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Measure and Integration: Mid-Term Exam April 19, 2005

1. Let $f, g : [a, b] \to \mathbb{R}$ be bounded Riemann integrable funcions. Show that fg is Riemann integrable. (Hint: express fg in terms of $(f + g)$ and $(f - g)$).

   Proof: First notice that f and g are Riemann integrable funcions, hence $f+g$ and $f-g$ are also Riemann integrable funcions. By problem 4 Exercises2, it follows that $(f+g)^2$ and $(f-g)^2$ are Riemann integrable funcions. Now,

$$fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2.$$

   Hence fg is Riemann integrable since it is the difference of two Riemann integrable funcions.

2. Consider the measure space $(\mathbb{R}, \overline{\mathcal{B}}_{\mathbb{R}}, \lambda)$, where $\overline{\mathcal{B}}_{\mathbb{R}}$ is the Lebesgue σ-algebra over $\mathbb{R}$, and λ is Lebesgue measure. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \cdot 1_{[k/2^n,(k+1)/2^n)},\ n \geq 1.$$

   (a) Show that f_n is measurable, and $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$.

   (b) Let $f(x) = \lim_{n\to\infty} f_n(x)$, for $x \in \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{R}$ is measurable.

   (c) Show that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)d\lambda(x) = \frac{1}{2}$.

   Proof (a): Since for each $n \geq 1$ and $k \leq 2^n - 1$ the set $[k/2^n, (k+1)/2^n)$ is Lebesgue measurable, it follows from problem 3 Exercises 8 that f_n is measurable. Now let $x \in \mathbb{R}$. If $x \geq 1$ or $x < 0$, then $f_n(x) = f_{n+1}(x) = 0$. Suppose $x \in [0, 1)$, then there exists a $k \leq 2^n - 1$ such that $k/2^n \leq x < (k+1)/2^n$ and hence $f_n(x) = k/2^n$. To determine $f_{n+1}(x)$ we divide the interval $[k/2^n, (k+1)/2^n)$ into two equal parts $[2k/2^{n+1}, (2k+1)/2^{n+1})$ and $[(2k+1)/2^{n+1}, (2k+2)/2^{n+1})$. If $x \in [2k/2^{n+1}, (2k+1)/2^{n+1})$, then $f_{n+1}(x) = 2k/2^{n+1} = k/2^n = f_n(x)$. If $x \in [(2k+1)/2^{n+1}, (2k+2)/2^{n+1})$, then $f_{n+1}(x) = (2k+1)/2^{n+1} > f_n(x)$. In all cases we see that $f_n(x) \leq f_{n+1}(x)$.

   Proof (b): Since for each $x \in \mathbb{R}$, $(f_n(x))$ is an increasing sequence, it follows that $f(x) = \lim_{n\to\infty} f_n(x) = \sup_n f_n(x)$. By problem 2 Exercises 8 and part (a), we see that f is measurable.

   Proof (c): Each f_n is a measurable simple function, hence

$$\int_{\mathbb{R}} f_n(x)d\lambda(x) = \sum_{k=0}^{2^n-1} \frac{k}{2^n}\lambda([k/2^n, (k+1)/2^n)) = \frac{1}{4^n}\sum_{k=0}^{2^n-1} k = \frac{(2^n-1)2^n}{2\cdot 4^n}.$$

Thus,

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)d\lambda(x) = \lim_{n\to\infty}\frac{(2^n-1)2^n}{2\cdot 4^n} = \frac{1}{2}.$$

3. Let $M \subset \mathbb{R}$ be a non-Lebesgue measurable set (i.e. $M \notin \overline{\mathcal{B}}_{\mathbb{R}}$.). Define $A = \{(x,x) \in \mathbb{R}^2 : x \in M\}$, and let $g : \mathbb{R} \to \mathbb{R}^2$ be given by $g(x) = (x,x)$.

   (a) Show that $A \in \overline{\mathcal{B}}_{\mathbb{R}^2}$. i.e. A is Lebesgue measurable. (Hint: use the fact that Lebesgue measure is rotation invariant).

   (b) Show that g is a Borel-measurable function, i.e. $g^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$ for each $B \in \mathcal{B}_{\mathbb{R}^2}$.

   (c) Show that $A \notin \mathcal{B}_{\mathbb{R}^2}$, i.e. A is not Borel measurable.

Proof (a): Notice that A is a subset of the diagonal line $L = \{(x,y) : y = x\}$. So L is obtained from the x-axis (i.e. $\mathbb{R}$) by rotating through an angle of $\pi/4$. Since Lebesgue measure is rotation invariant, and the Lebesgue measure of $\mathbb{R}$ (as a subset of $\mathbb{R}^2$) is zero, it follows that $|A|_e \leq |L|_e = 0$. Thus, A is Lebesgue measurable, i.e. $A \in \overline{\mathcal{B}}_{\mathbb{R}^2}$.

Proof (b): It is easy to see that the map g is continuous, and hence by Lemma 3.2.1 g is Borel-measurable, i.e. $g^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$ for each $B \in \mathcal{B}_{\mathbb{R}^2}$.

Proof (c): If $A \in \mathcal{B}_{\mathbb{R}^2}$, then by part (b) we would have that $M = g^{-1}(A) \in \mathcal{B}_{\mathbb{R}} \subset \overline{\mathcal{B}}_{\mathbb{R}}$, which is a contradiction.

4. Let $\mathcal{M} = \{E \subseteq \mathbb{R} : |A|_e = |A \cap E|_e + |A \cap E^c|_e \text{ for all } A \subseteq \mathbb{R}\}$, where $|A|_e$ denotes the outer Lebesgue measure of A.

   (a) Show that $\mathcal{M}$ is an algebra over $\mathbb{R}$. (Hint: $A \cap (E_1 \cup E_2) = (A \cap E_1) \bigcup (A \cap E_2 \cap E_1^c)$).

   (b) Prove by induction that if $E_1, \cdots, E_n \in \mathcal{M}$ are pairwise disjoint, then for any $A \subseteq \mathbb{R}$

   $$|A \cap (\bigcup_{i=1}^{n} E_i)|_e = \sum_{i=1}^{n} |A \cap E_i|_e.$$

   (c) Show that if $E_1, E_2, \cdots \in \mathcal{M}$ is a countable collection of disjoint elements of $\mathcal{M}$, then $\displaystyle\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

   (d) Show that $\mathcal{M}$ is a σ-algebra over $\mathbb{R}$.

   (e) Let $\mathcal{C} = \{(a,\infty) : a \in \mathbb{R}\}$. Show that $\mathcal{C} \subseteq \mathcal{M}$. Conclude that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}$, where $\mathcal{B}_{\mathbb{R}}$ denotes the Borel σ-algebra over $\mathbb{R}$.

Proof (a): It is clear from the definition of $\mathcal{M}$ that $\mathbb{R} \in \mathcal{M}$, and if $E \in \mathcal{M}$ then $E^c \in \mathcal{M}$, i.e. $\mathcal{M}$ is closed under complements. We show that $\mathcal{M}$ is closed under finite unions. Let $E_1, E_2 \in \mathcal{M}$, and A any subset of $\mathbb{R}$. We need to show that $|A|_e = |A \cap (E_1 \cup E_2)|_e + |A \cap (E_1 \cup E_2)^c|_e$. Since outer Lebesgue measure is subadditive, it follows that $|A|_e \leq |A\cap(E_1\cup E_2)|_e+|A\cap(E_1\cup E_2)^c|_e$. We now prove the other inequality.

$$\begin{aligned}|A\cap(E_1\cup E_2)|_e+|A\cap(E_1\cup E_2)^c|_e &\leq |A\cap E_1|_e+|A\cap E_1^c\cap E_2|_e+|A\cap E_1^c\cap E_2^c|_e \\ &= |A\cap E_1|_e+|A\cap E_1^c|_e \\ &= |A|_e.\end{aligned}$$

The first inequality follows from the hint and the subadditivity of the outer Lebesgue measure, the first equality follows from the fact that $E_2 \in \mathcal{M}$ and the second equality follows from the fact that $E_1 \in \mathcal{M}$.

Proof (b): The equality is trivial for $n = 1$. Suppose it is true for $i < n$, then

$$\begin{aligned}|A\cap(\bigcup_{j=1}^{i+1} E_j)|_e &= |A\cap(\bigcup_{j=1}^{i+1} E_j)\cap E_{i+1}|_e+|A\cap(\bigcup_{j=1}^{i+1} E_j)\cap E_{i+1}^c|_e \\ &= |A\cap E_{i+1}|_e+|A\cap(\bigcup_{j=1}^{i} E_j)|_e \\ &= |A\cap E_{i+1}|_e+\sum_{j=1}^{i}|A\cap E_j|_e \\ &= \sum_{j=1}^{i+1}|A\cap E_j|_e.\end{aligned}$$

The first equality follows from $E_{i+1} \in \mathcal{M}$, the second from the fact that $E_1, E_2, \cdots, E_{i+1}$ are pairwise disjoint and the third follows from our induction hypothesis.

Proof (c): Let $E = \bigcup_{i=1}^{\infty} E_i$, by subadditivity of the outer Lebesgue measure we only need to show that $|A\cap E|_e+|A\cap E^c|_e \leq |A|_e$ for any $A \subseteq \mathbb{R}$. Let $F_n = \bigcup_{i=1}^{n} E_i$, then by part (a) $F_n \in \mathcal{M}$. By part (b) and monotonicity of the outer Lebesgue measure, we have

$$|A|_e = |A\cap F_n|_e+|A\cap F_n^c|_e \geq \sum_{i=1}^{n}|A\cap E_i|_e+|A\cap E^c|_e.$$

Taking the limit as $n \to \infty$, we get by σ-subadditivity of the outer Lebesgue measure that

$$|A|_e \geq \sum_{i=1}^{\infty}|A\cap E_i|_e+|A\cap E^c|_e \geq |A\cap E|_e+|A\cap E^c|_e.$$

Proof (d): Let $F_1, F_2, \cdots \in \mathcal{M}$. Define $E_1 = F_1$ and $E_n = F_n \setminus \cup_{j=1}^{n-1} E_j$, $n \geq 2$. Then, $E_n \in \mathcal{M}$ (since $\mathcal{M}$ is an algebra) are pairwise disjoint, and $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i \in \mathcal{M}$ (by part (c)). Hence, $\mathcal{M}$ is a σ-algebra over $\mathbb{R}$.

Proof (e): Let $(a,\infty) \in \mathcal{C}$ and $A \subseteq \mathbb{R}$. We need to show that $|A|_e \geq |A\cap(a,\infty)|_e+|A\cap(-\infty,a]|_e$. If $|A|_e = \infty$, then the inequality is trivially true. Suppose that $|A|_e < \infty$ and let $\epsilon > 0$. There exists a countable collection of closed intervals

$[a_n, b_n]$ such that $A \subset \bigcup_{n=1}^{\infty}[a_n, b_n]$ and $\sum_{n=1}^{\infty}(b_n - a_n) \leq |A|_e + \epsilon$ (this follows from the definition of the outer Lebesgue measure). Let $I_n = [a_n, b_n] \cap (a, \infty)$ and $I'_n = [a_n, b_n] \cap (-\infty, a]$. Notice that I_n and I'_n are disjoint and $[a_n, b_n] = I_n \cup I'_n$. Now, $|A \cap (a, \infty)|_e \leq |\bigcup_{n=1}^{\infty} I_n|_e \leq \sum_{n=1}^{\infty} |I_n|_e$, and $|A \cap (-\infty, a]|_e \leq \sum_{n=1}^{\infty} |I'_n|_e$. Hence,

$$|A \cap (a, \infty)|_e + |A \cap (-\infty, a]|_e \leq \sum_{n=1}^{\infty}(|I_n|_e + |I'_n|_e) = \sum_{n=1}^{\infty}(b_n - a_n) \leq |A|_e + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $|A|_e \geq |A \cap (a, \infty)|_e + |A \cap (-\infty, a]|_e$. This shows that $\mathcal{C} \subseteq \mathcal{M}$. Since $\mathcal{B}_{\mathbb{R}}$ is the smallest σ-algebra generated by $\mathcal{C}$, it follows that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}$.