin(x)$, $\cos(x)$ *and* $1/(1-x)$.

Exercise 1.5.3 *Compute the third order Taylor approximation of* $\sqrt{1+x}$ *around* $x = 0$. *If you use this third order approximation to determine* $\sqrt{1.1}$, *how many correct digits do you expect? Do the evaluation.*

Exercise 1.5.4 *Compute the third order Taylor approximation of* $\sqrt{1+4x}$ *in two ways. First by computing the derivatives, secondly by taking the previous exercise and replacing* x *by* $4x$.

Exercise 1.5.5 *To see that Taylor approximations do not always a nice pattern evaluate the third order Taylor approximation of* $\tan(x)$ *around* $x = 0$.

1.6 Appendix

For those interested we prove Taylor's Theorem for the case $n = 1$:

Theorem 1.6.1 (Taylor for $n = 1$) *Let* $x \in (-p, p)$. *Then*

$$f(x) = f(0) + f'(0)x + Error$$

where

$$|Error| \leq \frac{|x|^2}{2} \max_{t \in I} |f''(t)|.$$

Here I *is the interval between* 0 *and* x.

Proof We rewrite $f(x)$ as

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

To check this, simply compute the integral on the right hand side. We now perform a partial integration with the underlying observation that the derivative of $(t - x)f'(t)$ with respect to t is $f'(t) + (t - x)f''(t)$.

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(t)dt \\ &= f(0) + [(t - x)f'(t)]_0^x - \int_0^x (t - x)f''(t)dt \\ &= f(0) + xf'(0) - \int_0^x (t - x)f''(t)dt \end{aligned}$$

You may recognize the first order Taylor approximation and the error term should be

$$Error = -\int_0^x (t - x)f''(t)dt.$$

This integral can be estimated by the maximal absolute value of $f''(t)$ over the interval between 0 and x (denoted by I), multiplied by the integral of

$t-x$. So,

$$\begin{aligned}
|Error| &= \left|\int_0^x (t-x)f''(t)dt\right| \\
&\leq \max_{t\in I}|f''(t)|\cdot\left|\int_0^x (t-x)dt\right| \\
&\leq \frac{|x|^2}{2}\max_{t\in I}|f''(t)|.
\end{aligned}$$

$\square$

Chapter 2

Solving equations

2.1 Introduction

In many situations the solution of a mathematical problem amounts to the solution of one or more equations in one or more unknowns. There is no general method to solve a system of equations and the approach will have to depend on the number of equations, the number of unknowns and also the complexity of the functions involved.

Another matter that needs to be addressed is the question of what you mean by a solution. For example, a solution to the equation $x^2 - 2 = 0$ is of course $\sqrt{2}$ (and $-\sqrt{2}$ the other solution). But what does the symbol $\sqrt{2}$ mean? A pocket calculator gives us 1.414213562373. But of course this is only a numerical approximation. This is handy if you want to have numerical answers in your computations, but the approximation will never allow you to verify with one hundred percent certainty that for example

$$\sqrt{3 + 2\sqrt{2}} = 1 + \sqrt{2}.$$

To verify this we simply square both sides and use the fact that $\sqrt{2}$ squared equals 2 precisely. In this course we shall mainly aim for numerical answers of our computations, but it is good to be aware of the phenomenon of exact versus approximate values. In particular because Mathematica and others good quality mathematical software packages make this distinction.

2.2 Equations in one variable

At high school you have probably solved such equations. For example $2x = 1, 3x - 2 = 0$, etc. Equations of the form

$$ax - b = 0$$

where a, b are given numbers, $a \neq 0$, and x is the unknown quantity to be solved. These equations are called *linear equations in one variable.* The word linear refers to the fact that the unknown occurs only to the first power in the equation. There is one solution x, which is of course

$$x = \frac{b}{a}.$$

You have probably also seen equations of the form

$$ax^2 + bx + c = 0$$

where a, b, c are given numbers with $a \neq 0$ and x is again the unknown. These equation are called *quadratic equations in one unknown* since x^2 is the highest power of x that occurs in the equation. There are two solutions x_1, x_2 given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $b^2 - 4ac > 0$, there is only one solution if $b^2 - 4ac = 0$ and there are no solutions if $b^2 - 4ac < 0$. This formula is probably familiar from high school and it is known since Babylonian times. When we speak of solutions here we think about solutions in $\mathbb{R}$, the real numbers. Later, we extend the world of real number by the symbol $\sqrt{-1}$ and obtain the so-called *complex numbers.* Quadratic equations always have solutions in the complex numbers.
Of course we can now proceed and consider equations of the form

$$ax^3 + bx^2 + cx + d = 0$$

where a, b, c, d are given, $a \neq 0$, and x is agian the unknown. This is called a *cubic or third degree equation* for obvious reasons. Its solution was found by the renaissance mathematician Tartaglia, but the formula is refered to as *Cardano's formula.* Mathematica has Cardano's formula in its database and if you issue the command

$$\texttt{Solve}[\texttt{a} * \texttt{x}^3 + \texttt{b} * \texttt{x}^2 + \texttt{c} * \texttt{x} + \texttt{d} == 0, \texttt{x}]$$

Mathematica spits out

$$\frac{-b}{3\,a}-\frac{2^{\frac{1}{3}}\,\left(-b^2+3\,a\,c\right)}{3\,a\left(-2\,b^3+9\,a\,b\,c-27\,a^2\,d+\sqrt{4\left(-b^2+3\,a\,c\right)^3+\left(-2\,b^3+9\,a\,b\,c-27\,a^2\,d\right)^2}\right)^{\frac{1}{3}}}$$

$$+\frac{\left(-2\,b^3+9\,a\,b\,c-27\,a^2\,d+\sqrt{4\left(-b^2+3\,a\,c\right)^3+\left(-2\,b^3+9\,a\,b\,c-27\,a^2\,d\right)^2}\right)^{\frac{1}{3}}}{3\cdot 2^{\frac{1}{3}}\,a}$$

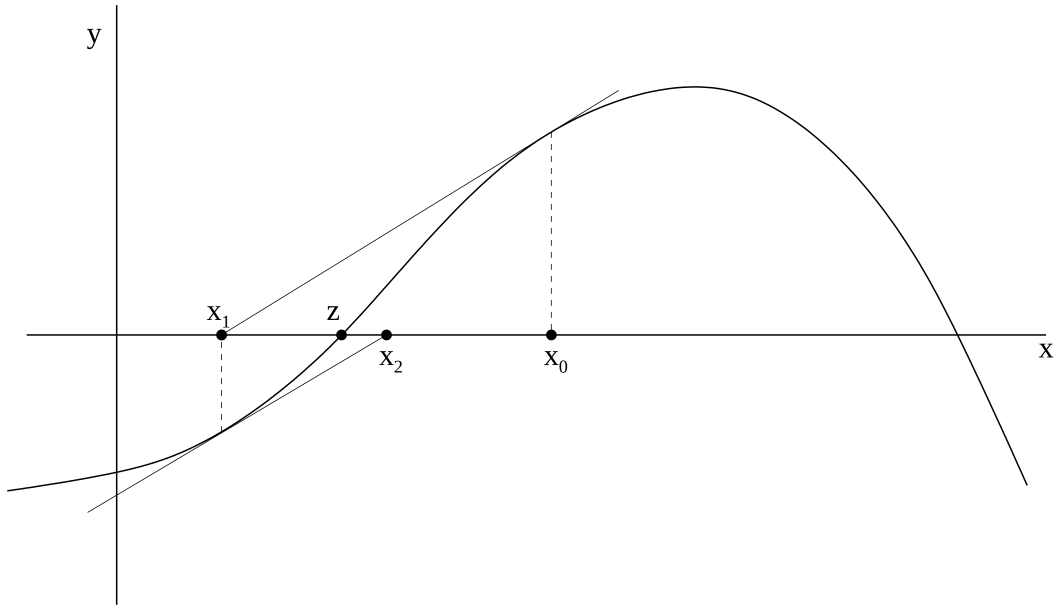
as one of the solutions. Of course this is an answer, but the question is whether this is a useful answer. If you look at it critically, its only merit is that the Cardano formula has expressed the solution of a cubic equation in terms of the root extractions $\sqrt{A}=A^{1/2}$ and $\sqrt[3]{A}=A^{1/3}$. But for numerical solution the formula is much too complicated.

There is a similar story for fourth degree equations. Again there is a formula, this time found by Ferrari in the 16th century. Ferrari's formula is even more complicated than Cardano's. It expresses the solution of the fourth degree equation in terms of root and cube root extractions. To make things worse, for fifth degree equations it is even impossible to express its solutions in terms of (higher) root extractions. This was shown by the Norwegian mathematician N.H.Abel in the beginning of the 19th century.

2.3 The Newton-Raphson method

Despite the fact that it becomes difficult, or even impossible, to write down formulae for the solution of higher order equations, they often do possess one or more solutions. In this section we discuss a numerical procedure which allows us to compute the solution of a one variable equation very quickly to any precision we like. It is called the *Newton-Raphson* method.

It is based on the following idea. Suppose we have an equation of the form $f(x)=0$, where $f(x)$ can be a polynomial of degree 3 or 4, but also a function like $\tan(x)-x$. By plotting the graph we may see several zeros of $f(x)$. We select the one that interests us and call it z (of **z**ero). Choose a first approximation x_0 to z. Now draw the tangent to the graph of $f(x)$ above the point x_0. This tangent intersects the x-axis in another point, which is often a better approximation to z. Call this point x_1 and repeat the operation, but now with x_1 instead of x_0. We get a new point x_2, and continue our procedure. Here is a picture of what we had in mind.



In this way we get a sequence of x-values $x_0, x_1, x_2, \ldots, x_n, \ldots$ that approximate the zero z better and better. The relation between the n-th approximation point x_n and $n+1$-st point x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This can be seen as follows. The tangent to the graph of f above the point x_n is given by

$$y = f(x_n) + f'(x_n)(x - x_n).$$

We intersect this line with the x-axis by putting $y = 0$. So we get

$$0 = f(x_n) + f'(x_n)(x - x_n).$$

Solving for x gives us

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

and this gives us precisely the desired formula for x_{n+1}.
Let us do a simple example with $f(x) = x^2 - 10$. We like to determine its positive zero by taking $x_0 = 2$ as a starting approximation. Notice that with

$f(x) = x^2 - 10$ we have

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 10}{2x} = x - \frac{x}{2} + \frac{10}{2x} = \frac{1}{2}\left(x + \frac{10}{x}\right).$$

So we get the recursive relation

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{10}{x_n}\right), \quad n \geq 0$$

Starting with $x_0 = 2$ we can compute $x_1, x_2, \ldots$ and we get for the first few values

n	x_n
0	$2.0000000000\ldots$
1	$3.5000000000\ldots$
2	$3.17857142857\ldots$
3	$3.16231942215\ldots$
4	$3.16227766044\ldots$
5	$3.16227766016\ldots$
6	$3.16227766016\ldots$

Note that after 6 steps the first ten digits of our approximation have stabilised. We can check that $\sqrt{10} = 3.1622776601683\ldots$. In fact, the Newton-Raphson method works so well that at every step the number of correct digits is roughly doubled. Such convergence speed is quite exceptional in the branch of numerical mathematics. Moreover, the beauty is that it works equally fast for the solution of third order and higher order equations, without ever knowing about Cardano's or Tartaglia's formulae. It also works very well for transcendental equations like $\tan(x) - x = 0$.
Although the source code of Mathematica is not public, it is very likely that the Mathematica function `FindRoot` works by using the Newton-Raphson method.

2.4 Systems of linear equations

In many applications we need to solve systems of equations in several unknowns. For example, if we are asked for a pair of numbers whose sum equals

100 and whose difference equals 40, this amounts to the set of equations

$$\begin{aligned} x + y &= 100 \\ x - y &= 40 \end{aligned}$$

This is an example of a system of two linear equations in two unkowns. The term *linear* refers to the fact that both unknowns x, y occur only to the first power in the equations. In general a system of two linear equations in two unknowns x, y has the form

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

where a, b, c, d, p, q are given numbers. Equations like this can be solved by using *elimination of variables*. In our first example we can express y in terms of x by using the first equation to get $y = 100 - x$. We can use this knowledge in the second equation $x - y = 40$ when we replace the y there by $100 - x$. We get $x - (100 - x) = 40$. After a little algebra we see that $2x = 140$ and so, $x = 70$. The value of y is then $100 - 70 = 30$. The elimination of y could have been done a bit more elegant by simply adding the two equations to obtain $2x = 140$. Here is another example of a system of 3 equations in 3 unknowns x, y, z,

$$\begin{aligned} -x + y + z &= -1 \\ 2x - y - z &= 0 \\ 3x + 2y - z &= 2 \end{aligned}$$

Elimination of variables is achieved by taking linear combination of the equation. Add the first equation twice to the second, and three times to the third. We find,

$$\begin{aligned} -x + y + z &= -1 \\ y + z &= -2 \\ 5y + 2z &= -1 \end{aligned}$$

Notice that the last two equations only contain the unknowns y and z. De unknown x has been eliminated form these equations. Now subtract the second five times from the third,

$$\begin{aligned} -x+y+z &= -1 \\ y+z &= -2 \\ -3z &= 9 \end{aligned}$$

From the last equation follows $z = -3$. From the second it follows that, $y = -2 - z = 1$ and from the first, $x = 1 + y + z = 1 + 1 - 3 = -1$. The solution reads, $x = -1,\ y = 1,\ z = -3$. Verify that this is indeed a solution to our original set of equations.

We can also write down the above operations in a schematic bookkeeping style. Denote the system of equations by the matrix

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 2 & -1 & -1 & 0 \\ 3 & 2 & -1 & 2 \end{array}\right)$$

Add the first row twice to the second and three times to the third,

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 5 & 2 & -1 \end{array}\right)$$

Subtract the second row 5 times from the third,

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -3 & 9 \end{array}\right)$$

The last row schematicaaly represents $-3z = 9$ and so we find $z = -3$ again. The value of x, y can be recovered similarly.

The systematic procedure we have just sketches is known as *Gaussian elimination*.

You might think that if we have two linear equations in two unknows, or three linear equations in three unknowns, the number of solutions is always one, or at least finite. However, this is not so, as we can see from the following example which has infinitely many solutions.

Example Solve,

$$\begin{array}{rrrcr} x & +y & -2z & = & 1 \\ 2x & +y & -3z & = & 2 \\ -x & +2y & -z & = & -1 \end{array}$$

Schematically

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 2 & 1 & -3 & 2 \\ -1 & 2 & -1 & -1 \end{array}\right)$$

Elimination of x from the second and third equation,

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right)$$

Elimination of y from the third equation,

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The last equation reads $0 = 0$ which is no restriction on x, y, z at all. So we can just drop the last equation to find

$$\begin{array}{rrrcr} x & +y & -2z & = & 1 \\ & y & -z & = & 0 \end{array}$$

From the second it follows that $y = z$ and from the first, $x = 1-y+2z = 1+z$. Apparently we can choose z arbitraily and the values of x, y just follow the choice of z. So we have an infinity of solutions.

It may also happen that there are no solutions at all.

Example We take the same equation as before, but replace the -1 on the bottom right into a -2. Schematically,

$$\begin{array}{rrrcr} x & +y & -2z & = & 1 \\ 2x & +y & -3z & = & 2 \\ -x & +2y & -z & = & -2 \end{array}$$

The elimination procedure is entirely similar, except that we end with

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

The last equation reads $0 = -1$. This can never happen, no matter how we choose x, y, z. We call the system *contradictory*.

2.5 A geometrical interpretation

Let us consider an arbitrary system of two linear equations in two unknowns x, y again.

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

In general such a system will have precisely one solution, but there are exceptions. Sometimes there are no solutions, sometimes infinitely many. A system which has no solutions is for example

$$\begin{aligned} x + y &= 1 \\ x + y &= 2 \end{aligned}$$

An example of a system with infinitely many solutions,

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

Both statements are very easy to check (Do this!). You may find the above examples a cheat, but actually all cases of systems of two linear equations with two unknowns, having infinitely many or no solutions at all, are of this level of difficulty.
To see this, we use a geometrical interpretation of the equations. The points (x, y) in the plane that satisfy an equation of the form $ax + by = p$ form a straight line, except when a, b are both zero. Let us assume that a, b are not both zero and call the line l. Another equation of the form $cx + dy = q$ also represents a straight line, which we denote by m. Finding a simultaneous solution for the system

$$ax + by = p, \qquad cx + dy = q$$

actually comes down to the determination of the point of intersection of l and m. Usually two lines l, m in the plane intersect in precisely one point. But there are two exceptions. The first is when l and m are distinct but parallel. Parallel lines do not intersect, so there is no solution for the corresponding system. The other exception is when l and m coincide. Then there are of course infinitely many points of intersection. The lines l, m coincide if the corresponding equations differ by a constant factor (as in our example).

Now you can also understand why three linear equations in two unknowns generally do not have a solution. Three lines in the plane generally do not meet in one point. If they do, we are in a very special situation.
A similar geometrical interpretation holds for equations in 3 unknowns x, y, z. The points (x, y, z) in three dimensional space that satisfy a linear equation $ax+by+cz = p$ form a two-dimensional plane (assuming that not all numbers a, b, c are zero). Solving three linear equations in three unknowns thus comes down to determining a common point of intersection of three planes in space. If the planes are in general position, they meet in exactly one point, and the corresponding linear equations have precisely one solution. But there are special configurations of the three planes, which have no common point or which have infinitely points in common. It is a very nice exercise to think of all such possible configurations.

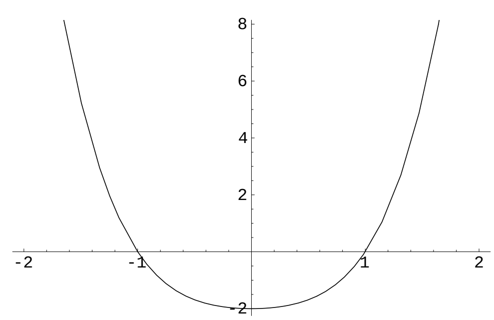
2.6 Arbitrary systems of equations

As we said before, there is no general method to solve systems van m equations in n unknowns. The general principle is to eliminate variables, but this may become quite complicated. So we just give a number of examples here.

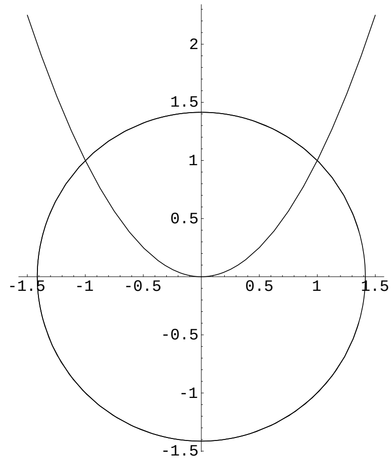
Example The system

$$\begin{aligned} y &= x^2 \\ x^2 + y^2 &= 2 \end{aligned}$$

We can eliminate y by substitution of $y = x^2$ in the second equation. We get $x^2 + x^4 = 2$. Usually we use the computer to solve such fourth degree equations, but here we are lucky to see that $x = 1$ and $x = -1$ are solutions. They are also the only solutions, which we can see by making a plot of $x^4 + x^2 - 2$,

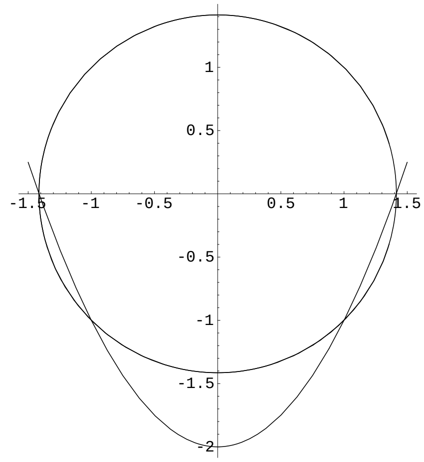


There is also geometrical interpretation of our system. The points (x, y) that satisfy $y = x^2$ form a parabola and $x^2 + y^2 = 2$ is a circle. Solving the system comes down to determining the intersection of these two curves. Here is a combined picture,



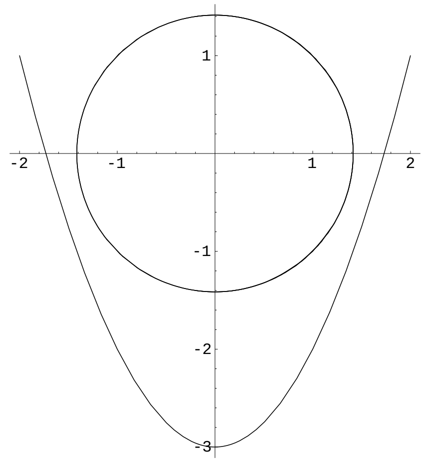
Note the two points of intersection corresponding to $x = \pm 1$ and $y = 1$. If we would sink the parabola a bit, there is a possibility that the parabola intersects the circle in four points. Here is a picture of the system

$$\begin{aligned} y &= x^2 - 2 \\ x^2 + y^2 &= 2 \end{aligned}$$



Substitution of $y = x^2 - 2$ in $x^2 + y^2 = 2$ gives $x^2 + (x^2 - 2)^2 = 2$. After evaluation of the square we get $x^4 - 3x^2 + 2 = 0$. Again we see the solutions $x = \pm 1$, but in addition we get $x = \pm\sqrt{2}$. The corresponding values of y are $y = -1$ and $y = 0$ respectively.

Sinking the parabola even lower, say $y = x^2 - 3$ would give us a system that has no solutions at all. Here is the combined plot of $y = x^2 - 3$ and $x^2 + y^2 = 2$,



2.7 Exercises

Exercise 2.7.1 *We determine* $\sqrt[3]{10}$ *via Newton's method, by determining the zero of* $x^3 - 10$*.*

1. *Show that the Newton iteration looks like*

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{10}{x_n^2}\right).$$

2. *Choose a starting value* x_0 *and make a table of the values* $x_0, x_1, \ldots, x_6$ *(you can use your pocket calculator to do the computations, without touching the cube root button though). Compare the final* x_6 *with the actual value of* $\sqrt[3]{10}$ *(now you may use the cube root button, or simply take the cube of* x_6*).*

Exercise 2.7.2 *We perform the same exercise, but now to determine the (unique) zero of* $x^3 - x - 1$*.*

1. *Show that the Newton iteration looks like*

$$x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 - 1}$$

2. *Choose a starting value x_0, say $x_0 = 2$, and make a table of the values $x_0, x_1, \ldots, x_6$ (you can use your pocket calculator to do the computations). To check how close x_6 is to the actual root, simply compute $x_6^3 - x_6 - 1$.*

Exercise 2.7.3 *Solve the following systems of linear equations*

$$\text{(a)} \quad \left\{ \begin{array}{rrcl} x_1 & -3x_2 & = & 5 \\ & x_2 & = & 4 \end{array} \right. \qquad \text{(b)} \quad \left\{ \begin{array}{rcl} x_1 \qquad\quad -3x_2 & = & 5 \\ -2x_1 + 5x_2 & = & 4 \end{array} \right.$$

$$\text{(c)} \quad \left\{ \begin{array}{rrrcl} x_1 & +3x_2 & -2x_3 & = & 4 \\ & x_2 & +5x_3 & = & 2 \\ & & x_3 & = & 2 \end{array} \right. \quad \text{(d)} \quad \left\{ \begin{array}{rrrcr} x_1 & -2x_2 & -x_3 & = & 5 \\ & x_2 & +3x_3 & = & 4 \\ & & x_3 & = & -2 \end{array} \right.$$

Exercise 2.7.4 *Solve the following systems of linear equations*

$$\text{(a)} \quad \left\{ \begin{array}{rrrcr} x_1 & +x_2 & & = & -7 \\ 2x_1 & +4x_2 & +x_3 & = & -16 \\ x_1 & +2x_2 & +x_3 & = & 9 \end{array} \right. \qquad \text{(b)} \quad \left\{ \begin{array}{rrrcl} x & +2y & +z & = & 0 \\ 3x & +2y & +z & = & 2 \\ 2x & +3y & +2z & = & 2 \end{array} \right.$$

Exercise 2.7.5 *Suppose we have three non-zero numbers x, y, z that add up to 1000. Suppose we also know that x is three times y and z two times $x - y$. Determine x, y, z.*

Exercise 2.7.6 *Certain rocks are composed of the minerals wollastonite $CaSiO_3$ (Wo), pyroxene $CaAl_2SiO_6$ (Py) and quartz SiO_2 (Qu). These minerals are all composed of the oxides SiO_2, Al_2O_3 and CaO. Analysis of such rock samples reveal the weight percentages of these oxides. In a certain sample, entirely composed of the above minerals, it is found that there is 63.8 weight percent SiO_2, 14.0 percent Al_2O_3 and 22.2 percent CaO. It is given that the atomic weights of O,Al,Si and Ca are 16,27,28 and 40 respectively.*

1. *Determine the molecular weights of the oxides and minerals.*

2. *Let x_1, x_2, x_3 be the weight percentages of Wo, Py, Qu present in the rock sample. Compute x_1, x_2, x_3.*

Exercise 2.7.7 *A greengrocer sells apples, bananas and oranges. Altogether he has 1500 pieces of fruit. The average weight of an apple, banana and orange are 120, 140 and 160 gram respectively. He sells the apples for 50 ct apiece, the bananas for 40 ct and the oranges for 60 cent. Suppose the total weight of the fruit is 208 kilogram and the greengrocer sells everything for 760 guilders. How many pieces of each fruit did he have at the beginning?*

Exercise 2.7.8 *Make a geometrical sketch of the following two systems of equations and then solve them.*

(a) $\left\{\begin{array}{rcl} x^2+y^2 &=& 2 \\ y &=& 2x+3 \end{array}\right.$ (b) $\left\{\begin{array}{rcl} x^2+y^2 &=& 2 \\ (x-1)^2+(y-2)^2 &=& 1 \end{array}\right.$

Exercise 2.7.9 *Solve the following two systems of equations.*

(a) $\left\{\begin{array}{rcl} x^2+2y^2 &=& 6 \\ y &=& zx \\ x &=& 4zy \end{array}\right.$ (b) $\left\{\begin{array}{rcl} x^2+y^2+z^2 &=& 10 \\ y &=& x+2 \\ z &=& 2x-y \end{array}\right.$

Chapter 3

Complex numbers

3.1 Introduction

From the moment the root function $\sqrt{x}$ was introduced, we were told that one cannot extract roots from negative numbers. In other words, the square root of a negative number does not exist. However, history of mathematics has taught us otherwise. It turns out that we can simply pretend that $\sqrt{-1}$ exists for example, and that we can just compute with it like any other number. It should only be realised then, that we are not working within the system of common numbers, or *real numbers* $\mathbb{R}$, but in the larger system of *complex numbers*. Before 1750 the use of complex numbers was restricted as a handy tool to determine (real) solutions of polynomial equations. After 1750 it became increasingly clear that complex numbers play a fundamental role in the study of functions and their integrals. Nowadays complex numbers are as indispensable in the exact sciences as the common real numbers.

3.2 Arithmetic

We shall denote the quantity $\sqrt{-1}$ by i. An expression of the form $a + bi$, where a and b are real numbers, is called a *complex number*. Examples: $2 + i$, $-1 + 3i$, $\pi - \sqrt{2}\,i$, $e + \pi i$, etcetera. The set of complex numbers is denoted by $\mathbb{C}$. It is obvious that the real numbers $\mathbb{R}$ form a subset of $\mathbb{C}$. For any complex number $z = a + bi$ we call a the *real part* of z and b the *imaginary part* of z. Complex numbers with zero imaginary part are of course real numbers. Numbers with zero real part, i.e. of the form bi, are

called *imaginary numbers.*
Within the complex numbers we can do arithmetic in the usual way. We have

Addition and subtraction. In general,

$$a + bi \ + \ c + di = a + c + (b + d)i$$

and

$$a + bi \ - \ (c + di) = a - c + (b - d)i$$

Example: $3+2i + (-5+3i) = 3-5+(2+3)i = -2+5i$ en $3+2i - ((-5)+3i) = 3 + 5 + (2 - 3)i = 8 - i$.

Multiplication. In general:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = ac - bd + (ad + bc)i$$

Example: $(3 + 2i)(-5 + 3i) = -15 + 9i - 10i + 6i^2 = -21 - i$. Notice in particular that $(a + bi)(a - bi) = a^2 + b^2$.

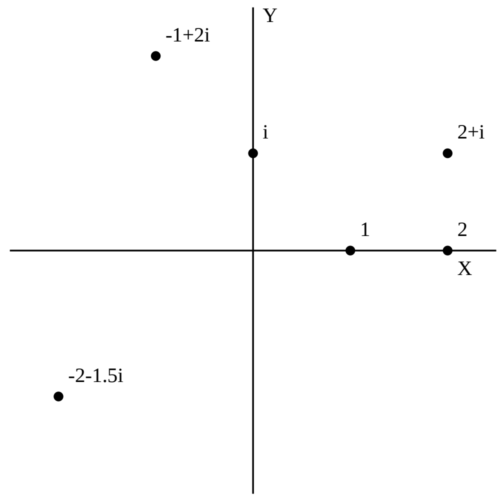
Division.In general:

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

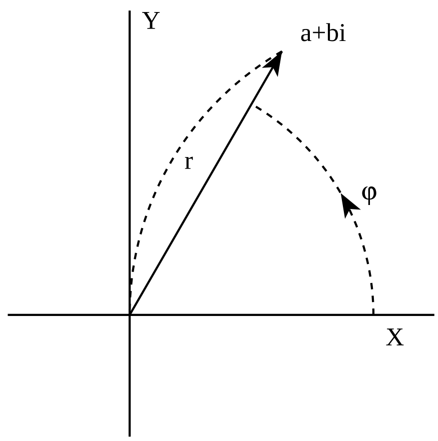
In the first step we multiplied numerator and denominator by $c - di$. In this way the numbers becomes equal to $c^2 + d^2$, which is a real number. Example:

$$\frac{3 + 2i}{-5 + 3i} = \frac{(3 + 2i)(-5 - 3i)}{(-5 + 3i)(-5 - 3i)} = \frac{-15 + 6 - (9 + 10)i}{25 + 9} = -\frac{9}{34} - \frac{19}{34}i.$$

Complex numbers can be represented in the XY-plane. We simply associate the point (a, b) to the complex number $a + bi$. In this way we can think of the entire XY-plane as being occupied by complex numbers. We therefore speak of the *complex plane*. The real number are then located precisely on the X-axis. The numbers on the Y-axis are the *imaginary numbers*.

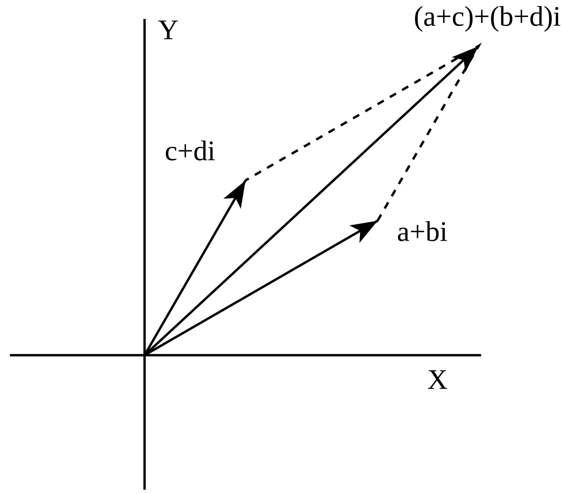


To any complex number $z = a + bi$ we assign the *absolute value* or *modulus* r and an *argument* ϕ as in the following picture. Notation, $r = |z|$, $\phi = \arg(z)$.



Convince yourself that a complex number is uniquely determined by its absolute value and argument. So the numbers r and ϕ give a second characterisation of complex numbers. They are often called the *polar coordinates*. There are two subtleties though. One is that the argument is not uniquely determined, but only up to multiples of 2π. To any argument we can add 2π radians (or 360 degrees), and the complex number doesn't change. A second, smaller subtlety is dat the argument of $z = 0$ is completely undetermined. However, having an argument in this case is not really necessary since $|z| = 0$ already determines z uniquely as $z = 0$.

Using the complex plane representation we can illustrate addition and multiplication. Addition:



It turns out that addition of the complex numbers $a+bi$ and $c+di$ corresponds simply to the vector addition of the vectors (a, b) and (c, d).

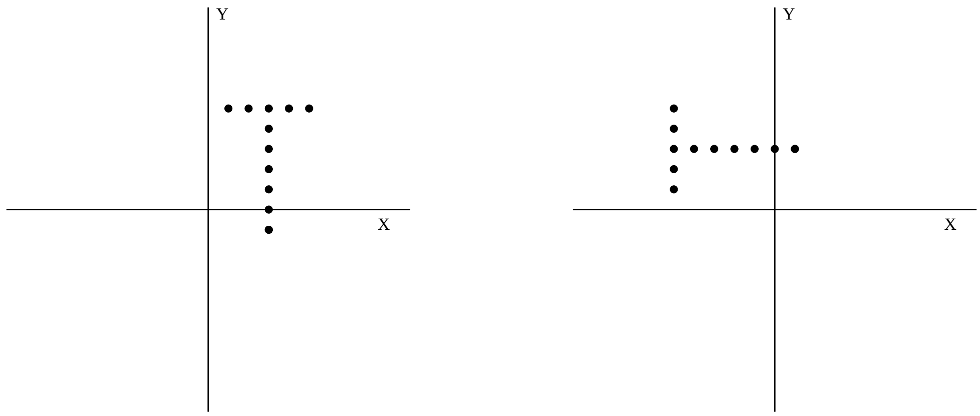
For multiplication we have the following useful rules.

Theorem 3.2.1 *Let z_1, z_2 be any complex numbers. Then,*

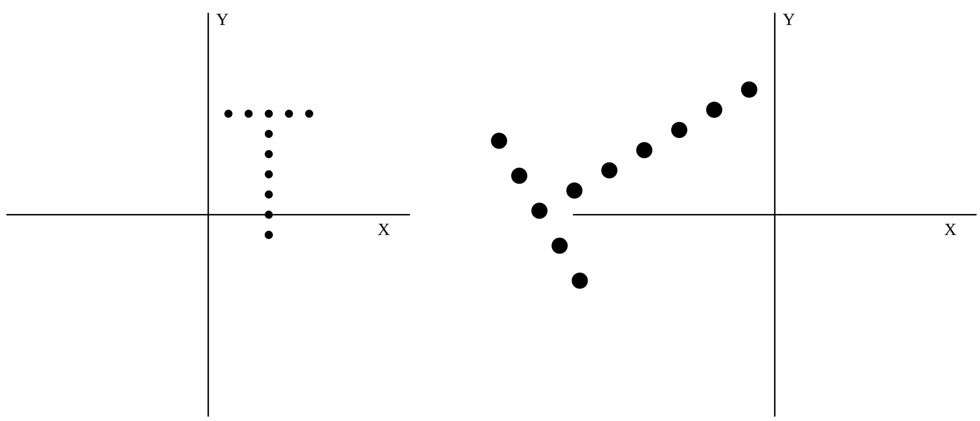
1. *i)* $|z_1 z_2| = |z_1||z_2|$ *('The absolute value of a product equals the product of the absolute values').*

2. *ii)* $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ *('The argument of a product equals the sum of the arguments').*

In the last section of this chapter we shall show prove this theorem. We illustrate multiplication by i with two pictures. In the left hand picture we have selected a number of points in the complex plane. In the right hand picture you see these points, all multiplied by i.



Multiplication by i causes a 90 degree anti-clockwise rotation. The explanation is simple. For any complex number z we have $|iz| = |i| \cdot |z| = |z|$ and $\arg(iz) = \arg(i) + \arg(z) = \pi/2 + \arg(z)$. In other words, iz has been rotated 90 degrees (=$\pi/2$ radians) with respect to z.
Another example, multiplication by $-1 + \sqrt{3}\,i$. We have $|-1 + \sqrt{3}\,i| = 2$ and $\arg(-1 + \sqrt{3}\,i) = 2\pi/3$. This means that multiplication by $-1 + \sqrt{3}\,i$ causes a magnification by a factor 2 and rotation over an angle $2\pi/3$ (=120 degrees).



3.3 The exponential notation

It is not very hard to establish the connection between the rectangular coordinates a, b of a complex number and its polar coordinates r, ϕ. Convince yourself that thsi relation is given by

$$a = r\cos\phi, \qquad b = r\sin\phi.$$

So we get

$$a + bi = r(\cos\phi + i\sin\phi).$$

Complex numbers of the form $\cos\phi + i\sin\phi$ lie on the unit circle in the complex plane (i.e. their absolute value is 1, and their argument is equal to ϕ. Using this representation it is not hard to prove Theorem 3.2.1 about the rules of multiplication for complex numbers. Suppose that we have two complex numbers z_1, z_2 whose absolute values are r_1, r_2 and with arguments ϕ_1, ϕ_2. Then we have

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\phi_1 + i\sin\phi_1)(\cos\phi_2 + i\sin\phi_2) \\ &= r_1 r_2(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 + \\ &\quad i(\sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2)) \end{aligned}$$

Now we use the standard addition formulae for sine and cosine,

$$\begin{aligned} \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 &= \cos(\phi_1 + \phi_2) \\ \sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2 &= \sin(\phi_1 + \phi_2) \end{aligned}$$

to get

$$z_1 z_2 = r_1 r_2(\cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2)).$$

From this formula we can read off that the absolute value of $z_1 z_2$ is $r_1 r_2 = |z_1| \cdot |z_2|$ and that the argument of $z_1 z_2$ is equal to $\phi_1 + \phi_2 = \arg(z_1) + \arg(z_2)$. On might say that the argument of a complex number acts as if it is some sort of logarithm with respect to multiplication. An immediate consequence of the addition property of arguments is the following.

Theorem 3.3.1 (De Moivre's theorem) *We have for any integer n and any real number ϕ that*

$$(\cos\phi + i\sin\phi)^n = \cos(n\phi) + i\sin(n\phi).$$

Now comes an interesting step. We put

$$e^{i\phi} = \cos\phi + i\sin\phi.$$

Then De Moivre's formula can be rewritten as

$$(e^{i\phi})^n = e^{n\phi},$$

a very natural looking formula. With this notation we also get the natural looking formula

$$e^{i\phi_1}e^{i\phi_2} = e^{i(\phi_1+\phi_2)}.$$

In fact, the connection between the exponential function e^z at imaginary values z and $\cos, \sin$ becomes even more apparent if we look at Taylor series expansions. Notice that

$$e^{i\phi} = 1 + \frac{i\phi}{1!} + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^3}{3!} + \frac{(i\phi)^4}{4!} + \frac{(i\phi)^5}{5!} + \frac{(i\phi)^6}{6!} + \cdots$$

First we collect the even powers of $i\phi$. We get

$$1 + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^4}{4!} + \frac{(i\phi)^6}{6!} + \cdots = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \cdots$$

You probably recognize the Taylor series for the cosine function. If you don't believe, just compute a few more terms. If we collect the odd powers of $i\phi$ we get

$$\frac{i\phi}{1!} + \frac{(i\phi)^3}{3!} + \frac{(i\phi)^5}{5!} + \frac{(i\phi)^7}{7!} + \cdots = i\frac{\phi}{1!} - i\frac{\phi^3}{3!} + i\frac{\phi^5}{5!} - i\frac{\phi^7}{7!} + \cdots$$

This time you may recognize i times the Taylor series of the sine function. So we conclude that, once again,

$$e^{i\phi} = \cos\phi + i\sin\phi.$$

This relation was discovered by the famous 18-th century mathematician Leonhard Euler. In particular, if we take $\phi = \pi$ we get

$$e^{\pi\sqrt{-1}} = -1,$$

a formula which gives a mysterious looking connection between the three equally mysterious numbers $e, \pi, \sqrt{-1}$.

From now on, when we have a complex number z with absolute r and argument ϕ, we write

$$z = re^{i\phi}.$$

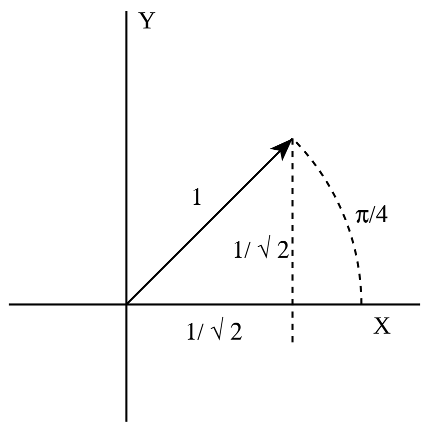
3.4 Polynomial equations

We have seen that complex arose because we extended the real numbers $\mathbb{R}$ by the new quantity $i = \sqrt{-1}$. Notice now that the roots of all negative numbers belong to $\mathbb{C}$. For example, $\sqrt{-2} = \sqrt{2}\, i$,
$sqrt{-9} = 3i$ etc. Notice by the way that $-i,\ -\sqrt{2}\, i,\ -3i$ can be considered as roots of $-1, -2, -9$ equallly well. This corresponds to the fact that $x^2 = -1,\ x^2 = -2,\ x^2 = -9$ etc. has two solutions. For square roots of positive numbers you are probably used to taking the positive value, but of course we could also have taken the negative root.

The nice thing about complex numbers is, that square roots of complex numbers also exist. An example, solve $X^2 = i$ (i.e. determine $\sqrt{i}$). The easiest way to do this is to use the exponential notation and writing $i = e^{\pi i/2}$. So we must solve

$$X^2 = e^{\pi i/2}$$

This gives us $X = e^{\pi i/4}$. This solution is a complex number with a 45 degree argument and absolute value 1. In a picture,

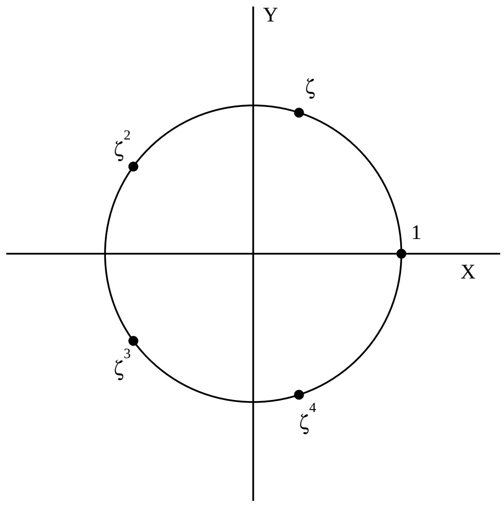


We see that $1/\sqrt{2} + i/\sqrt{2}$ is a solution. There is also a second solution which we get if we write $i = e^{(2+1/2)\pi i}$. Solving $X^2 = e^{(5/2)\pi i}$ gives us $X = e^{5\pi i/4} = e^{\pi i} e^{\pi i/4} = -e^{\pi i/4}$ as was to be expected. The second solution is opposite, namely $-1/\sqrt{2} - i/\sqrt{2}$.

But there is more. For example, consider the equation $X^5 = 1$. One solution is clear, $X = 1$ and this is the only real solution in $\mathbb{R}$. In $\mathbb{C}$ there are more solutions however. We can see them if we write $1 = e^{2\pi k i}$ where k is an integer. Solving $X^5 = e^{2\pi ki}$ gives us

$$X = 1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$$

In principle we could continue with $e^{10\pi i/5}$ but this equals $e^{2\pi i} = 1$ and the sequence of solutions repeats itself. So we have found five distinct solutions whose absolute values are 1 and with arguments $0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5$. Here is a picture in the complex plane,



Moreover, these are all solutions and we have that

$$X^5 - 1 = (X-1)(X-\zeta)(X-\zeta^2)(X-\zeta^3)(X-\zeta^4).$$

Perhaps you find it a number like $e^{2\pi i/5}$ a bit unsatisfactory. It is possible to write its real and imaginary part. We would get

$$\zeta = -\frac{1}{4} + \frac{\sqrt{5}}{4} + i\sqrt{\frac{5+\sqrt{5}}{8}}.$$

Not very illuminating. Also, writing down its numerical value does not reveal much,

$$\zeta = 0.3090169943\cdots + (0.9510565162\cdots)i.$$

The important thing to remember is that $e^{2\pi i/5}$ is a complex number, which is uniquely determined by its absolute value and argument.
One of the amazing things about complex numbers is that they do not only allow us to take suare roots, or higher roots, but also allow us to solve any polynomial equation.

Theorem 3.4.1 (Main Theorem of algebra) *Consider any polynomial $P(X)$ of degree n given by*

$$X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_1X + a_0$$

where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$. Then there exist precisely n complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$P(X) = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_n).$$

In particular, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are solutions of $P(X) = 0$.

Roughly speaking, this theorem tells us that an n-th degree polynomial equation has n solutions in $\mathbb{C}$ ($\alpha_1, \alpha_2, \ldots, \alpha_n$, to wit). Proving this theorem is not very easy. Although complex numbers were used in the 18-th century, it was not until the beginning of the 19-th century that the famous mathematician C.F.Gauss convincingly showed that the theorem is true under all circumstances.

3.5 Exercises

Exercise 3.5.1 *Determine the sum and product of the following pairs of numbers*

$$\text{(a)} \begin{matrix} 2+i \\ 3+2i \end{matrix} \quad \text{(b)} \begin{matrix} 1-2i \\ 1+3i \end{matrix} \quad \text{(c)} \begin{matrix} 2i \\ 3+7i \end{matrix} \quad \text{(d)} \begin{matrix} i+\sqrt{3} \\ 1+i\sqrt{3} \end{matrix} \quad \text{(e)} \begin{matrix} 5+3i \\ 5-3i \end{matrix}.$$

Exercise 3.5.2 *Determine the following quotients,*

$$\text{(a)} \ \frac{2+i}{3+2i} \quad \text{(b)} \ \frac{2+3i}{1+2i}, \quad \text{(c)} \ \frac{2i}{2-5i} \quad \text{(d)} \ \frac{5}{2+i} \quad \text{(e)} \ \frac{3+i}{1+i}.$$

Exercise 3.5.3 *Write the following complex numbers in the polar form $re^{i\phi}$,*

$$\text{(a)} \ 1+i \quad \text{(b)} \ 1+i\sqrt{3} \quad \text{(c)} \ -3$$
$$\text{(d)} \ 4i \quad \text{(e)} \ \sqrt{3}-i \quad \text{(f)} \ -1-i.$$

Exercise 3.5.4 *Put the following complex numbers in the rectangular form $a + bi$,*

$$\text{(a)} \ e^{3\pi i} \quad \text{(b)} \ e^{2\pi i/3} \quad \text{(c)} \ 3e^{i\pi/4} \quad \text{(d)} \ \pi e^{-i\pi/3}.$$
$$\text{(e)} \ e^{2\pi i/6} \quad \text{(f)} \ 2e^{-i\pi/2} \quad \text{(g)} \ e^{-i\pi} \quad \text{(h)} e^{-5i\pi/4}.$$

Exercise 3.5.5 *Let α be a complex number $\neq 0$. Show that there are two distinct complex numbers whose square is α.*

Exercise 3.5.6 *Let $a + bi$ be a complex number. Find real numbers x, y so that*

$$(x + iy)^2 = a + bi$$

by expressing x, y in terms of a, b.

Exercise 3.5.7 *Find the real and imaginary part of $i^{1/4}$, with argument lying between 0 and $\pi/2$.*

Exercise 3.5.8 *Using De Moivre's formula for $n = 3$, show that*

$$\cos(3\phi) = (\cos\phi)^3 - 3\cos\phi(\sin\phi)^2.$$

Chapter 4

Functions in several variables

In practical applications we often encounter functions in several variables of which we need to determine (local) maxima and minima.

First we say what is meant by a function in several variables. Let D be a region of $\mathbb{R}^n$. A function f which assigns to every point in D a real number value, is called a function in n variables. Examples of two variable functions are

$$f(x,y) = x^2 + y^2 \text{ on } D = \mathbb{R}^2$$

$$f(x,y) = y/x \text{ on } D = \mathbb{R}^2 \text{ minus y} - \text{axis}$$

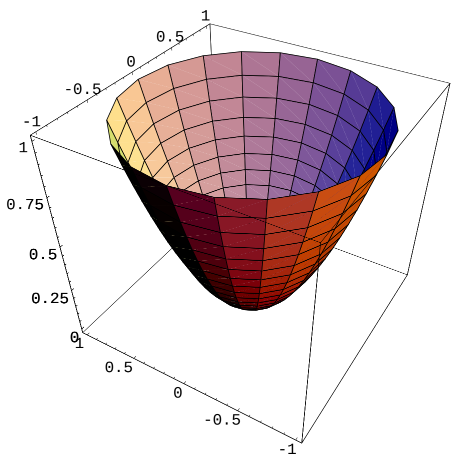
$$f(x,y) = \frac{1}{1 - x^2 - y^2} \text{ on } x^2 + y^2 < 1.$$

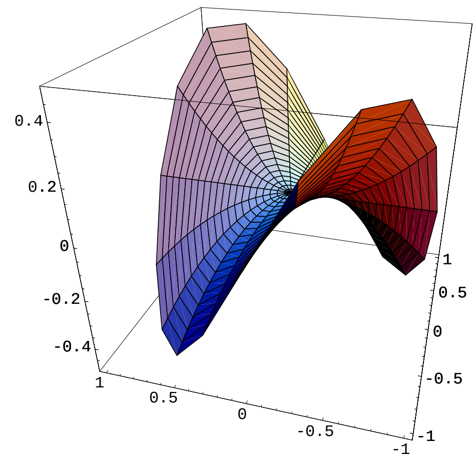
Examples of three variable functions,

$$f(x,y,z) = x^2 + y^2 + z^2, \quad f(x,y,z) = \frac{\sin(z)}{x + y + 1}, \quad f(x,y,z) = \log(x - yz).$$

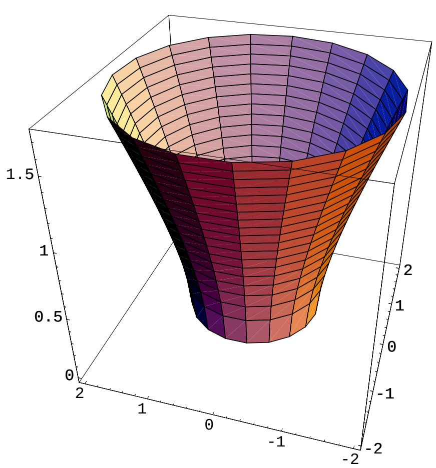
In the remainder of this chapter we will restrict our attention to two-variable functions. However, most of the things we say, almost immediately carries over to n-variable functions.

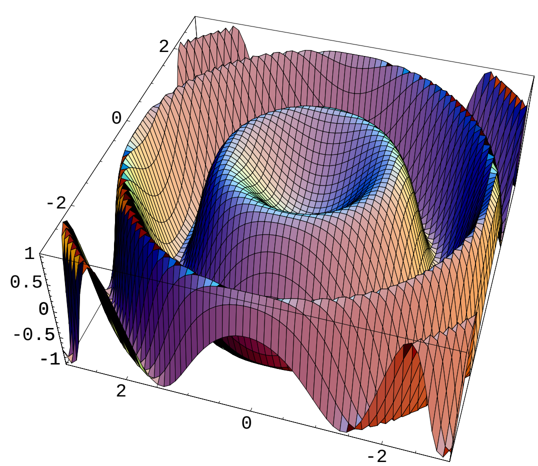
Functions of two variables can be plotted in three dimensions. Here are two examples, the functions $f(x,y) = x^2 + y^2$ and $f(x,y) = xy$ both defined on $D = \mathbb{R}^2$.



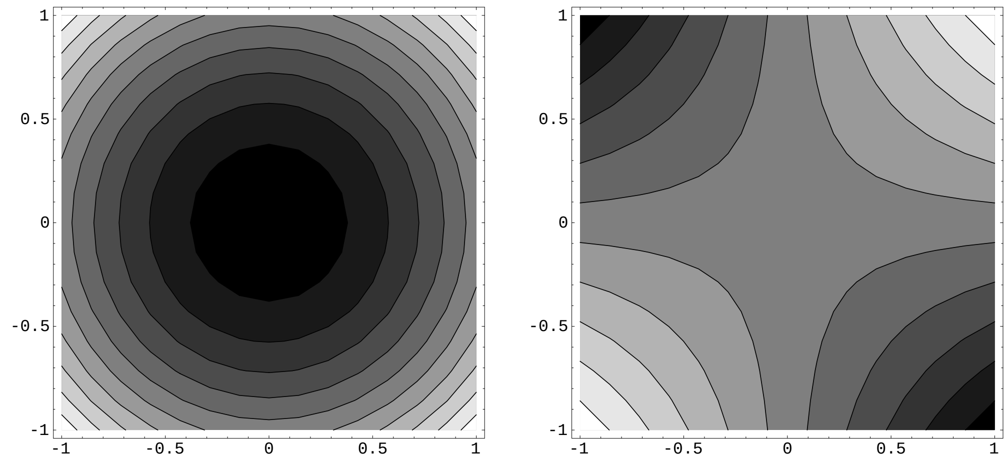


Another two examples are $f(x, y) = \sqrt{x^2 + y^2 - 1}$ defined on $x^2 + y^2 \geq 1$ and $\sin(x^2 + y^2)$ defined on $\mathbb{R}^2$.





As you see, the three-dimensional graphs of functions of two variables look like mountaineous landscapes. This immediately suggests another way of plotting two-variable functions. By drawing the curves of equal height we get a so-called *contour plot* of our function. This is nothing else than the lines of altitude that one can see plotted on maps of mountain ranges. Or, if you want, the isobaric lines on a weather map. Here are the contourplots of $f(x, y) = x^2 + y^2$ and $f(x, y) = xy$,



Strictly speaking, in a contourplot we plot the so-called *level lines* of the function $f(x, y)$. The level line of $f(x, y)$ corresponding to the value c is the curve in the x, y-plane given by the implicit equation $f(x, y) = c$.
From your experience with mountain maps you can undoubtedly see that the point $(0, 0)$ in the first contour plot indicates either a mountain peak or the bottom of a valley (which one is it?). The second plot, that of $f(x, y) = xy$, strongly indicates a mountain pass at the point $(0, 0)$ (compare with the 3D-plot). In the next section we shall continue the discussion of local maxima and minima for functions in several variables. For the moment we need a few more facts on functions in several variables.

4.1 Partial derivatives

Functions in several variables can be differentiated, just like functions of one variable. But since there are more variables we can also choose with respect to which variable we like to differentiate. Let $f(x, y)$ be a function of two variables. The *partial derivative* with respect to x is the derivative we get if we treat y as an ordinary constant. Notations: $\partial_x f$, $\frac{\partial f}{\partial x}$ or f_x. Similarly, if we treat the x-variable as a constant we get the partial derivative with respect to y. Notations: $\partial_y f$, $\frac{\partial f}{\partial y}$ or f_y. Of course we can also repeat differentiation to get second and higher derivatives. But there are several kinds of them, since we can choose a variable at each differentiation. Here are the second derivatives,

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}.$$

In alternative notations

$$\partial_x^2 f, \quad \partial_x\partial_y f, \quad \partial_y\partial_x f, \quad \partial_y^2 f$$

and

$$f_{xx}, \quad f_{xy}, \quad f_{yx}, \quad f_{yy}.$$

Of course these differentiations can only take place under the assumption that the derivatives actually exist. Here are some partial derivatives of $x\sin(x^2+y^2)$.

$$\frac{\partial f}{\partial x} = 2x^2\cos(x^2+y^2)+\sin(x^2+y^2), \quad \frac{\partial f}{\partial y} = 2xy\cos(x^2+y^2)$$

and the second derivatives

$$\frac{\partial^2 f}{\partial x^2} = 6x^2\cos(x^2+y^2)-4x^3\sin(x^2+y^2), \quad \frac{\partial^2 f}{\partial y^2} = 2xy\cos(x^2+y^2)-4xy^2\sin(x^2+y^2)$$

$$\frac{\partial^2 f}{\partial y\partial x} = 2y\cos(x^2+y^2)-4x^2y\sin(x^2+y^2), \quad \frac{\partial^2 f}{\partial x\partial y} = 2y\cos(x^2+y^2)-4x^2y\sin(x^2+y^2).$$

Carry out the computation yourself (by hand!). Notice that the last two mixed derivatives are equal. You may not have expected this while doing the calculation, but this equality is no coincidence. It works for any function f that is sufficiently often differentiable.

Theorem 4.1.1 *Let $f(x,y)$ be a function of two variables in a region $D\subset\mathbb{R}^2$. Suppose that the first and second order derivatives of f exist and are continuous functions in the interior of D. Then we have*

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 f}{\partial x\partial y}$$

in every point (x,y) in the interior of D.

Roughly speaking, when doing calculations with repeated differentiation, the ordering of the variables in the differentiation does not matter. Just as with one variable functions, we have also Taylor approximations for functions in several variables. Suppose we have a function $f(x,y)$ in two variables defined in the neighbourhood of a point (a,b) and whose partial derivatives of sufficient high order also exist and are continuous. Then the linear Taylor approximation of $f(x, y)$ near the point (a, b) is given by

$$f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

This is a function of the form $Ax + By + C$, a linear function of x, y, and its three dimensional graph is a plane. In fact, this plane is precisely the tangent plane of the graph of $f(x, y)$ above the point (a, b).
There is also a second order Taylor approximation which reads,

$$\begin{aligned} & f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + \frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2. \end{aligned}$$

In less elaborate notation,

$$\begin{aligned} & f(a,b) + f_x(a,b)(x-a) + f_y(a,b) \\ & + \frac{1}{2} f_{xx}(a,b)(x-a)^2 + f_{xy}(x-a)(y-b) + \frac{1}{2} f_{yy}(a,b)(y-b)^2. \end{aligned}$$

There also exist higher order Taylor approximations, but they become progressively more complicated. The general shape of the n-th order term is like this. Write down all possible terms of the form

$$\frac{1}{n!}\frac{\partial^n f}{\partial x_1 \partial x_2 \cdots \partial x_n}(a,b)(x_1 - a_1)(x_2 - a_2)\cdots(x_n - a_n)$$

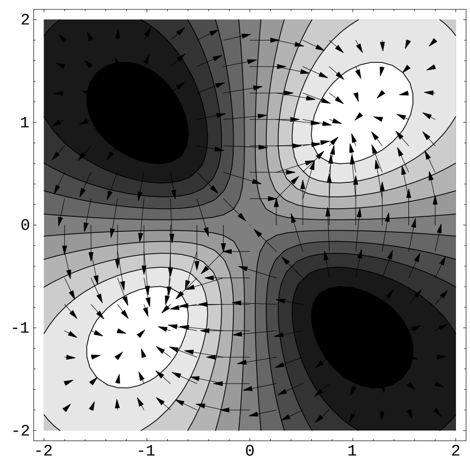
where, for each i the variables x_i, a_i are either x, a or y, b. So for each i we have 2 choices, which gives us a total of 2^n terms. The sum of all these terms is the n-th term in a Taylor approximation.
For example, for $n = 2$ the four terms read

$$\begin{aligned} & \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2, \quad \frac{1}{2}\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) \\ & \frac{1}{2}\frac{\partial^2 f}{\partial y \partial x}(a,b)(y-b)(x-a), \quad \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2 \end{aligned}$$

Since the order of differentiation does not matter, the middle two terms are the same and we can group them together to get the three second order terms you see in the second order Taylor approximation.

4.2 The gradient

Suppose we have a function of two variables defined in a region $D \subset \mathbb{R}^2$. In every point $(a,b) \in D$ we can consider the 2-vector $(f_x(a,b), f_y(a,b))$. This vector is called the *gradient* of the function f at the point (a,b). It is denoted by ∇f or by grad$f$. The gradient vectors form an examples of a so-called *vector field*. At every point we can draw an arrow corresponding to the gradient. Here is a picture of the contourlines of the function $xy/(2+x^2+y^2)^2$ together with the gradient vectors at a large number of points.



Notice that the gradient vectors are all perpendicular to the level lines. In fact, the direction of a gradient vector in a point (a,b) is precisely the direction of steepest ascent of f at the point (a,b). An explanation for this phenomenon can be found in the rest of this section.

Theorem 4.2.1 *Suppose we have a curve C in the plane given by the implicit equation $g(x,y) = 0$. Let (a,b) be a point on C. Then the line in the plane which is tangent to C in the point (a,b) is given by the equation*

$$g_x(a,b)(x-a) + g_y(a,b)(y-a) = 0.$$

As an example consider the circle $x^2 + y^2 = 5$. It contains the point $(2,1)$. Taking $f(x,y) = x^2 + y^2 - 5$ we get $f_x(2,1) = 4$, $f_y(2,1) = 2$. So the tangent in $(2,1)$ of the circle is given by

$$4(x-2) + 2(y-1) = 0.$$

In simplified form, $4x + 2y - 10 = 0$ or, if you want, $y = 5 - 2x$.

The idea of the proof is simply that the tangent line is actually the 0-level line of the linear approximation of $g(x, y)$ at the point (a, b). The linear approximation of g is given by $g_x(a, b)(x - a) + g_y(a, b)(y - b)$ and its 0-level line by $0 = g_x(a, b)(x - a) + g_y(a, b)(y - b)$. This is precisely the equation of our tangent.
As an application to the level lines of a function we have,

Consequence 4.2.2 *Let $f(x, y)$ be a function defined in a region $D \subset \mathbb{R}^2$ and (a, b) an interior point of D. Then the tangent to the level line of f through the point (a, b) is given by $f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0$.*

Suppose $f(a, b) = C$. Then the level line of f through (a, b) is given by $f(x, y) = C$ or, $f(x, y) - C = 0$. Now apply Theorem 4.2.1 to the function $g(x, y) = f(x, y) - C$. Since $g_x = f_x$ and $g_y = f_y$, our consequence follows immediately.
The explanation why the gradient is perpendicular to the level line is now simple. Suppose we have a straight line given by the equation $Ax + By = C$ with $(A, B) \neq (0, 0)$. Then the vector (A, B) is perpendicular to this line. For a derivation see the Chapter on Vectors. In particular, we know that the level line of the function f through (a, b) has tangent given by $f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0$. Hence the vector $(f_x(a, b), f_y(a, b))$ is perpendicular to it and thus perpendicular to the level line.

4.3 Exercises

Exercise 4.3.1 *Determine $f_x(x, y), f_y(x, y)$ when*

$$\text{(a) } f(x, y) = e^{xy}, \quad \text{(b) } f(x, y) = x \sin y, \quad \text{(c) } f(x, y) = y \log x.$$

Exercise 4.3.2 *Determine $f_{xx}(x, y), f_{xy}(x, y), f_{yx}(x, y), f_{yy}(x, y)$ when*

$$\text{(a) } f(x, y) = e^{xy}, \quad \text{(b) } f(x, y) = x \sin y, \quad \text{(c) } f(x, y) = y \log x, \quad \text{(d) } f(x, y) = xe^{xy+y}.$$

Exercise 4.3.3 *An important equation in mathematical physics is the heat equation. The unknown function is a function of two variables, t (time) and x (place). It reads $u_t = Du_{xx}$, where D is some material constant. Show that each of the following functions is a solution*

1. $u(x, t) = x^4 + 12Dx^2t + 12D^2t^2$

2. $u(x,t) = e^{4Dt-2x}$
3. $u(x,t) = e^{-4Dt}\cos(2x)$
4. $u(x,t) = e^{-x^2/4Dt}/\sqrt{t}$

Exercise 4.3.4 *Determine the equation of the plane in $\mathbb{R}^3$ which is tangent to the surface $z = F(x,y)$ in the point P for the following choices of F and P,*

1. $F(x,y) = x^3 + 2xy^2 - 6xy$ *and* $P = (1,1,-3)$
2. $F(x,y) = \log(1 + x^2 + 2y^2)$ *and* $P = (0,0,0)$
3. $F(x,y) = \sqrt{6 - x^2 - y^2}$ *and* $P = (1,1,2)$

Exercise 4.3.5 *Determine the equation of the line in $\mathbb{R}^2$ which is tangent to the curve $g(x,y) = 0$ in the point P for the following choices of g and P,*

1. $g(x,y) = x^3 - 3x^2 + 4y^2$ *and* $P = (-1,1)$
2. $g(x,y) = y^2 + xy^2 + x - 1$ *and* $P = (0,1)$

For each of these two cases also determine all points of the curve where the tangent is horizontal respectively vertical.

Exercise 4.3.6 *Determine the second order Taylor approximation of*

$$\frac{e^{xy}}{1 + x^2 + y^2}$$

around the point $(0,0)$.

Exercise 4.3.7 *Let us determine the third order Taylor approximation of a function in the point (a,b). We have already seen the shape of the linear and quadratic part of this approximation from the second order Taylor approximation. Determine the third order part we write down all possible terms*

$$\frac{1}{3!}\frac{\partial^3 f}{\partial x_1 \partial x_2 \partial x_3}(a_1,a_2)(x_1 - a_1)(x_2 - a_2)(x_3 - a_3)$$

where (x_i, a_i) runs over both choices (x,a) and (y,b) for $i = 1,2,3$. In total there are 8 such terms.

1. *Write down all these terms.*
2. *Group these terms together, using the fact that the order of differentiation does not matter.*

Chapter 5

Maxima and minima

5.1 Introduction

The problem of maxima and minima is something that keeps virtually everybody busy in daily life. For example, where do I buy the cheapest car, how do I invest my money at maximal profit, what is the shortest path from the bus stop to the lecture room, and so on. You can probably think of a host of similar problems. Questions like these may arise on a personal scale, a corporate scale, national scale or even international scale. An example of the latter is how to maximize world wide economic growth while keeping the CO_2-level at an accepted rate. Although mathematics cannot solve all these questions for us, there are certainly mathematical principles underlying these problems. Ever since the Second World War a new branch of mathematics, called optimisation and control theory, has evolved. It was stimulated precisely by optimisation problems in economics both on a micro and macroscopical scale. In this chapter we shall discuss the theory of maxima and minima of functions in one and several variables and have a brief meeting with so-called *Lagrange multipliers*. These Lagrange multipliers play an all pervasive role in optimisation theory.

5.2 Local maxima and minima in one variable

Let us consider a function $f(x)$ on an interval V. Let a be a point in V. We say that $f(x)$ has a local maximum in a if there is a small open interval $D = (a - \epsilon, a + \epsilon)$ around a, such that $f(a) \geq f(x)$ for every point in $V \cap D$.

Similarly, $f(x)$ has a local minimum in a if there is a small open interval $D = (a - \epsilon, a + \epsilon)$ around a, such that $f(a) \le f(x)$ for every point in $V \cap D$. There is a very nice criterion to determine local maxima and minima of a function $f(x)$. However, it only works for functions that are differentiable. To make thing mathematically correct we have to speak of the interior of interval. By this we mean all points of V minus its possible boundary points. So the interior of the interval $1 \le x \le 2$ is $1 < x < 2$, and the interior of the set of points with $x \ge 5$ is $x > 5$.

Theorem 5.2.1 *Let $f(x)$ be a function defined on an interval V, and let a be a point in the interior of V. Assume that $f(x)$ can be differentiated on the interior of V and that $f'x)$ is continuous. Suppose that $f(x)$ has a local maximum or minimum in a. Then $f'(a) = 0$.*

We do not give a rigorous proof, but informally it is not hard to see why this theorem is true. Above the point a the function $f(x)$ can be approximated by its linear approximation $l(x) = f(a) + f'(a)(x - a)$. The graph of this linear approximation is a straight line, tangent to the graph of $f(x)$ above $x = a$. If $f'(a) > 0$ or $f'(a) < 0$ the tangent line has positive, resp. negative slope. This means that the linear approximation is increasing resp. decreasing in a, and the same holds for $f(x)$. So if $f'(a) > 0$ or $f'(a) < 0$, the function $f(x)$ cannot have a local extreme in a. The only conclusion left is that $f'(a)$ should be zero.

So, to find local extrema of $f(x)$ we have to determine the zeros of $f'(x)$. The zeros of $f'(x)$ are called the *stationary points* of $f(x)$. However, not every stationary point is a point where $f(x)$ is locally extreme. For example, $f(x) = x^3$. Notice that $f'(0) = 0$, but $f(x)$ has no local extreme in $x = 0$ (explain why!). So, if we have found a stationary point of $f(x)$ we need some additional verification to see if we are dealing with a local extreme. There are various ways of doing this, but the most automatic one is by using the following Theorem.

Theorem 5.2.2 *Let $f(x)$ be a function defined on an interval I. Assume that $f(x)$ is at least twice differentiable in the interior of I and that $f''(x)$ is continuous on I. Let a be a stationary point of $f(x)$. Then,*

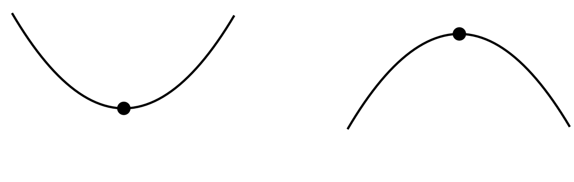
i) $f''(a) > 0 \Rightarrow f(x)$ *has a local minimum in a.*

ii) $f''(a) < 0 \Rightarrow f(x)$ *has a local maximum in a.*

Again we do not give a rigorous proof. But we can get some insight of why this theorem is true by looking at the second order Taylor approximation of $f(x)$ around $x = a$. Because $f'(a) = 0$ this second order approximation has the form

$$q(x) = f(a) + \frac{1}{2}f''(a)(x-a)^2.$$

When $f''(a) \neq 0$ the graph of this quadratic function is a parabola of one of the following two forms



The first picture corresponds to $f''(a) > 0$, the second to $f''(a) < 0$. The graph of $f(x)$ has a similar shape around $x = a$, and so we recognize which is the local maximum and which the local minimum.
When $f''(a) = 0$ the matter is undecided. For example, consider the functions x^3 and x^4 around $x = 0$. They both have first and second order derivative at $x = 0$. But x^3 has no local extremum at $x = 0$, whereas x^4 has a local minimum at $x = 0$. In cases like that, further investigation is required. We shall not do that however, since cases like these rarely occur in practice.

5.3 Local maxima and minima in several variables

In many practical circumstances one is asked to find the (local) maxima and minima of functions of several variables. In this section we shall discuss a technique to compute them. You will probably see that many of the things that in this section are almost the exact analogue of the one variable case.
First we need to say what we mean by a local minimum or maximum. Let $f(x, y)$ be a function of two variables which is defined in a region $V \subset \mathbb{R}^2$. We say that f assumes a *local maximum* in a point $(a, b) \in V$ if there is a small disk D, with positive radius and (a, b) as center, such that $f(x, y) \leq f(a, b)$

for all points $(x, y) \in V \cap D$. Similarly, f assumes a *local minimum* in (a, b) if there is a small disk D such that $f(x, y) \geq f(a, b)$ for all $(x, y) \in V \cap D$.
Just as in the one variable case there is a criterion which enables one to compute local maxima and minima. This criterion holds for interior points of V. A point (a, b) of a domain $V \subset \mathbb{R}^2$ is called *interior point* if there is a small disk around (a, b) with positive radius, which is completely contained in V. Roughly speaking, a point $(a, b) \in V$ is called interior if it is completely surrounded by points of V. The set of all interior points of V is simply called the interior of V, i.e. V with its boundary points stripped.

Theorem 5.3.1 *Let $f(x, y)$ be a function defined in a domain $V \subset \mathbb{R}^2$ and suppose its first derivatives exist in the interior of V and are continuous there. Suppose (a, b) is an interior point of V where f assumes a local maximum or minimum. Then*

$$\frac{\partial f}{\partial x}(a, b) = 0 \qquad \frac{\partial f}{\partial y}(a, b) = 0.$$

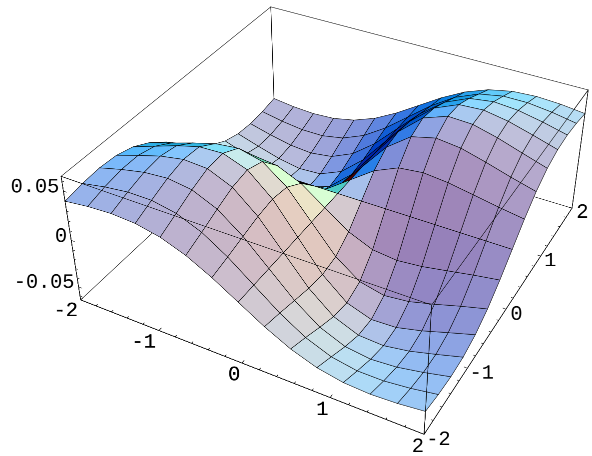
Again we do not give an exact proof. Intuitively one sees that the tangent plane of the graph of $f(x, y)$ in a local extremum should be horizontal. The tangent plane is given by $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$, which is precisely horizontal if

$$f_x(a, b) = f_y(a, b) = 0.$$

A point (a, b) where both first partial derivatives vanish is called a *stationary point*. So to compute local extrema, we must compute stationary points of $f(x, y)$.
Example. Compute the local extrema of

$$f(x, y) = \frac{xy}{(2 + x^2 + y^2)^2}.$$

Here is a Mathematica plot of the three dimensional graph.



To compute the stationary points we set the partial derivatives equal to 0.

$$\frac{y(2-3x^2+y^2)}{2+x^2+y^2} = 0, \quad \frac{x(2-3y^2+x^2)}{2+x^2+y^2} = 0.$$

Hence we must solve the system

$$y(2-3x^2+y^2) = 0, \quad x(2-3y^2+x^2) = 0.$$

The solutions read

$$(x,y) = (0,0), (1,1), (1,-1), (-1,1), (-1,-1)$$

and these are the stationary points of f (Check!). From the 3D plot we can decide in which points f has a local maximum or minimum. We find that $(1,1)$ and $(-1,-1)$ are local maxima and $(1,-1),(-1,1)$ local minima. The point $(0,0)$ does not correspond to a local extremum. In the neighbourhood of $(0,0)$ we see that the graph of $f(x,y)$ is saddle shaped, or a mountain pass. Apparently such points are also stationary.
Besides looking at graphs plotted by Mathematica there is another way to decide whether a stationary point corresponds to a local maximum, minimum or saddle point which involves second order derivatives of f. It is the analogon of the second order criterion for one variable functions.

Theorem 5.3.2 *Let $f(x,y)$ be a function of two variables with continuous first and second order derivatives in some region V. Suppose $(a,b) \in V$ is a stationary point of f. Define*

$$H = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2.$$

Then

1. *If* $H < 0$ *then the graph of* f *has a saddle shape above* (a, b).

2. *If* $H > 0$ *and* $f_{xx}(a, b) > 0$ *then* f *has a local minimum in* (a, b).

3. *If* $H > 0$ *and* $f_{xx}(a, b) < 0$ *then* f *has a local maximum in* (a, b).

By looking at the second order Taylor approximation we can see why this theorem holds. In a stationary point the first partial derivates are zero and the second order Taylor approximation has the form

$$f(x,y) = f(a,b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b)\frac{1}{2}f_{yy}(a,b)(y-b)^2.$$

So around the stationary point (a, b) the second order part of the Taylor approximation is responsible for the behaviour of $f(x, y)$ near the point (a, b). An expression of the form $AX^2 + 2BXY + CY^2$ is called a quadratic form in X, Y. The numbers A, B, C are given. When we take $A = f_{xx}(a, b)/2,\ B = f_{xy}(a, b),\ C = f_{yy}(a, b)$ and $X = x - a, Y = y - b$ we get the quadratic part of the Taylor approximation as an example. Let us define $D = AC - B^2$. Quadratic forms with $D \neq 0$ occur in different types.

1. Forms which assume both positive and negative values. These are called *indefinite forms* and are characterised by $D < 0$.

2. Forms which assume only values ≥ 0. These are called *positive definite forms*, they are characterised by $D > 0,\ A > 0$.

3. Forms which assume only values ≤ 0. These are called *negative definite forms*, they are characterised by $D > 0,\ A < 0$.

Check for yourselves that these characterisations give rise to the classification of stationary points given in Theorem 5.3.2.
To see why the characterisation of quadratic forms is true we remark that

$$AX^2 + 2BXY + CY^2 = Y^2(At^2 + 2Bt + C),$$

where $t = X/Y$. Since $Y^2 > 0$, the sign of $At^2 + 2Bt + C$ is the same as that of the quadratic form. From the theory of quadratic polynomials we know that $At^2 + 2Bt + C$ assumes both positive and negative values if it has real zeros. I.e. $0 < (2B)^2 - 4AC = -4D$, from which we get $D < 0$.
When $(2B)^2 - 4AC < 0$ (i.e. $D > 0$) the function $At^2 + 2Bt + C$ has no zeros, so all of its values have the same sign. This sign is positive when $A > 0$

and negative when $A < 0$. This explains the characterisation of the three different classes.

Example. We use Theorem 5.3.2 to determine the nature of the stationary points $(0,0)$ and $(1,1)$ of the function

$$f(x,y) = \frac{xy}{(2+x^2+y^2)^2}$$

(see example above). After elaborate computation (do it!) of the second partial derivatives we find that

$$f_{xx}(0,0) = 0, \quad f_{xy}(0,0) = 1/4, \quad f_{yy}(0,0) = 0.$$

Hence $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = -1/16 < 0$ and our criterion implies that $(0,0)$ is a saddle point. For the point $(1,1)$ we get

$$f_{xx}(1,1) = -3/32, \quad f_{xy}(1,1) = 1/32, \quad f_{yy}(1,1) = -3/32.$$

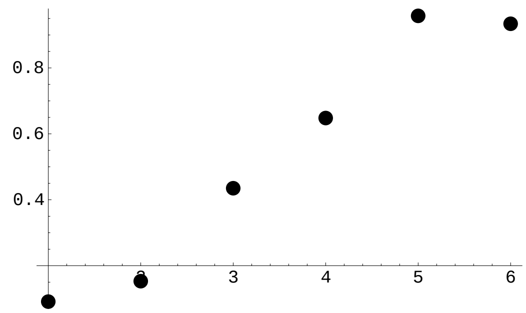
Hence $f_{xx}(1,1)f_{yy}(1,1) - f_{xy}(1,1)^2 = 1/128 > 0$. Since, in addition $f_{xx}(1,1) < 0$, Theorem 5.3.2 implies that f has a local maximum at $(1,1)$.

5.4 Method of least squares

A very important and often quoted example of optimization in several variables is the following problem. It occurs when we carry out an experiment to determine the dependence of a certain quantity y as function of another quantity x. For example, x could be the temperature and y the length of a metal bar with temperature x. We assume that the dependence of y on x is linear, i.e. of the form $y = ax + b$. The only problem is to determine a, b experimentally. We do this by a number of measurements in which we measure the value of y corresponding to a sequence of values of x, which we denote by $x_1, x_2, \ldots, x_n$. The corresponding values of y are denoted by $y_1, y_2, \ldots, y_n$. Here is a sample list of results with $n = 6$.

n	x_n	y_n
1	1	0.091
2	2	0.153
3	3	0.435
4	4	0.648
5	5	0.958
6	6	0.934

Here is a plot of the results



The question is now to determine a, b in such a way that the graph of the function $y = ax + b$ fits these data as best as possible. This is usually done in such a way that the sum of the squares of the differences $y_i - ax_i - b$ is minimized. I.e. we must minimize

$$\Delta(a,b) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

This is called the *method of least squares*. In our example we must minimize

$$\begin{aligned} & (0.091 - a - b)^2 + (0.153 - 2a - b)^2 + (0.435 - 3a - b)^2 \\ & +(0.648 - 4a - b)^2 + (0.958 - 5a - b)^2 + (0.934 - 6a - b)^2 \\ = \; & 91\,a^2 + 42\,a\,b + 6\,b^2 - 29.376\,a - 6.438\,b + 2.43094 \end{aligned}$$

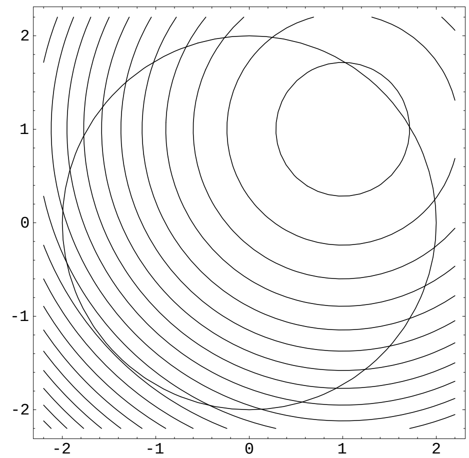
To determine a, b which minimizes $\Delta(a,b)$, we take the partial derivatives and set them equal to zero. In our specific example we get

$$\begin{aligned} 182a + 42b - 29.376 &= 0 \\ 42a + 12b - 6.438 &= 0 \end{aligned}$$

The solution of this system is $a = 0.195$, $b = -0.148$. In this case it is obvious that the solution corresponds to the minimal value. The function $\Delta(a,b)$ is always ≥ 0. So it must have a minimum. This minimum is also a local minimum, which can be found by setting the partial derivatives equal to zero. Since we have found only one solution in this way, the point we found must correspond to our minimum.

5.5 Lagrange multiplier method

In the previous section we have concentrated on the determination of local maxima and minima which could be found by means of Theoren 5.3.1. But this theorem deals only with interior points of a domain V. The boundary points of V need to be analysed separately. Here is an example of what we mean. Consider the function $f(x,y) = (x-1)^2 + (y-1)^2$ restricted to the disc $x^2 + y^2 \leq 4$. Here is a plot of the level lines together with the circle C given by $x^2 + y^2 = 4$.



We only consider the function values of points within the circle C and wonder where the minimum and the maximal function value occur. From the picture this is hopefully clear. The minimum value 0 is attained at $x = 1, y = 1$, which is inside C, the maximal value is attained at $x = -\sqrt{2}, y = -\sqrt{2}$, which is on the circle C.

Theorem 5.3.1 only gives us the minimum since this is an interior point of the inside of C. Putting the two partial derivatives equal to zero gives us

$$2(x-1) = 0 \qquad 2(y-1) = 0,$$

hence $x = 1$ and $y = 1$. Notice that we did not get the point $(-\sqrt{2}, -\sqrt{2})$ this way. The reason that f has a maximum at this point is that we restricted our scope of attention to the disc $x^2 + y^2 \leq 4$. In order to find such a point we have to determine the local maxima and minima of $f(x,y)$ restricted to the circle C. There are two ways to do this.

The first is by choosing a parametrisation of C, for example $x = 2\cos t, y = 2\sin t$, substitute this into $f(x,y) = (x-1)^2+(y-1)^2$ and then determine t for which this new function is minimal or maximal. First of all $f(2\cos t, 2\sin t) = (2\cos t - 1)^2 + (2\sin t - 1)^2$. After elaboration we get $4\cos^2 t + 4\sin^2 t - 4\cos t - 4\sin t + 2 = 6 - 4\cos t - 4\sin t$. The local extrema of this function can determined by setting its derivative equal to 0. So $4\sin t - 4\cos t = 0$.

We find that $t = \pi/4, 5\pi/4$ up to multiples of 2π. The function values of $f(2\cos t, 2\sin t)$ for $t = \pi/4, 5\pi/4$ are respectively $6-4\sqrt{2}$ and $6+4\sqrt{2}$. The latter value is the maximal value of f we were looking for.
The second method is based on an observation. Consider the picture once more. It is clear that the maximum on the boundary occurs precisely at the place where the level lines of $f(x, y)$ are tangent to the circle C. This means that the gradient vectors of f and $x^2 + y^2$ have the same direction in this point of maximum. The gradient of f is $(2(x-1), 2(y-1))$, the gradient of $x^2 + y^2$ is $(2x, 2y)$. These gradients have the same (or opposite) direction if there exists $\lambda \in \mathbb{R}$ such that $(2(x-1), 2(y-1)) = \lambda(2x, 2y)$. So we get three equations,

$$\begin{array}{rcl} 2(x-1) & = & 2\lambda x \\ 2(y-1) & = & 2\lambda y \\ x^2+y^2 & = & 4 \end{array}$$

The last equation expresses the fact that the point we look for must be on C. First note that x cannot be 0. Because if $x = 0$, the first equation would imply $x - 1 = 0$. But this is impossible, x cannot be 1 and 0 at the same time. We can now divide the first equation by x to get $\lambda = (x-1)/x$. Use this to eliminate λ from the second equation. We find

$$y - 1 = \frac{x-1}{x} y.$$

Multiply on both sides by x to get $x(y-1) = y(x-1)$. Hence $y = x$. Use this in the third equation to get $2x^2 = 4$. So $x = \pm\sqrt{2}$. Since $y = x$ we find the points $\pm(\sqrt{2}, \sqrt{2})$, the same ones we found before.
The latter computation is based on the following general rule.

Theorem 5.5.1 (Lagrange multiplier method) *Consider the function $f(x, y)$ restricted to a curve C in $\mathbb{R}^2$ given by the implicit equation $g(x, y) = 0$. Suppose that f restricted to the curve C has a local maximum or minimum in a point (a, b). Then there exists $\lambda \in \mathbb{R}$ such that*

$$\begin{array}{rcl} f_x(a,b) & = & \lambda g_x(a,b) \\ f_y(a,b) & = & \lambda g_y(a,b) \\ g(a,b) & = & 0 \end{array}$$

A similar theorem also holds for functions in 3 variables restricted to a surface in space given by an equation $g(x, y, z) = 0$. More generally, the theorem can also be formulated for functions in n variables on subspaces of $\mathbb{R}^n$ given by several equations.

5.6 Variational calculus

Although optimisation theory is a relatively new field in mathematics, it has its sources in the 17th century, at the same time when differential and integral calculus was developed. Two classical problems are the *brachistochrone problem* and the problem of the *suspended chain*. These problems were solved by Bernoulli and Newton in the 17-th century. These are examples of optimisation problems where an infinite number of variables is involved.
A similar famous optimisation phenomenon is Fermat's principle, which states that light always follows the path with shortest optical length. This was one the first times when it was realised that nature also works according to some optimisation scheme. In the 18-th and 19-th century mathematical physicists discovered that the laws of classical mechanics can be formulated in terms of optimisation principles of certain action integrals. This approach to mechanics has become know as the Lagrange formalism and a related version, the Hamilton formalism. The mathematical analysis of such optimisation problems has become known as *variational calculus*, first developed by Euler and Lagrange, and which is still of utmost importance today.
The Hamiltonian version of classical mechanics is usually taken as a point of departure for quantisation of mechanical systems.

5.7 Exercises

Many of the problems below have been taken from M.de Gee, Wiskunde in Werking, Epsilon Uitgaven 1994, Utrecht.

Exercise 5.7.1 *For each of the following functions F, determine the stationary points, and the nature of these stationary points.*

1. $F(x, y) = x^3 - 3x^2y/2 + 6x^2 + y^2/2$
2. $F(x, y) = x^2y + xy^2 - x^2 - xy - y^2 + 3$

3. $F(x, y) = x^3y + xy^2 - 5xy$

Exercise 5.7.2 *Consider the function* $F(x, y) = xy(8x - x^2 - y^2)$.

1. *Make a sketch of the 0-level line of F. Also indicate in your sketch for which points (x, y) in the plane F has positive, resp. negative values. Can you infer from this picture where the stationary points of F are located and what their nature is?*

2. *Compute the stationary points of F and determine their nature.*

3. *Answer the two above questions for the function* $F(x, y) = \sin 2\pi x \sin 2\pi y$.

Exercise 5.7.3 *An enterprise manufactures the same article in two different factories A and B. The production cost to manufacture a quantity q in these two factories are different. In A the cost of production is* $20q^2 - 60q + 100$, *in factory B it is* $10q^2 - 40q + 90$.

1. *Suppose one unit produced can be sold at the price $p = 220$. Determine the production levels q_A, q_B in each factory to achieve maximal profit.*

2. *Suppose now that the price per unit depends on the supply as follows $p = 520 - 10q$. Now determine the production levels which yield maximal profit.*

Exercise 5.7.4 *A manufacturer produces one product, which is marketed on two different markets A and B. His total production q is split into two parts, q_A and q_B which is sold on market A resp. B. The prices p_A and p_B per item depend on the quantity offered on each market as follows,*

$$p_A = 57 - 5q_A, \qquad p_B = 40 - 7q_B.$$

The total cost of production is given by $K(q) = 3q^2 - 5q + 16$. Of course the manufacturer aims for maximal profit. In order to do so, how much should he produce and how should this production be divided over the two markets? What is the maximal profit?

Exercise 5.7.5 *A rectangular tank, open at the top, has a volume of* $4\ \mathrm{m}^3$. *If the base measurements (in m) are x by y, show that the surface area (in* m^2*) is given by*

$$A = xy + \frac{8}{x} + \frac{8}{y}.$$

For which x, y is A minimal, and what is the corresponding value of A?

Exercise 5.7.6 *Determine the local extrema of* $F(x,y) = x^2 - y^2$ *restricted to the points* (x,y) *with* $x^2 + (y+2)^2 = 4$. *Also determine the nature of these local extrema.*

Exercise 5.7.7 *The equation* $5x^2 + 6xy + 5y^2 = 8$ *represents an ellipse whose centre is at the origin. By considering the extrema of* $x^2 + y^2$, *obtain the lengths of the semi-axes of the ellipse.*

Exercise 5.7.8 *Which point on the sphere* $x^2 + y^2 + z^2 = 1$ *is at the greatest distance from the point* $(1,2,2)$ *?*

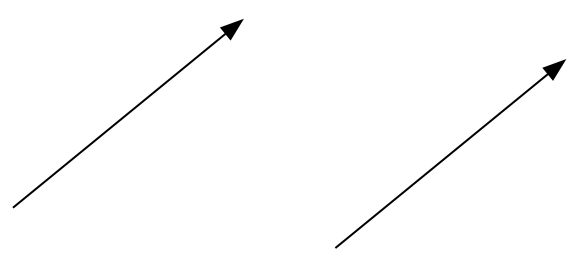
Exercise 5.7.9 *A student buys* x *items of product* A *at price* 1 *per item and* y *items of product* B *at price* 4 *per item. The total budget available is* 36. *The benefit that the student has from his purchases can be made quantitative. It is* $N(x,y) = x^2y^2$. *Which quantities of each product should the student buy to have maximal benefit?*

Chapter 6

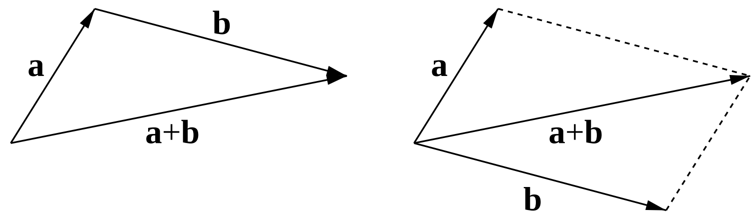
Vectors

6.1 Intuitive introduction

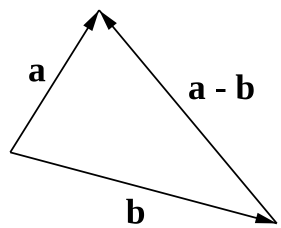
We have all seen vectors, either with mathematics or with physics. In this chapter a vector will be a displacement or, if you want, a direction and a length. A displacement, also called translation, can be pictured by an arrow, but not in a unique way. Below you see two arrows and it is hopefully clear that they represent the same vector, since direction and length of the arrows are the same.



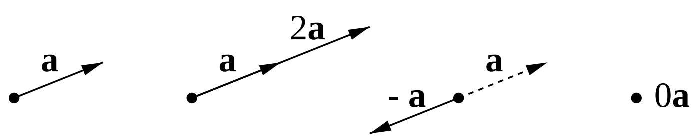
later we shall adopt the habit to let the arrows begin in a specifically chosen point, called the origin. In this section arrows representing vectors may be drawn anywhere. The length of the arrow will be called the *length* of the vector. The length of the vector $\mathbf{v}$ is denoted by $|\mathbf{v}|$ (all our vectors will be written in bold face, in physics one would write $\vec{v}$). There is one vector whose length is zero, which is called the *zero vector*, denoted by $\mathbf{0}$.
Suppose we have two vectors $\mathbf{a}$ en $\mathbf{b}$. We define the *sumvector* as the translation we get by first performing the translation $\mathbf{a}$ and then $\mathbf{b}$. This sequence of translations is again a translation which we denote by $\mathbf{a}+\mathbf{b}$. In a picture,



In the right hand picture we have chosen the arrow representing **b** by letting it start in the same point where the arrow for **a** starts. Hopefully it is clear from the picture why vector addition is done using the so-called *parallelogram law*. We also have the *difference vector* $\mathbf{a} - \mathbf{b}$. This is precisely the vector which, added to **b**, gives **a**. In a picture,

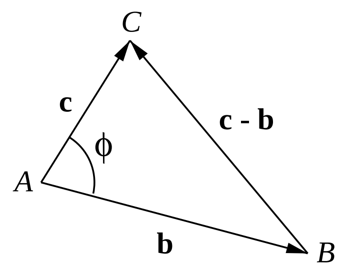


Next to addition of vectors there is also *scalar multiplication*. Choose a vector **a** and a real number λ. Suppose $\lambda > 0$. The vector which has the same direction as **a**, but whose length is λ times the length of **a** is called the scalar product of **a** by λ. When $\lambda < 0$, the scalar product of λ and **a** is the vector with direction opposite to **a** and whose length is $|\lambda| \cdot |\mathbf{a}|$. We denote the scalar product by $\lambda\mathbf{a}$. Finally we always take $0\mathbf{a} = \mathbf{0}$. Here is an illustration,



An important concept for vectors is the *inner product*. Let $\mathbf{a}, \mathbf{b}$ be two non-

zero vectors and ϕ the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$. The inner product of $\mathbf{a}$ and $\mathbf{b}$ is the number $|\mathbf{a}||\mathbf{b}|\cos\phi$. Notation: $\mathbf{a}\cdot\mathbf{b}$. Because of this notation the inner product is also called *dot product*. When $\mathbf{a}$ or $\mathbf{b}$ is the zero vector, we define $\mathbf{a}\cdot\mathbf{b}=\mathbf{0}$.
The inner product arises naturally with the *cosine rule* for triangles. Consider a triangle whose sides are formed by arrows which represent the vectors $\mathbf{a},\mathbf{b},\mathbf{a}-\mathbf{b}$.



According to the cosine rule we have that

$$|\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2-2|\mathbf{a}||\mathbf{b}|\cos\phi.$$

This can be seen as a generalisation of Pythagoras' theorem. The latter corresponds to the special case $\phi=\pi/2$, where we get that $|\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2$. With our notation for inner product we can also formulate the cosine rule as

$$|\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}. \tag{6.1}$$

6.2 Coordinates

The easiest way to compute with vectors is to use their coordinates. In the previous section we could either have worked in space or in the plane. In this section we agree to work in the plane, the three dimensional case being similar.
Choose a vector $\mathbf{e}_1$ of length 1 pointing to the right, and a vector $\mathbf{e}_2$ perpendicular to it, pointing upward. Let us consider the vector $\mathbf{e}_1$ as the translation "one step to the right" and $\mathbf{e}_2$ as "one upward step" in the plane. Any translation in the plane can be carried out by taking a number of steps to the right (a negative number of steps means "to the left") and a number of upward steps ("downward" for a negative number of steps). More precisely this means that any vector $\mathbf{v}$ in the plane can be written as a combination

$x\mathbf{e}_1 + y\mathbf{e}_2$ (i.e. x steps to the right and y steps upward). The numbers x, y are called the *coordinates* of the vector $\mathbf{v}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2$. Let us now assume we have a fixed basis $\mathbf{e}_1, \mathbf{e}_2$. Then vectors can be characterised by their coordinates and vice versa. So we can also denote the vector $\mathbf{v}$ by the pair of numbers x, y which is, for historical and future practical reasons, written as a column of numbers $\begin{pmatrix} x \\ y \end{pmatrix}$. Although writing coordinates in the form of columns is the official policy, it is not very handy typographically. That is why we shall very often deviate from this policy by writing coordinates in row-form. However, when we get to matrix multiplication it is wise to write coordinates in column from.
Scalar multiplication and vector addition can also be carried out on the coordinates. We have

$$\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}.$$

The space of ordered pairs of numbers $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is denoted by $\mathbb{R}^2$ (pronounce: R-two)
As a final step we now make a picture of $\mathbb{R}^2$. Fix a point in the plane and call it O, the origin. We now make a one-to-one identification of points in the plane and vectors. To a vector $\mathbf{v}$ we associate the point gotten by translation of O over $\mathbf{v}$. We call this point $\mathbf{v}$ again, or denote it by its coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$.
The set of points corresponding to the multiples of $\mathbf{e}_1$ is called the x-axis, the points corresponding to the multiples of $\mathbf{e}_2$ is called the y-axis. In this way we have made an identification of the points in the plane with $\mathbb{R}^2$, which is probably already very familiar to you. Note that this identification depends on the choice of O and the basis vectors $\mathbf{e}_1, \mathbf{e}_2$. But usually we do not worry about this in practice.
A vector $\mathbf{v}$ having coordinates x, y has length $\sqrt{x^2 + y^2}$. This is a consequence of Pythagoras' theorem. Furthermore between points $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ we have a distance which is simply the length of the difference vector,

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

It turns out that the inner product can also be expressed in terms of coordinates.

Theorem 6.2.1 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two vectors having coordinates* x_1, y_1 *and* x_2, y_2 *respectively. Then,*

$$\mathbf{a} \cdot \mathbf{b} = x_1 x_2 + y_1 y_2.$$

It may be a bit mysterious why the definition of inner product, which involves a cosine, has the nice looking form $x_1x_2+y_1y_2$. The reason is that the cosine rule is behind it. We know from (6.1) that

$$|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}.$$

So,

$$2\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 \tag{6.2}$$

We now note that

$$|\mathbf{a}-\mathbf{b}|^2 = (x_1-x_2)^2 + (y_1-y_2)^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2x_1x_2 - 2y_1y_2.$$

Moreover,

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2.$$

Therefore,

$$\begin{aligned} 2\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 \\ &= (x_1^2 + x_2^2 + y_1^2 + y_2^2) - (x_1^2 + x_2^2 + y_1^2 + y_2^2) + 2x_1x_2 + 2y_1y_2 \\ &= 2(x_1x_2 + y_1y_2) \end{aligned}$$

After division by 2 we get the desired $\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2$. □

Using this formula for the inner product it is very simple to compute angles between vectors.

Example. Compute the angle ϕ between $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Solution: The lengths of $\mathbf{a}$ and $\mathbf{b}$ are $\sqrt{2^2+1^2} = \sqrt{5}$ and $\sqrt{1^2+3^2} = \sqrt{10}$ respectively. We get

$$\cos\phi = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2\cdot 1 + 1\cdot 3}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}}.$$

From $\cos\phi = 1/\sqrt{2}$ it follows that ϕ is 45 degrees (check this by making an accurate drawing of the vectors).

Two vectors **a** and **b** will be called *perpendicular*, or *orthogonal*, if we have $\mathbf{a}\cdot\mathbf{b} = 0$. When both **a** and **b** are non-zero vectors this makes sense, because then $|\mathbf{a}||\mathbf{b}|\cos\phi = 0$. And thus $\cos\phi = 0$, i.e. ϕ is 90 degrees. If either of the vectors is the zero, we cannot speak of an angle, but we formally declare the vectors perpendicular.

It is now very easy to see that the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are orthogonal because their inner product is $1\cdot(-2) + 2\cdot 1 = 0$.

6.3 Higher dimensions

In three dimensions the story is completely similar. We can identify the space vectors with $\mathbb{R}^3$ by choosing a basis of vectors of length 1 which are mutually orthogonal (these will be the future x-,y-,z-direction). The spatial points are then identified with $\mathbb{R}^3$ if we also choose an origin O. The length of a vector with coordinates x, y, z is $\sqrt{x^2+y^2+z^2}$. The inner product of two vectors in terms of coordinates is

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2.$$

So we see that it is possible to associate to the plane and space an algebraic description in terms of $\mathbb{R}^2$ and $\mathbb{R}^3$. Of course it is not possible to picture oneself four dimensional or higher dimensional space. But in mathematics we do speak about higher dimensional space. This is done by simply extending the algebraic description. For example, four dimensional space is simply $\mathbb{R}^4$, the space of 4-tuples x, y, z, u with distance function

$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 + (u_1-u_2)^2}.$$

As soon as we have a space with a distance function we have geometry, even if we cannot make a mental picture of it.

6.4 Equations of lines and planes

As an application of Theorem 6.2.1 and its analogue in three dimensional space we consider the equations of lines and planes.
As we all know, the equation of a line in the plane has the form $ax + by = c$ where a, b, c are fixed real numbers with $(a, b) \neq (0, 0)$. Let us call the line l. We assert:

The vector (a, b) is perpendicular to the line l.

We can see this easily as follows. Take a fixed point (p, q) on the line l. So we have $ap+bq = c$. Take any other point (x, y) on the line l. We get $a(x-p)+b(y-q) = ax+by-ap-bq = c-c = 0$. This equality can be interpreted as saying that the vectors (a, b) and $(x-p, y-q)$ are perpendicular. The latter vector is the difference of the two points, hence its direction is the direction of the line. So we conclude that (a, b) is perpendicular to l.
In a similar way we can show that the vector (a, b, c) is perpendicular to the plane in three dimensional space given by the equation $ax + by + cz = d$.
Suppose we want to find the equation of the plane passing through the points $(1, 0, 2), (0, 1, 1), (-1, 1, -2)$. In order to do so we must solve a, b, c, d from

$$\begin{array}{rrrcl} a & & +2c & = & d \\ & b & +c & = & d \\ -a & +b & -2c & = & d \end{array}$$

Bring the unknown d to the left hand side to get

$$\begin{array}{rrrrl} a & & +2c & -d & = 0 \\ & b & +c & -d & = 0 \\ -a & +b & -2c & -d & = 0 \end{array}$$

We solve this system by our matrix bookkeeping system

$$\left(\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ -1 & 1 & -2 & -1 & 0 \end{array}\right).$$

Elimination of a yields

$$\left(\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 \end{array}\right).$$

Elimination of b yields

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right).$$

From the last equation we conculde that $c = -d$. From the second last we get $b = d - c = d - (-d) = 2d$ and form the first $a = -2c + d = -2(-d) + d = 3d$. To get an equation of our plane it suffices to pick one value of d. All other choices will give the same equation up to a common factor. So let us take $d = 1$. Then $c = -1, b = 2$ and $a = 3$. So the equation of our plane is

$$3x + 2y - z = 1.$$

In particular, the vector $(3, 2, -1)$ is perpendicular to our plane.

6.5 Exercises

Exercise 6.5.1 *Let* $\mathbf{u} = (-1, 3, -2), \mathbf{v} = (4, 0, -1), \mathbf{w} = (-3, -1, 2)$. *Compute the following linear combinations*

1. $3\mathbf{u} - 2\mathbf{v}$
2. $\mathbf{u} + 2(\mathbf{v} - 4\mathbf{w})$
3. $\mathbf{u} + \mathbf{v} - \mathbf{w}$
4. $4(3\mathbf{u} + 2\mathbf{v} - 5\mathbf{w})$

Exercise 6.5.2 *Let* $\mathbf{u} = (-1, 3, 4), \mathbf{v} = (2, 1, -1), \mathbf{w} = (-2, -1, 3)$. *Compute the following quantities,*

1. $|\mathbf{u}|$ *(i.e. the length of* $\mathbf{u}$*)*
2. $|-\mathbf{v}|$
3. $|\mathbf{u} + \mathbf{v}|$
4. $|\mathbf{v} - 2\mathbf{u}|$
5. *The unit vector (=vector of length 1) with the same direction as* $\mathbf{u}$.

6. *The unit vector with a direction opposite to* $\mathbf{w}$*.*
7. $\mathbf{u} \cdot \mathbf{v}$
8. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$
9. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$
10. *The angle between* $\mathbf{u}$ *and* $\mathbf{v}$
11. *The angle between* $\mathbf{u}$ *and* $\mathbf{w}$
12. *The value of* x *such that* $(x, -3, 5)$ *is perpendicular to* $\mathbf{u}$
13. *The value of* y *such that* $(-3, y, 10)$ *is perpendicular to* $\mathbf{u}$
14. *A nonzero vector perpendicular to both* $\mathbf{u}$ *and* $\mathbf{v}$
15. *A nonzero vector perpendicular to both* $\mathbf{u}$ *and* $\mathbf{w}$

Exercise 6.5.3 *Determine an equation of the line in* $\mathbb{R}^2$ *passing through the following pairs of points*

1. $(1, 2),\ (2, 1)$
2. $(1, -1),\ (3, 2)$
3. $(0, 3),\ (1, 1)$

Exercise 6.5.4 *Determine an equation of the plane in* $\mathbb{R}^3$ *passing through the following triples of points*

1. $(1, 0, 1),\ (2, -1, 1),\ (0, 1, 2)$
2. $(1, 1, 1),\ (3, 0, 1),\ (0, 0, 1)$
3. $(1, -1, 2),\ (2, 0, 1),\ (0, 1, 1)$

Chapter 7

Matrices

A *matrix* is a set of numbers arranged in a rectangular form. For example,

$$M = \begin{pmatrix} 3 & -1 & 2 & 0 \\ 1 & 4 & -2 & 1 \\ 2 & -1 & -5 & 2 \end{pmatrix}$$

If the number of rows is r and number of columns equal to k we speak of a $r \times k$-matrix. In the above example M is a 3×4-matrix. The element in the i-th row and j-th column is denoted by M_{ij}. For example $M_{23} = -2$. A matrix consisting of one column is called a *column vector*, a matrix consisting of one row is called *row vector*. A matrix with the number of rows equal to the number of columns is called a *square matrix*. A matrix with 0 at every place is called *null matrix*. A square $m \times m$-matrix with 1 on the diagonal places and 0 elsewhere is called an *identity matrix*. Notation: I_m. Example,

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7.1 Basic operations

Addition of two matrices A and B is only possible if A and B have the same rectangular format. Addition is performed bij simply adding the corresponding matrix elements. For example,

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 5 \\ 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 5 \\ 3 & 5 & -1 \end{pmatrix}$$

Notation: $A+B$.
Scalar multiplicationg of a matrix A with a real number λ is performed by multiplying each entry of A by λ. For example,

$$3 \cdot \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 0 \\ 6 & 6 & -3 \end{pmatrix}$$

Notation: λA.
Transposition of a matrix A is performed by turning rows into columns and vice versa. For example,

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ -1 & 2 \\ 0 & -1 \end{pmatrix}$$

Notation: A^T.
Matrix multiplication of an $r \times k$-matrix A and a $s \times l$-matrix B is only possible if $s = k$. The matrix product AB is an $r \times l$-matrix which is determined as follows. The entry of AB on position i, j is determined bij multiplication of every element from the i-th row of A by the corresponding elements of the j-th column of B and then adding all these products together. For example,

$$\begin{pmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{2} & -3 & 1 & 0 \\ \mathbf{1} & 1 & 2 & -2 \\ \mathbf{2} & 0 & 3 & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & -4 & -1 & 2 \\ 4 & -4 & 3 & -3 \end{pmatrix}.$$

The element 1 from the product matrix arises by the computation $1 \cdot 2 + (-1) \cdot 1 + 0 \cdot 2 = 1$. The other entries of AB are determined similarly. Note that BA does not exist because the number of columns of B is 4 and the number of rows of A is 2.
Some important rules for matrix multiplication,

$$\begin{aligned} A(B+C) &= AB + AC \\ (A+B)C &= AC + BC \\ (AB)C &= A(BC) \\ (AB)^T &= B^T A^T \end{aligned}$$

Finally, when A is an $m \times n$-matrix,

$$I_m A = A \qquad AI_n = A.$$

If it so happens that both AB and BA exist, then most of the time they are *not equal*. For example,

$$A = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Then,

$$AB = \begin{pmatrix} -1 \end{pmatrix} \qquad BA = \begin{pmatrix} 3 & 6 & -3 \\ -1 & -2 & 1 \\ 2 & 4 & -2 \end{pmatrix}$$

Another example,

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$$

Then,

$$AB = \begin{pmatrix} 3 & -5 \\ 0 & 2 \end{pmatrix} \qquad BA = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

Verify these examples!

7.1.1 Examples of matrix multiplication

Until now matrix multiplication has been a purely formal operation without any meaning. It turns out that in many cases matrix multiplication can be considered as bookkeeping device which arises very naturally. This is why matrix multiplication was introduced in the first place. In this section we give two real life situations where matrix multiplication occurs.

Vitamines eating habits. Three food articles S_1, S_2 en S_3 contain the vitamines A,B,C en D in the following quantities (in vitamine units per gram)

	S_1	S_2	S_3
V_A	$0,5$	$0,3$	$0,1$
V_B	$0,5$	$0,0$	$0,1$
V_C	$0,0$	$0,2$	$0,2$
V_D	$0,0$	$0,1$	$0,5$

Suppose that one day mr.Bean consumes 200, 100 and 100 gram of S_1 S_2 and S_3. How much of each vitamine does he consume? For vitamine A this is $A = 0,5 \times 200 + 0,3 \times 100 + 0,1 \times 100 = 140$. Analogous computation for the other vitamines gives us,

$$\begin{aligned} A &= 0,5 \times 200 + 0,3 \times 100 + 0,1 \times 100 = 140 \\ B &= 0,5 \times 200 + 0,0 \times 100 + 0,1 \times 100 = 110 \\ C &= 0,0 \times 200 + 0,2 \times 100 + 0,2 \times 100 = 40 \\ D &= 0,0 \times 200 + 0,1 \times 100 + 0,5 \times 100 = 60 \end{aligned}$$

Notice that this nothing but the matrix multiplication

$$\begin{pmatrix} 0,5 & 0,3 & 0,1 \\ 0,5 & 0,0 & 0,1 \\ 0,0 & 0,2 & 0,2 \\ 0,0 & 0,1 & 0,5 \end{pmatrix} \begin{pmatrix} 200 \\ 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 140 \\ 110 \\ 40 \\ 60 \end{pmatrix}$$

Lesliematrices As a second concrete example we consider the population of terns on Texel in the Dutch Waddenzee. This population has fluctuated in the 1960's due to toxic waste, which originated from the river Rhine.

jaar	1940	1955	1960	1965	1966	1967	1968	1969	1970
breedingcouples	4000	2600	1200	65	170	100	150	200	250

Closer study reveals that there are three stages in a tern's life. The phase until the egg hatches, the phase in which young terns become adults, and the phase of an adult tern. The second phase is the learning phase, a hazardous one in which the tern has to learn about the hardships of life. We limit our consideration to the first three years and to the female population. In this way we avoid bookkeeping of mixed second/third year couples. In a healthy population the following occurs. From the first year, between hatching and learning phase, a fourth survives. Half of those survivors live to see the end of the second year We consider the tern population at the end of year n. Suppose $N_1(n)$ is the number of chickens that hatched, $N_2(n)$ the number of terns who just survived the learning phase, and $N_3(n)$ the number of terns that are at the end of their third year. So we have $N_2(n+1) = N_1(n)/4$ and $N_3(n+1) = N_2(n)/2$. For the number of female offspring we take from observations that $N_1(n+1) = 2N_2(n) + 4N_3(n)$. Schematically,

$$\begin{aligned} N_1(n+1) &= 2N_2(n) + 4N_3(n) \\ N_2(n+1) &= N_1(n)/4 \\ N_3(n+1) &= N_2(n)/2 \end{aligned}$$

Again we see a matrix multiplication,

$$\begin{pmatrix} N_1(n+1) \\ N_2(n+1) \\ N_3(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} N_1(n) \\ N_2(n) \\ N_3(n) \end{pmatrix}.$$

A large drop in the number of newborns, due to toxic waste, could be given by the matrix

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

With a lesser toxic concentration the eggs may hatch, but the young could very vulnerable. This could be described by another matrix

$$\begin{pmatrix} 0 & 2 & 4 \\ 1/16 & 0 & 0 \\ 0 & 1/4 & 0 \end{pmatrix}.$$

The question is what happens to the tern population in the long run, i.e. when $n \to \infty$. We abbreviate the numbers $N_1(n), N_2(n), N_3(n)$ by the column 3-vector $\mathbf{N}(n)$ and the matrix

$$\begin{pmatrix} 0 & 2 & 4 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

by L. Then, $\mathbf{N}(n+1) = L\mathbf{N}(n)$. This is a so-called recursive equation for the population vector $\mathbf{N}$. Suppose we know the population at time $n = 0$, say $\mathbf{N}(0)$.Then

$$\mathbf{N}(1) = L\mathbf{N}(0),\ \mathbf{N}(2) = L \cdot L\mathbf{N}(0) = L^2\mathbf{N}(0),\ \ldots,\ \mathbf{N}(n) = L^n\mathbf{N}(0)$$

So the behaviour of the matrix L^n as $n \to \infty$ is important for the future of the tern colony. The concepts eigenvector, eigenvalue, to be discussed later, is of crucial importance here.

7.2 Geometrical interpretation of matrix multiplication

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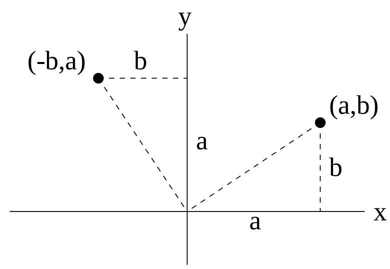
In the 2-dimensional plane, denoted by $\mathbb{R}^2$, every point is characterised by its x and y coordinate. Often we write these coordinates as a column matrix, $\begin{pmatrix} x \\ y \end{pmatrix}$.

Rotations

Suppose we perform in $\mathbb{R}^2$ an anti-clockwise rotation of 90^o. As center of rotation we take the point $(0,0)$. What happens to the coordinates of the point (a,b) if we rotate it in this way? The result can be seen in the following picture,



The point (a,b) turned into $(-b,a)$ after rotation. Using the column notation for vectors we can write the relation between the two points in matrix form.

$$\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

In other words, the coordinates of the rotated point can be obtained by multiplication of the original point by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We call $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the *rotation matrix* with angle $\pi/2$.

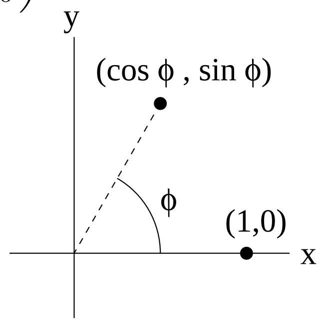
For rotations around other angles we have a similar story. Consider an anti-clockwise rotation about the angle ϕ and again $(0,0)$ as center. To it, there corresponds another matrix, namely

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

To see that this is the correct matrix we apply it to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The result,

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

From the picture below we see indeed that $\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$ has been rotated over angle ϕ with respect to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$,



Construct a similar picture, but now with the vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
A very nice consequece is the following. Suppose we first rotate about an angle α and then about the angle β, the total result will be a rotation about $\alpha+\beta$. In matrix Form,

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

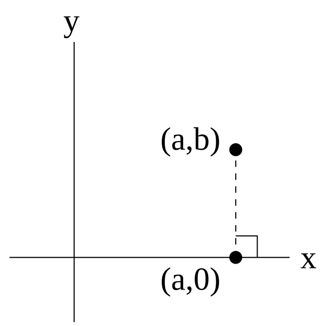
We work out the right hand matrix multiplication and only write down the first column of the product,

$$\begin{pmatrix} \sin(\alpha+\beta) \\ \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \sin\beta\cos\alpha + \cos\beta\sin\alpha \\ \cos\beta\cos\alpha - \sin\beta\sin\alpha) \end{pmatrix}$$

We have discovered the addition laws of cosine and sine!

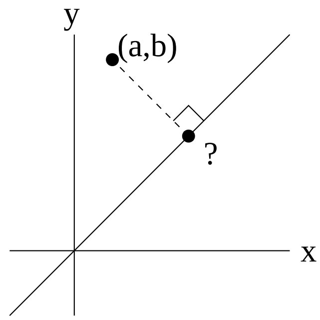
Projections

Suppose we project a point (a, b) perpendicular on the x-axis. This is an example of an orthogonal projection. The projected point has coordinates $(a, 0)$.



Note that $\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$. Orthogonal projection on the x-axis is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Now we take a slightly. Let l be the straight line given by $y = x$. Choose a point (a, b) and project it orthogonally on the line l. What are the coordinates of the projection.



Our projection can be seen as a concatenation three manipulations: clockwise rotation around 45^o, projection onto the x-axis and finally a counter clockwise rotation around 45^o. The corresponding matrix becomes,

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

As a sanity check we verify that the projection o $(1, 1)$ is the point itself and that the projection of $(1, -1)$ is the zero vector $(0, 0)$. And indeed,

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Linear maps

Rotation and projection are examples of linear maps from vector spaces to it itself. In general, linear maps are maps that are given by matrix multiplication.

Let M be an $n \times n$-matrix. We denote the elements of $\mathbb{R}^n$ in column form. For any $\mathbf{x} \in \mathbb{R}^n$ we have of course $M\mathbf{x} \in \mathbb{R}^n$ The map $\mathbf{x} \mapsto M\mathbf{x}$ is called a *linear map*.

7.3 Exercises

Exercise 7.3.1 *Let*

$$A = \begin{pmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 \\ 0 & 6 \\ -3 & 2 \end{pmatrix} \quad D = \begin{pmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{pmatrix}$$

Compute the following quantities, if they are defined,

1. $3A$
2. $A + B$
3. $B + C$
4. $C - D$
5. $4A - 2B$
6. AB
7. $(CD)^T$
8. A^2
9. $(AC)^2$
10. ADB
11. $(A^T)A$
12. BC *and* CB

Exercise 7.3.2 *Let*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1. *Find* A^2
2. *Find* A^7

Exercise 7.3.3 *Let*

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

1. *Find* A^2

2. *Find* A^7

Exercise 7.3.4 *Consider*

$$x = (-2, 3, -1), \qquad y = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

Compute xy *and* yx*.*

Exercise 7.3.5 *Give the standard matrix representation of the rotation of the plane counterclockwise about the origin through an angle of*

1. 30^o

2. 90^o

3. 135^o

Exercise 7.3.6 *Give the standard matrix representation of the rotation of the plane clockwise about the origin through an angle of*

1. 45^o

2. 60^o

3. 150^o

Exercise 7.3.7 *Use the rotation matrix for a general rotation through an angle of* θ *to derive formulas which express* $\cos 3\theta$ *and* $\sin 3\theta$ *in terms of* $\cos\theta$ *and* $\sin\theta$*.*

Exercise 7.3.8 *Find the matrix representation for the orthogonal reflection in the plane in the line* $y = 2x$*.*

Exercise 7.3.9 *Let* m *be a real number. Find the matrix representation for the orthogonal reflection in the plane in the line* $y = mx$*.*

Chapter 8

Determinants

8.1 Determinants and equation solving

For simple systems of linear equations it is possible to give general formulas for its solution. The simplest is one equation in one unknown,

$$ax = b.$$

The solution is of course $x = b/a$ *provided that* $a \neq 0$. The latter condition is very important. We now have only one solution, namely $x = b/a$. What happens if $a = 0$?. Our equation becomes $0x = b$. There are two possibilities, either the value of b is 0 and thus we find that every x is a solution of $0x = 0$. The other possibility is that $b \neq 0$, in which case $0x = b$ has no solution at all. This may seem all a bit trivial, but the above phenomena also happen, in a disguised form, for n equations in n variables..
We restrict ourselves to 2 equations in 2 unknowns,

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

where a, b, c, d, p, q are given and x, y the unknowns. Take d times the first equation and subtract b times the second equation. We get $(ad - bc)x = dp - bq$. Now take a times the second equation and subtract c times the first equation. We find $(ad - bc)y = aq - cp$. Let us suppose that $ad - bc \neq 0$. Then we get

$$x = \frac{dp - bq}{ad - bc}, \qquad y = \frac{aq - cp}{ad - bc}.$$

So there is precisely one solution, like in the case $a \neq 0$ of an equation in one unknown.
What happens if $ad-bc=0$? To fix ideas, let us assume that $a \neq 0$. Multiply the first equation by c/a. We find,

$$\frac{c}{a}(ax+by) = \frac{c}{a}p \Rightarrow cx + \frac{bc}{a}y = \frac{pc}{a}$$

Since $bc = ad$ the equation goes over into $cx + dy = pc/a$. We now have a new system of equations,

$$\begin{array}{rcl} cx+dy &=& pc/a \\ cx+dy &=& q \end{array}.$$

We see that we can only have solutions when $pc/a = q$. In other words, when $ad - bc = 0$ there can only be solutions if the equations differ by a factor (in the case at hand, c/a). We call the equations *dependent* For the solution it then suffices to solve only one of the equations, e.g. $ax + by = p$, which has an infinite number of solutions.
When $ad - bc = 0$ and $pc/a \neq q$ we have no solutions at all and we call the system of equations *contradictory*.

Example Consider the system

$$\begin{array}{rcl} 9x+6y &=& 1 \\ 6x+4y &=& 2 \end{array}$$

Note that $ad-bc = 9\cdot4-6\cdot6 = 0$ in this case. Multiply the first equation by $2/3$. We get $6x + 4y = 2/3$. The left hand side of this equation is the same as that of the second equation. The right hand sides are not equal however. Therefore the system is contradictory.
Consider the system

$$\begin{array}{rcl} 9x+6y &=& 0 \\ 6x+4y &=& 0 \end{array}$$

Again multiply the first equation by $2/3$. We get $6x + 4y = 0$, and this precisely the second equation. The equation are dependent and thus it suffices to solve only one of them, i.e. $9x + 6y = 0$.

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponding to the system

$$\begin{array}{rcl} ax+by &=& p \\ cx+dy &=& q \end{array}$$

is called the *coefficient matrix*. The quantity $ad-bc$ is called the *determinant* of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A common notation for the determinant is

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

From the discussion above we get

Theorem 8.1.1 *If $ad - bc \neq 0$, the system*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

has the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

We also have

Theorem 8.1.2 *When $ad - bc = 0$ and $p = q = 0$, the system of equations*

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

is dependent and we have infinitely many solutions.

We call $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ the *inverse matrix* of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The inverse matrix is characterised by the fact that

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $ad - bc = 0$, such an inverse matrix does not exist.
Another important property of determinants is the multiplicative property which states that for any two 2×2-matrices A, B we have $\det(AB) = \det(A)\det(B)$.

A similar discussion can be held for 3 equations in 3 unknown. We shall only give the answers here. Consider the system of equations given by,

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= p \\ a_{21}x + a_{22}y + a_{23}z &= q \\ a_{31}x + a_{32}y + a_{33}z &= r \end{aligned}$$

Define $\Delta = a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}$. If $\Delta \neq 0$, the the system has precisely one solution, namely

$$\begin{aligned} x &= (pa_{22}a_{33} + qa_{23}a_{31} + ra_{21}a_{32} - qa_{21}a_{33} - ra_{22}a_{31} - pa_{23}a_{32})/\Delta \\ y &= (qa_{11}a_{33} + ra_{12}a_{31} + pa_{13}a_{32} - pa_{12}a_{33} - qa_{13}a_{31} - ra_{11}a_{32})/\Delta \\ z &= (ra_{11}a_{22} + pa_{12}a_{23} + qa_{13}a_{21} - ra_{12}a_{21} - pa_{13}a_{22} - qa_{11}a_{23})/\Delta \end{aligned}$$

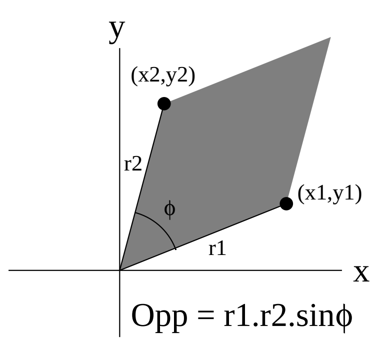
Again, if $\Delta = 0$, the system is either contradictory, or it has infinitely many solutions. We call Δ the *determinant* of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

In general we can write down the general solution of n linear equations in n unknowns, in which the determinant of the *coefficientmatrix* plays an important role. As n increases these determinants will become progressively more complicated. The determinant of a 4×4-matrix contains 24 termen, and the determinant of a 6×6-matrix has 720 terms! We shall not go into these interesting matters, but content ourselves with determinants of 2×2-matrices.

8.1.3 Geometrical interpretation of the determinant

Besides the theory of linear equations, determinants also arise ina geometrical way. Consider two vectors (x_1, y_1), (x_2, y_2) in $\mathbb{R}^2$, together with the parallellogram spanned by these vectors.
Question: what is the surface area of this parallellogram? A familiar theorem from plane geometry tells us that this area equals the product of the two sides times the sine of the enclosed angle.



Let r_1, r_2 be the lengths of the vectors (x_1, y_1), (x_2, y_2) and ϕ_1, ϕ_2 their angle with the positive x-axis. We have,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} r_1 \cos\phi_1 \\ r_1 \sin\phi_1 \end{pmatrix} \qquad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} r_2 \cos\phi_2 \\ r_2 \sin\phi_2 \end{pmatrix}.$$

The enclosed angle is equal to $|\phi_2 - \phi_1|$. The surface area O can be computed as follows,

$$r_1 r_2 \sin|\phi_2 - \phi_1| = r_1 r_2 |\sin(\phi_2 - \phi_1)|$$

Apply the difference formula of the sine,

$$O = |r_1 r_2 (\sin\phi_2 \cos\phi_1 - \cos\phi_2 \sin\phi_1)|.$$

After using $x_i = r_i \cos\phi_i,\ y_i = r_i \sin\phi_i$ for $i = 1, 2$ we find that

$$O = |x_1 y_2 - x_2 y_1|.$$

In other words, the surface area of the parallellogram spanned by the vectors (x_1, y_1) en (x_2, y_2) is equal to the absolute value of $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.
In particular we see that, with non-zero $(x_1, y_1), (x_2, y_2)$ the determinant can only be zero if the vectors (x_1, y_1) and (x_2, y_2) point in the same, or the opposite direction.
A similar derivation can be given in dimension three. A triple of vectors (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) in $\mathbb{R}^3$ spans a block B. A more complicated calculation then shows that the volume of B equals the absolute value of the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Chapter 9

Eigenvectors and eigenvalues

9.1 An example

Consider the 2×2-matrix

$$\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$$

If we apply this matrix to an arbitrary vector, you will not see much similarity between the vector we started with and the result after multiplication by the matrix. For example,

$$\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \qquad \begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

But there are exceptions, for example,

$$\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

The vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ change into a scalar multiple after multiplication by the matrix $\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$. One might wonder if there are more of such vectors. In other words, do there exist x, y, not both zero, and a number λ such that

$$\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

We call such vector $\begin{pmatrix} x \\ y \end{pmatrix}$ *eigenvectors* with *eigenvalue* λ.
To get back to our example, we must solve the explicit system of equations

$$\begin{aligned} 5x - 4y &= \lambda x \\ 3x - 2y &= \lambda y \end{aligned}$$

Of course we immediately see the solutions $x = y = 0$. But that does not give us eigenvectors. So we want non-trivial solutions, i.e. solutions where x, y are not both zero. We rewrite our equations as

$$\begin{aligned} (5 - \lambda)x - 4y &= 0 \\ 3x + (-2 - \lambda)y &= 0 \end{aligned}$$

For any λ this set of equations is a homogeneous set of two linear equations in two unknowns. In Theorem 8.1.2 we have seen that such systems can only have non-trivial solutions x, y if the coefficient determinant is zero.
So we conclude that

$$(5 - \lambda)(-2 - \lambda) + 12 = 0$$

This equation is known as the *eigenvalue equation*. The possible eigenvalues of our matrix are solutions to this equation. After elaboration we find $\lambda^2 - 3\lambda + 2 = 0$ which has the solutions 1 and 2. These are precisely the scalar multiples we saw above. Now we determine the eigenvectors. We do this by solving our system of linear equations above.
When $\lambda = 1$ we get,

$$\begin{aligned} 4x - 4y &= 0 \\ 3x - 3y &= 0 \end{aligned}$$

These equations are of course dependent. Let us choose y equal to an arbitrary number, say t. Then $y = t$ and from our equations it follows that $x = y = t$. The full solution set to equation (E) with $\lambda = 1$ reads,

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

When $\lambda = 2$ we get,

$$\begin{aligned} 3x - 4y &= 0 \\ 3x - 4y &= 0 \end{aligned}$$

We get $3x - 4y = 0$ twice. Let $y = t$ arbitrary, then $x = 4y/3 = 4t/3$. The solution set of (E) with $\lambda = 2$ reads,

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$$

With the choice $t = 3$ we find the above mentioned vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

9.2 In general

Let M be a general $n \times n$-matrix. A non-trivial column vector $\mathbf{v}$ with n components is called an *eigenvector* of M if there exists a number λ such that $M\mathbf{v} = \lambda\mathbf{v}$. The number λ is called the *eigenvalue* of M for the vector v. For computation of eigenvectors and eigenvalues we only give the recipe for 2×2 matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. First we form the *eigenvalue equation*,

$$(a - \lambda)(d - \lambda) - bc = 0.$$

For each solution λ we solve the linear system of equations

$$\begin{aligned} (a - \lambda)x + by &= 0 \\ cx + (d - \lambda)y &= 0 \end{aligned}$$

An important remark, the eigenvalue equation has degree 2. So, for a 2×2 there can be at most two distinct eigenvalues. Similarly, for an $n \times n$-matrix there can be at most n distinct eigenvalues.

9.3 An application of eigenvectors

Eigenvalues and eigenvectors play a very important role in many areas of mathematics. In this section we give an application to a population growth model in theoretical biology. There are also applications to probablity theory in the form of Markov chains, to be treated in the next chapter. Another application we will see is to systems of linear differential equations.

Example Consider a rabbit population which can be divided into two age groups. The first year, in which the young rabbit is born and becomes fertile.

The second year, in which the animal still produces offspring. We assume that animals older than two years do not contribute to new rabbits. Let $N_1(n)$ be the number of rabbits in group 1 in year n and $N_2(n)$ the number of rabbits in group 2 in year n. We assume that half of the yearlings makes it to year two. The offspring production rate of the population is modelled according to

$$\begin{pmatrix} N_1(n+1) \\ N_2(n+1) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} N_1(n) \\ N_2(n) \end{pmatrix}.$$

Suppose that $N_1(0) = 60$ and $N_2(0) = 0$. What happens to the population growth and the ratio of the groups in the long run?

To answer these questions we determine the eigenvectors and eigenvalues of the so-called *Lesliematrix* $\begin{pmatrix} 1 & 4 \\ 0.5 & 0 \end{pmatrix}$. The eigenvalues turn out to be $\lambda = 2, -1$ corresponding to the eigenvectors $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ respectively.

The eigenvector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ has a nice interpretation. Suppose that in year $n = 0$ the ratio one:two year olds equals $4 : 1$. So $N_1(0) = 4N$ and $N_2(0) = N$ for some N. Then

$$\begin{pmatrix} N_1(1) \\ N_2(1) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} N_1(0) \\ N_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} 4N \\ N \end{pmatrix} = \begin{pmatrix} 8N \\ 2N \end{pmatrix}$$

In other words, the population numbers in both groups have doubled and the ratio one:two year olds stayed the same as in the previous year. Of course this happens also in the next years and we find that the ratio $N_1(n) : N_2(n)$ is always $4 : 1$.

If the ratio $N_1(0) : N_2(0)$ does not equal $4 : 1$, the ratio $N_1(1) : N_2(1)$ will generally differ from $N_1(0) : N_2(0)$. However, it turns out that in the long run we still have $\lim_{n\to\infty} N_1(n)/N_2(n) = 4$, independent of the choice of $N_1(0), N_2(0)$!.

We can see this by taking the example $N_1(0) = 60,\ N_2(0) = 0$. Write the vector $\begin{pmatrix} 60 \\ 0 \end{pmatrix}$ as linear combination of the eigenvectors,

$$\begin{pmatrix} 60 \\ 0 \end{pmatrix} = 10 \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 10 \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

We know that

$$\begin{pmatrix} N_1(n) \\ N_2(n) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0.5 & 0 \end{pmatrix}^n \begin{pmatrix} 60 \\ 0 \end{pmatrix}$$

Because $\begin{pmatrix} 1 & 4 \\ 0.5 & 0 \end{pmatrix}^n \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 2^n \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 4 \\ 0.5 & 0 \end{pmatrix}^n \begin{pmatrix} 2 \\ -1 \end{pmatrix} = (-1)^n \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ we get,

$$\begin{pmatrix} N_1(n) \\ N_2(n) \end{pmatrix} = 10 \cdot 2^n \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 10 \cdot (-1)^n \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Divide both sides by 2^n and take the limit $\lim_{n\to\infty}$,

$$\lim_{n\to\infty} 2^{-n} \begin{pmatrix} N_1(n) \\ N_2(n) \end{pmatrix} = 10 \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

In particular it follows that $\lim_{n\to\infty} N_1(n)/N_2(n) = 4$.

We hope that you have understood enough of the above example to appreciate the following theorem

Theorem 9.3.1 *Let M be an $m \times m$ matrix and suppose that it has a positive eigenvalue l which is strictly larger than the absolute values of all other eigenvalues (including the complex ones). Suppose also that l is not a multiple solution of the eigenvalue equation.*

Let $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$ *be the corresponding eigenvector. Consider the recursion*

$$\begin{pmatrix} x_1(n+1) \\ \vdots \\ x_m(n+1) \end{pmatrix} = M \begin{pmatrix} x_1(n) \\ \vdots \\ x_m(n) \end{pmatrix}$$

Write

$$\mathbf{x}(n) = \begin{pmatrix} x_1(n) \\ \vdots \\ x_m(n) \end{pmatrix}$$

Then,

$$\lim_{n\to\infty} l^{-n}\mathbf{x}(n) = a\mathbf{v}$$

for a certain constant a.

Remark 1 A positive eigenvalue l, which is not a multiple eigenvalue, and which is strictly larger than the absolute value of all other (complex and real) eigenvalues is called *dominant*.

Remark 2 The constant a can be computed as follows. Let $\mathbf{w}$ be the eigenvector with eigenvalue l of the transpose matrix M^T. Then $a = (\mathbf{w}\cdot\mathbf{x}(0))/(\mathbf{w}\cdot\mathbf{v})$. In particular, the theorem is only of interest when $a \neq 0$, i.e. $\mathbf{w}\cdot\mathbf{x}(0) \neq 0$. This happens most of the time.
Remark 3 When $a \neq 0$ we see that the ratios $x_1(n) : x_2(n) : \cdots : x_m(n)$ converge to the ratios $v_1 : v_2 : \cdots : v_m$ as $n \to \infty$. Furthermore we see that if $l < 1$ the $x_i(n)$ all tend to 0 as $n \to \infty$.
Remark 4 When M is a Leslie matrix "the population becomes extinct" when $l < 1$. If $l = 1$ the population survives and when $l > 1$ it increases exponentionally. The components of the eigenvector corresponding to the dominant eigenvalue are sometimes called the *stable distribution*.
Remark 5 We call l the *growth factor* of our recursion.

Example We take the example of the tern population from Chapter 7. We had a recursion of the form,

$$\begin{pmatrix} N_1(n+1) \\ N_2(n+1) \\ N_3(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} N_1(n) \\ N_2(n) \\ N_3(n) \end{pmatrix}.$$

To determine the growth behaviour we determine the largest eigenvalue of the Leslie matrix. We do this by direct solution of the equations

$$\begin{pmatrix} 0 & 2 & 4 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

In the case of Leslie matrices this goes very easily by simply writing out the equations. We find, $2y + 4z = \lambda x,\ x/4 = \lambda y,\ y/2 = \lambda z$. From this follows $y = 2\lambda z,\ x = 4\lambda y = 8\lambda^2 z$ and by substitution into the first equation, $4\lambda z + 4z = 8\lambda^3 z$. After division by z we are left with $8l^3 = 4l + 4$. This is the eigenvalue equation, sometimes called Lotka-equation in the case of Leslie-matrices. The solutions of our equation are $1, (-1 \pm i)/2$. We see that $l = 1$ is the largest eigenvalue and it is positive. So our population persists and the ratios of the age groups go to $8 : 2 : 1$ because $\begin{pmatrix} 8 \\ 2 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 1$

9.4 Exercises

The exercises below are taken from M.de Gee, Wiskunde in Werking, Epsilon Uitgaven, Utrecht.

Exercise 9.4.1 *We are given the* 3×3*-matrix*

$$A = \begin{pmatrix} 3 & -1 & 4 \\ -1 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

1. *Find out if the vector*
$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
is an eigenvector or not.

2. *From the matrix* A *we can immediately see that there is an eigenvector having eigenvalue* 0*. Why? Determine an eigenvector.*

3. *Consider the vectors*
$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$
Show that they are eigenvectors of A *and determine their eigenvalues.*

Exercise 9.4.2 *Determine eigenvectors and eigenvalues of the following matrices*

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 3 \\ 2 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} -6 & 3 \\ -4 & 1 \end{pmatrix}.$$

Exercise 9.4.3 *For the population growth of a certain bird species three Leslie matrices are given,*

$$L = \begin{pmatrix} 0 & 2 & 4 \\ 1/16 & 0 & 0 \\ 0 & 1/4 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 2 & 4 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix},$$

which correspond to the following situations,

S_1 *: Good environmental conditions*

S_2 *: Presence in the environment of toxic waste that diminishes fertility.*

S_3 *: Presence in the environment of toxines that kill some of the living birds.*

1. *Which matrix corresponds to which situation and why?*
2. *Determine the stable distribution of each matrix (stable distribution is the distribution over the age classes in which the ratios do not change over the years).*
3. *How does the bird population develop in each situation when we take a stable distribution as starting point.*

Exercise 9.4.4 *The female population of the Birma-beetle is divided into three age classes:*

K_1 *0-1 year*

K_2 *1-2 year*

K_3 *2-3 year*

Only the beetles from class K_3 *produce offspring, about 24 young beetles per individual. Of class* K_1 *only 2/3 survives, of class* K_2 *half survives.*

1. *Write down the Leslie matrix* L *corresponding to this model.*
2. *The initial population distribution is represented by the vector* $\mathbf{n}(0)$*:*

$$\mathbf{n}(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

Compute $\mathbf{n}(1), \mathbf{n}(1), \mathbf{n}(2)$.

3. *Show that* L *has one real eigenvalue and compute it. Is this a dominant eigenvalue?*
4. *Determine the stable population distribution.*

Exercise 9.4.5 *In a closed and isolated vessel there is a mixture of gass and liquid of the same compound S. At time t the (weight) fraction of liquid is $l(t)$ and the fraction of gass is $g(t)$. So we have $l(t)+g(t)=1$ at all times. The time t is measured in minutes.*
Through evaporation 2% of the liquid turns into gass and 6% of the gass turns into liquid though condensation. Furthermore, $l(0)=0.1$.

1. *Show that the change per minute is described by*

$$\begin{pmatrix} l(t+1) \\ g(t+1) \end{pmatrix} = A \begin{pmatrix} l(t) \\ g(t) \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 0.98 & 0.06 \\ 0.02 & 0.94 \end{pmatrix}.$$

2. *Compute the distribution gass/liquid after 1 minute.*

3. *Compute the eigenvectors and eigenvalues of A. Is there a dominant eigenvalue?*

4. *What will be the ratio $l(t) : g(t)$ in the long run?*