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Measure and Integration Solutions 12

1. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to \mathbb{R}$ a sequence of measurable real valued functions on $(E, \mathcal{B}, \mu)$. Suppose $f, g : E \to \mathbb{R}$ are measurable functions such that $f_n \to f$ in μ-measure and $f_n \to g$ μ a.e. Show that $f = g$ μ a.e.

   Proof Since $f_n \to f$ in μ-measure, then by Theorem 3.3.10, there exists a subsequence (f_{n_j}) such that $f_{n_j} \to f$ μ a.e. Furthermore, $f_n \to g$ μ a.e. implies $f_{n_j} \to g$ μ a.e. Let $A = \{x \in E : \lim_{j\to\infty} f_{n_j}(x) = f(x)\}$ and $B = \{x \in E : \lim_{j\to\infty} f_{n_j}(x) = g(x)\}$. Then $\mu(A^c) = \mu(B^c) = 0$. For each $x \in A \cap B$, we have $f(x) = g(x)$ (since limits of real valued sequences are unique), and $\mu((A \cap B)^c) \leq \mu(A^c) + \mu(B^c) = 0$, it follows that $f = g$ μ a.e.

2. Consider the measure space $([0, \infty), \mathcal{B}, \lambda)$, where $\mathcal{B}$ and λ are the restriction of the Borel σ-algebra and Lebesgue measure to the interval $[0, \infty)$. Define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } n \leq x \leq n + \frac{1}{n} \\ 0 & \text{elsewhere} . \end{cases}$$

   (a) Prove that $f_n \to 0$ λ a.e. and in λ-measure.

   (b) Prove that condition (3.3.8) of Theorem 3.3.7 does not hold, λ i.e. **it is not true** that

$$\lim_{m\to\infty} \lambda(\sup_{n\geq m} |f_n| \geq \epsilon) = 0 \quad \text{for all } \epsilon > 0.$$

   Proof (a) For any $x \geq 0$, there exists an integer N such that $x < N$. Then, for any $n \geq N$, $x \notin [n, n + 1/n]$ and hence $f_n(x) = 0$ for all $n \geq N$. This show that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \geq 0$, in particular, λ a.e.. Now, for any $\epsilon > 0$,

$$\lambda(|f_n| \geq \epsilon) = \lambda([n, n + 1/n]) = 1/n \to 0 \text{ as } n \to \infty.$$

   Thus, $f_n \to 0$ in λ measure

   Proof (b) For any $m \geq 1$,

$$\lambda(\sup_{n\geq m} |f_n| \geq \epsilon) = \lambda(\bigcup_{n=m}^{\infty} [n, n + 1/n]) = \sum_{n=m}^{\infty} 1/n = \infty.$$

   So, condition (3.3.8) of Theorem 3.3.7 does not hold.

3. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to \mathbb{R}$ a sequence of measurable real valued functions on $(E, \mathcal{B}, \mu)$. Let (ϵ_n) be a sequence of positive real numbers such that $\sum_n \epsilon_n < \infty$. Prove that if $\sum_{n=0}^{\infty} \mu(|f_{n+1} - f_n| \geq \epsilon_n)) < \infty$, then there exists a measurable function $g : E \to R$ such that $f_n \to g$ in μ-measure and μ a.e.

   Proof Let $\epsilon > 0$. There exists an integer $N \geq 1$ such that $\sum_{n=m}^{\infty} \epsilon_n < \epsilon$ for all $m \geq N$. We first show that for any $m \geq N$ if $\sup_{n \geq m} |f_n(x) - f_m(x)| > \epsilon$, then $|f_{n+1}(x) - f_n(x)| \geq \epsilon_n$ for some $n \geq m$. This is proved by contradiction. Assume that $\sup_{n \geq m} |f_n(x) - f_m(x)| > \epsilon$ but $|f_{n+1}(x) - f_n(x)| < \epsilon_n$ for all $n \geq m$. Then there exists an integer $n_0 \geq m$ such that $|f_{n_0}(x) - f_m(x)| > \epsilon$. Then,

$$\epsilon < |f_{n_0}(x) - f_m(x)| \leq \sum_{n=m}^{n_0-1} |f_{n+1}(x) - f_n(x)| < \sum_{n=m}^{n_0-1} \epsilon_n < \sum_{n=m}^{\infty} \epsilon_n < \epsilon$$

   which is a contradiction. Hence, for all $m \geq N$,

$$\mu(\sup_{n \geq m} |f_n - f_m| \geq \epsilon) \leq \mu\left(\bigcup_{n=m}^{\infty} \{|f_{n+1} - f_n| \geq \epsilon_n\}\right) \leq \sum_{n=m}^{\infty} \mu(|f_{n+1} - f_n| \geq \epsilon_n).$$

   This shows that

$$\lim_{m \to \infty} \mu(\sup_{n \geq m} |f_n - f_m| \geq \epsilon) \leq \lim_{m \to \infty} \sum_{n=m}^{\infty} \mu(|f_{n+1} - f_n| \geq \epsilon_n) = 0.$$

   By Theorem 3.3.7, there exists a measurable function $g : E \to R$ such that $f_n \to g$ in μ-measure and μ a.e.

4. Let f and $\{f_n\}$ be measurable real valued functions on a measure space $(E, \mathcal{B}, \mu)$ such that $f_n \to f$ in μ-measure, and $\sup_{n \geq 1} ||f_n||_{L^1(\mu)} < \infty$. Show that f is μ-integrable, and

$$\lim_{n \to \infty} \big| ||f_n||_{L^1(\mu)} - ||f||_{L^1(\mu)} - ||f_n - f||_{L^1(\mu)} \big| = || \, |f_n| - |f| - |f_n - f| \, ||_{L^1(\mu)} = 0.$$

   Conclude that if $||f_n||_{L^1(\mu)} \to ||f||_{L^1(\mu)} \in \mathbb{R}$, then $||f_n - f||_{L^1(\mu)} \to 0$.

   Proof Choose a subsequence $\{f_{n_m}\}$ such that

$$\lim_{m \to \infty} || \, |f_{n_m}| - |f| - |f_{n_m} - f| \, ||_{L^1(\mu)} = \limsup_{n \to \infty} || \, |f_n| - |f| - |f_n - f| \, ||_{L^1(\mu)}.$$

   Since $f_{n_m} \to f$ in μ-measure, it follows from Theorem 3.3.10 that there exists a subsequence $\{f_{n_{m_i}}\}$ of $\{f_{n_m}\}$ such that $f_{n_{m_i}} \to f$ μ a.e. Then by Fatou's Lemma,

$$\int |f| \, d\mu = \int \liminf_{i \to \infty} |f_{n_{m_i}}| \, d\mu \leq \liminf_{i \to \infty} \int |f_{n_{m_i}}| \, d\mu \leq \sup_n \int |f_n| \, d\mu < \infty.$$

   Thus, f is μ-integrable. By Theorem 3.3.5,

$$\limsup_{n \to \infty} || \, |f_n| - |f| - |f_n - f| \, ||_{L^1(\mu)} = \lim_{i \to \infty} || \, |f_{n_{m_i}}| - |f| - |f_{n_{m_i}} - f| \, ||_{L^1(\mu)} = 0.$$

Thus, $\lim_{n\to\infty} || \, |f_n| - |f| - |f_n - f| \, ||_{L^1(\mu)} = 0$. Since

$$\Big| ||f_n||_{L^1(\mu)} - ||f||_{L^1(\mu)} - ||f_n - f||_{L^1(\mu)} \Big| \leq || \, |f_n| - |f| - |f_n - f| \, ||_{L^1(\mu)}$$

for all n, it follows that

$$\lim_{n\to\infty} \Big| ||f_n||_{L^1(\mu)} - ||f||_{L^1(\mu)} - ||f_n - f||_{L^1(\mu)} \Big| = \lim_{n\to\infty} || \, |f_n| - |f| - |f_n - f| \, ||_{L^1(\mu)} = 0.$$

Finally, if $||f_n||_{L^1(\mu)} \to ||f||_{L^1(\mu)} \in \mathbb{R}$, then $||f_n||_{L^1(\mu)} - ||f||_{L^1(\mu)} \to 0$ and hence $||f_n - f||_{L^1(\mu)} \to 0$.