Universiteit Utrecht | Universiteit Utrecht | Boedapestlaan 6
Mathematisch Instituut | 3584 CD Utrecht

Measure and Integration Exercises 10

1. Let $(E, \mathcal{B}, \mu)$ be a measure space. Let (f_n) be a sequence of non-negative measurable functions.

   (a) Prove that
   $$\int_E \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu.$$

   (b) Let (g_n) be a sequence of μ-integrable functions on E such that $\sum_{n=1}^{\infty} \int_E |g_n| \, d\mu < \infty$. Show that $\sum_{n=1}^{\infty} g_n$ is finite μ almost everywhere, and
   $$\int_E \sum_{n=1}^{\infty} g_n \, d\mu = \sum_{n=1}^{\infty} \int_E g_n \, d\mu.$$

   (c) Let f be a non-negative integrable function on E. Define ν on $\mathcal{B}$ by
   $$\nu(A) = \int_A f \, d\mu.$$
   Show that ν is a finite measure on $\mathcal{B}$.

proof (a): Let $h_n = \sum_{m=1}^{n} f_m$, then (h_n) is an increasing sequence of non-negative measurable functions converging to $\sum_{n=1}^{\infty} f_n$. By the Monotone Convergence Theorem,
$$\lim_{n\to\infty} \int_E h_n \, d\mu = \int_E \lim_{n\to\infty} h_n \, d\mu = \int_E \sum_{n=1}^{\infty} f_n d\mu.$$
By the linearity of the integral, $\int_E h_n \, d\mu = \sum_{m=1}^{n} \int_E f_m \, d\mu$, and hence $\lim_{n\to\infty} \int_E h_n \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu$. Thus,
$$\int_E \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu.$$

proof (b): By part (a), $\int_E \sum_{n=1}^{\infty} |g_n| \, d\mu = \sum_{n=1}^{\infty} \int_E |g_n| \, d\mu < \infty$, hence $\sum_{n=1}^{\infty} |g_n|$ is μ-integrable. By Theorem 3.2.8, $\sum_{n=1}^{\infty} |g_n|$ is finite μ almost everywhere. Since $|\sum_{n=1}^{\infty} g_n| \leq \sum_{n=1}^{\infty} |g_n|$, it follows that $\sum_{n=1}^{\infty} g_n$ is finite μ almost everywhere. Let $h_n = \sum_{m=1}^{n} g_m$, then (h_m) converges to $\sum_{n=1}^{\infty} g_n$ μ a.e. Furthermore, $|h_n| \leq \sum_{n=1}^{\infty} |g_n|$, thus by the Dominated Convergence Theorem,
$$\sum_{n=1}^{\infty} \int_E g_n \, d\mu = \lim_{n\to\infty} \int_E h_n \, d\mu = \int_E \lim_{n\to\infty} h_n \, d\mu = \int_E \sum_{n=1}^{\infty} g_n \, d\mu.$$

proof (c): Clearly, $\nu(\emptyset) = 0$ and $\nu(E) < \infty$. We only need to show that ν is σ-additive. Let $\{B_n\}$ be pairwise disjoint, then $f \cdot 1_{\cup_{n=1}^{\infty} B_n} = \sum_{n=1}^{\infty} f \cdot 1_{B_n}$. By part (a),

$$\nu(\cup_{n=1}^{\infty} B_n) = \int_E f \cdot 1_{\cup_{n=1}^{\infty} B_n} \, d\mu = \sum_{n=1}^{\infty} \int_E f \cdot 1_{B_n} = \sum_{n=1}^{\infty} \nu(B_n).$$

2. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where μ is the counting measure on $\mathcal{P}(\mathbb{N})$, i.e. $\mu(A)$ is equal to the number of elements in A.

   (a) Show that for any $f : \mathbb{N} \to [0, \infty]$, one has

$$\int_{\mathbb{N}} f \, d\mu = \sum_{k=1}^{\infty} f(k).$$

   (b) For each $n \geq 1$, let $(a_k^n)_k$ be a sequence of real numbers such that $0 \leq a_k^n \leq a_k^{n+1}$ for all k and n. Show that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_k^n = \sum_{k=1}^{\infty} \lim_{n\to\infty} a_k^n.$$

proof (a): Notice that if $f = 1_A$, the indicator function of a measurable set A, then

$$\int_{\mathbb{N}} 1_A \, d\mu = \mu(A) = \sum_{k=1}^{\infty} 1_A(k).$$

If f is a non-negative simple function, then $f = \sum_{m=1}^{n} \alpha_i 1_{A_i}$, where A_i are measurable sets. By the linearity of the integal, we have

$$\int_{\mathbb{N}} f \, d\mu = \sum_{m=1}^{n} \alpha_i \int_{\mathbb{N}} 1_{A_i} \, d\mu = \sum_{m=1}^{n} \alpha_i \sum_{k=1}^{\infty} 1_{A_i}(k) = \sum_{k=1}^{\infty} \sum_{m=1}^{n} \alpha_i 1_{A_i}(k) = \sum_{k=1}^{\infty} f(k).$$

Finally, let f be a non-negative measurable function. Let

$$g_n(k) = \begin{cases} f(k) & \text{if } k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

Then, (g_n) is a sequence of non-negative simple functions, $g_n \leq g_{n+1} \leq f$ and $\lim_{n\to\infty} g_n(k) = f(k)$ for all $k \geq 1$. Moreover, $\int_{\mathbb{N}} g_n \, d\mu = \sum_{k=1}^{n} f(k)$. By the Monotone convergence Theorem,

$$\int_{\mathbb{N}} f \, d\mu = \lim_{n\to\infty} \int_{\mathbb{N}} g_n \, d\mu = \sum_{k=1}^{\infty} f(k).$$

proof (b): Let $f_n(k) = a_k^n$. Then, $f_n \leq f_{n+1}$ for all $n \geq 1$. By the Monotone Convergence Theorem and part (a), we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_k^n = \lim_{n\to\infty} \int_{\mathbb{N}} f_n \, d\mu = \int_{\mathbb{N}} \lim_{n\to\infty} f_n \, d\mu = \sum_{k=1}^{\infty} \lim_{n\to\infty} a_k^n.$$

3. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f : E \to [0, \infty]$ a measurable function.

(a) Show that if $\int_E f\,d\mu < \infty$, then $\lim_{n\to\infty} n\mu(f \geq n) = 0$.

(b) Suppose that $\mu(E) < \infty$. Show that

$$\int_E f\,d\mu < \infty \textbf{ if and only if } \sum_{n=0}^{\infty} \mu(f > n) < \infty.$$

proof (a): Suppose $\int_E f\,d\mu < \infty$, then $\mu(f = \infty) = 0$, and

$$n\mu(f \geq n) = \int_E n \cdot 1_{\{f\geq n\}}\,d\mu \leq \int_E f \cdot 1_{\{f \geq n\}}\,d\mu.$$

Now, $(f \cdot 1_{\{f\geq n\}})$ is a sequence of non-negative functions converging to $f \cdot 1_{\{f=\infty\}}$. Since, $f \cdot 1_{\{f\geq n\}} \leq f$, and f is μ-integrable, it follows by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_E f \cdot 1_{\{f\geq n\}}\,d\mu = \int_E f \cdot 1_{\{f=\infty\}}\,d\mu = \int_{\{f=\infty\}} f\,d\mu = 0.$$

Thus,

$$\lim_{n\to\infty} n\mu(f \geq n) \leq \lim_{n\to\infty} \int_E f \cdot 1_{\{f\geq n\}}\,d\mu = 0.$$

proof (b): Assume $\mu(E) < \infty$. Suppose $\int_E f\,d\mu < \infty$, then $\mu(f = \infty) = 0$ and using the same proof as in part (a) $\lim_{N\to\infty} N\mu(f > N) = 0$. By the Lebesgue Dominated Convergence Theorem $\int_E f\,d\mu = \lim_{N\to\infty} \int_E f \cdot 1_{\{f\leq N\}}\,d\mu$.

$$\begin{aligned}
\int_E f \cdot 1_{\{f\leq N\}}\,d\mu &= \int_E \sum_{n=0}^{N-1} f 1_{\{n<f\leq n+1\}}\,d\mu \\
&> \sum_{n=0}^{N-1} \int_E n 1_{\{n<f\leq n+1\}}\,d\mu \\
&= \sum_{n=0}^{N-1} n\mu(n < f \leq n+1) \\
&= \sum_{n=0}^{N-1} (n\mu(f > n) - (n+1)\mu(f > n+1) + \mu(f > n+1)) \\
&= -N\mu(f > N) + \sum_{n=1}^{N} \mu(f > n).
\end{aligned}$$

Notice that $\mu(n < f \leq n+1) = \mu(f > n) - \mu(f > n+1)$ since $\mu(E) < \infty$. Taking the limit as $N \to \infty$, we get

$$\sum_{n=1}^{\infty} \mu(f > n) \leq \int_E f\,d\mu < \infty.$$

Since $\mu(f>0)<\infty$, it follows that $\sum_{n=0}^{\infty}\mu(f>n)<\infty$.

Conversely, suppose $\sum_{n=0}^{\infty}\mu(f>n)<\infty$. From this it follows that $\mu(f=\infty)=\lim_{n\to\infty}\mu(f>n)=0$. For each $N\geq 1$,

$$\begin{aligned}
\int_E f\cdot 1_{f\leq N\}}\,d\mu &= \int_E \sum_{n=0}^{N-1} f1_{\{n<f\leq n+1\}}\,d\mu \\
&= \int_E \sum_{n=0}^{N-1} f1_{\{n<f\leq n+1\}}\,d\mu \\
&\leq \int_E \sum_{n=0}^{N-1}(n+1)1_{\{n<f\leq n+1\}}\,d\mu \\
&\leq \int_E \sum_{n=0}^{N-1} 1_{\{f>n\}}d\mu \\
&= \sum_{n=0}^{N-1}\mu(f>n).
\end{aligned}$$

By the Monotone Covergence Theorem, we get

$$\int_E f\,d\mu = \lim_{N\to\infty}\int_E f\cdot 1_{\{f\leq N\}}\,d\mu \leq \sum_{n=0}^{\infty}\mu(f>n)<\infty.$$