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Measure and Integration Solutions 1

1. Let J be a rectangle in $\mathbb{R}^n$, $c, d \in \mathbb{R}$, and $f, g : J \to \mathbb{R}$ Riemann integrable functions. Show that $cf + dg$ is Riemann Integrable on J.

   Proof. Notice that for any exact non-overlapping finte cover $\mathcal{C}$ of J and any choice function ξ, we have

   $$\mathcal{R}(cf + dg; \mathcal{C}, \xi) = c\mathcal{R}(f; \mathcal{C}, \xi), d\mathcal{R}(g; \mathcal{C}, \xi).$$

   Given $\varepsilon > 0$, there exists $\delta > 0$ such that

   $$|\mathcal{R}(f; \mathcal{C}, \xi) - (R)\int_J f(x)dx| < \varepsilon, \text{ and } |\mathcal{R}(g; \mathcal{C}, \xi) - (R)\int_J g(x)dx| < \varepsilon$$

   for all exact non-overlapping finte covers $\mathcal{C}$ of J such that $\|\mathcal{C}\| < \delta$ and all choice functions ξ. Then,

   $$|\mathcal{R}(cf + dg; \mathcal{C}, \xi) - c(R)\int_J f(x)dx + d(R)\int_J g(x)dx| < \varepsilon(|c| + |d|).$$

   Thus, $cf + dg$ is Riemann Integrable on J.

2. Let J be a rectangle in $\mathbb{R}^n$, and $f : J \to \mathbb{R}$ a bounded function. Show that f is Riemann Integrable on J **if and only if** for every $\epsilon > 0$, there exists a finite non-overlapping exact cover $\mathcal{C}$ of J such that

   $$\mathcal{U}(f; \mathcal{C}) - \mathcal{L}(f; \mathcal{C}) < \epsilon.$$

   Proof. Suppose f is Riemann integrable. Then,

   $$\lim_{\|\mathcal{C}\| \to 0} \mathcal{U}(f; \mathcal{C}) = \lim_{\|\mathcal{C}\| \to 0} \mathcal{L}(f; \mathcal{C}) = (R)\int_J f(x)dx.$$

   Given $\varepsilon > 0$ there exists a $\delta > 0$ such that

   $$|\mathcal{U}(f; \mathcal{C}) - (R)\int_J f(x)dx| < \varepsilon/2, \text{ and } |\mathcal{L}(f; \mathcal{C}) - (R)\int_J f(x)dx| < \varepsilon/2$$

   for all exact non-overlapping finte covers $\mathcal{C}$ of J such that $\|\mathcal{C}\| < \delta$. Choose any such cover $\mathcal{C}$, then $\mathcal{U}(f; \mathcal{C}) - \mathcal{L}(f; \mathcal{C}) < \varepsilon$.

   Conversely, Let $\varepsilon > 0$ and suppose $\mathcal{C}$ is a finite non-overlapping exact cover of J such that

   $$\mathcal{U}(f; \mathcal{C}) - \mathcal{L}(f; \mathcal{C}) < \epsilon.$$

   Then,

   $$\inf_{\mathcal{C}'} \mathcal{U}(f; \mathcal{C}') - \sup_{\mathcal{C}'} \mathcal{L}(f; \mathcal{C}') \le \mathcal{U}(f; \mathcal{C}) - \mathcal{L}(f; \mathcal{C}) < \epsilon.$$

   Since $\varepsilon > 0$ is arbitrary, we get $\inf_{\mathcal{C}'} \mathcal{U}(f; \mathcal{C}') = \sup_{\mathcal{C}'} \mathcal{L}(f; \mathcal{C}')$. By Theorem 1.1.8, f is Riemann integrable.

3. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded monotone function. Show that f is Riemann Integrable.

Proof. Let $\varepsilon > 0$, choose n sufficiently large so that $(f(b) - f(a))(b-a)/n < \varepsilon$. Let

$$\mathcal{C} = \{[a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}] : i = 1, 2, \cdots, n\}.$$

Then, $\mathcal{C}$ is a finite non-overlapping exact cover of $[a, b]$. Assume with no loss of generality that f is non-decreasing, then

$$\mathcal{U}(f; \mathcal{C}) = \sum_{i=1}^{n} \frac{b-a}{n} f(a + i\frac{b-a}{n}),$$

and

$$\mathcal{L}(f; \mathcal{C}) = \sum_{i=1}^{n} \frac{b-a}{n} f(a + (i-1)\frac{b-a}{n}).$$

Thus,

$$\mathcal{U}(f; \mathcal{C}) - \mathcal{L}(f; \mathcal{C}) = (f(b) - f(a))\frac{b-a}{n} < \varepsilon.$$

By exercise (2) above, it follows that f is Riemann integrable.

4. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and suppose that f is continuous except at the points $t_1 < t_2 < \cdots < t_n$. Show that f is Riemann Integrable.

Proof. Let $\varepsilon > 0$ and define $I_j = (s_j - \varepsilon/2n, s_j + \varepsilon/2n)$ for $j = 1, 2, \cdots, n$. Then, $\sum_{j=1}^{n} \text{vol}(I_j) = \varepsilon$. Let $\mathcal{B} = \{\overline{I}_j = [s_j - \varepsilon/2n, s_j + \varepsilon/2n] : j = 1, 2, \cdots, n\}$ and $K = [a, b] - \cup_{j=1}^{n} I_j$. Then K is compact, and is the disjoint union of $n + 1$ closed intervals. By the uniform continuity of f on K, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in K$ with $|x - y| < \delta$. Now, divide each interval making up K into closed intervals of length less than δ, and let $\mathcal{G}$ be the collection of closed intervals thus obtained. Consider the cover $\mathcal{C} = \mathcal{B} \cup \mathcal{G}$. Then, $\mathcal{C}$ is a finite non-overlapping exact cover of $[a, b]$, and

$$\begin{aligned}
\mathcal{U}(f; \mathcal{C}) - \mathcal{L}(f; \mathcal{C}) &= \sum_{I \in \mathcal{G}} (\sup_{x \in I} f(x) - \inf_{x \in I} f(x)) \text{vol}(I) + \sum_{I \in \mathcal{B}} (\sup_{x \in I} f(x) - \inf_{x \in I} f(x)) \text{vol}(I) \\
&< \varepsilon(b-a) + 2\|f\|_u \varepsilon,
\end{aligned}$$

where, $\|f\|_u = \sup_{x \in [a,b]} |f(x)|$. By exercise 2 above, it follows that f is Riemann integrable.